# Maximal subsemigroups of finite semigroups

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# Maximal subgroups and maximal subsemigroups

## Definition (maximal subgroup)

Let $G$ be a group and let $H$ be a subgroup of $G$. Then $H$ is *maximal* if:

- $H \neq G$.
- $H \lneq U \leq G \Rightarrow U = G$.

## Definition (maximal subsemigroup)

Let $S$ be a semigroup and let $T$ be a subsemigroup of $S$. Then $T$ is *maximal* if:

- $T \neq S$.
- $T \lneq U \leq S \Rightarrow U = S$.

# A more practical definition (computationally)

**Definition (maximal subsemigroup)**

Let $S$ be a semigroup and let $T \leq S$. Then $T$ is *maximal* if:

- $S \neq T$.
- For all $x \in S \setminus T$: $\langle T, x \rangle = S$.

# Getting familiar

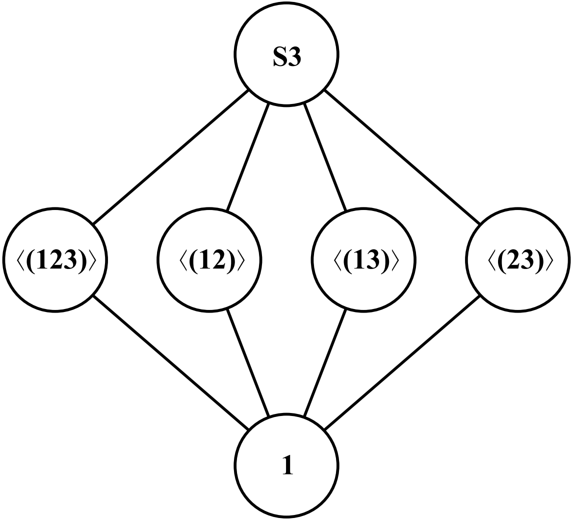
- Maximal sub(semi)groups are *as big as possible* in some sense.
- They let you find all sub(semi)groups.

# Let's make some observations

- Any subsemigroup lacking just a single element is maximal.
- There can be lots.
- Their sizes can differ.
- They exist (at least for finite semigroups*).

# Our first maximal subgroups

Let $G = S_3 = \langle (12), (123) \rangle$.



# Index

Subgroups of prime index are maximal.

# Our first maximal subsemigroups

Let $S = \{0, 1\}$, with multiplication modulo $2$.

# A free semigroup with n generators

Let $S = F_X$, where $|X| = n$.

For example if $X = \{a, b\}$, then $F_X = \{a, b, aa, bb, ab, ba, aaa, bab, ...\}$.

# A null semigroup

Let $N_n$ be the null semigroup with $n$ elements (i.e. $a \cdot b = 0$ for all $a, b$).

# A finite group

A subsemigroup of a finite group is a subgroup*.

So the maximal subsemigroups of a finite group are its maximal subgroups.

Now some pre-requisites.

# An idempotent is idempotent

**Definition (idempotent)**

An element $x$ of a semigroup is *idempotent* if $x^2 = x$. We call such an element *an idempotent*.

# Green's relations

Need to introduce Green's $\mathscr{R}, \mathscr{L}, \mathscr{H}, \mathscr{J}$ relations for a semigroup $S$.

These are equivalence relations defined on the set $S$ as follows:

- $x \mathscr{R} y$ if and only if $xS^1 = yS^1$.
- $x \mathscr{L} y$ if and only if $S^1x = S^1y$.
- $\mathscr{H} = \mathscr{R} \cap \mathscr{L}$.

- $x \mathscr{J} y$ if and only if $S^1xS^1 = S^1yS^1$.

# Egg-box diagram of a semigroup (I)

For an element $x$ in a semigroup $S$, we write $W_x$ to be the $\mathscr{W}$-class of $x$.

- A $\mathscr{J}$-class is *regular* if it contains an idempotent. Else non-regular.
- $\mathscr{J}$-classes form a partition.
- $\mathscr{J}$-classes are unions of $\mathscr{R}$- and $\mathscr{L}$-classes.
- $\mathscr{R}$- and $\mathscr{L}$-classes intersect in $\mathscr{H}$-classes.
- $\mathscr{J}$-classes can be partially ordered:

$$J_x \leq J_y \Leftrightarrow S^1xS^1 \subseteq S^1yS^1$$

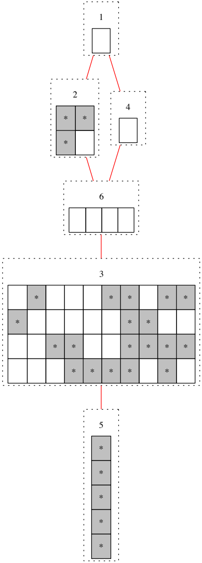
Also note for later that $S^1(xy)S^1 = S^1x(yS^1) \subseteq S^1xS^1$. Hence:

- $J_{xy} \leq J_x$.
- $J_{xy} \leq J_y$ (shown similarly).

# Egg-box diagram of a semigroup (II)

The diagram of the semigroup generated by these three transformations:

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 2 & 5 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 4 & 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 5 & 2 & 5 & 5 \end{pmatrix}$.



# Rees 0-matrix semigroups

Let:

- $I$, $\Lambda$ be finite index sets,
- $T$ be a semigroup,
- $P = (p_{\lambda i})_{\lambda \in \Lambda, i \in I}$ be a $|\Lambda| \times |I|$ matrix over the set $T \cup \{0\}$.

Let $\mathscr{M}^0(T; I, \Lambda; P)$ be the set $(I \times T \times \Lambda) \cup \{0\}$ with multiplication:

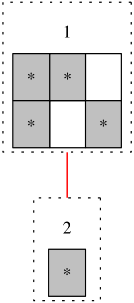
$$(i, s, \lambda) \cdot (j, t, \mu) = \begin{cases} (i, sp_{\lambda j}t, \mu) & \text{if } p_{\lambda j} \neq 0. \\ 0 & \text{otherwise.} \end{cases}$$

and $0 \cdot$ anything is $0$.

Then $\mathscr{M}^0(T; I, \Lambda; P)$ is a *Rees 0-matrix semigroup*.

# An example

$$\mathscr{M}^0[C_2; \{1,2\}, \{1,2,3\}; P].$$



# The principal factor $J^*$

If $J$ is a $\mathscr{J}$-class of a semigroup, define $J^*$, the *principal factor* of $J$, to be the semigroup $J \cup \{0\}$, with multiplication:

$$x * y = \begin{cases} xy & \text{if } x, y, xy \in J. \\ 0 & \text{otherwise.} \end{cases}$$

*The punchline:* if $J$ is a regular $\mathscr{J}$-class, then $J^*$ is (isomorphic to) a Rees 0-matrix semigroup where the underlying semigroup is a group.

# Graham, Graham and Rhodes 1968

Graham, N. and Graham, R. and Rhodes J.
*Maximal Subsemigroups of Finite Semigroups*.
Journal of Combinatorial Theory, 4:203-209, 1968.

Collaboration

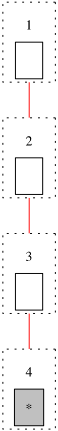
# The main results of GGR '68

Let $M$ be a maximal subsemigroup of a finite semigroup $S$.

1. $M$ contains all but one $\mathscr{J}$-class of $S$, $J$.
2. $M$ intersects every $\mathscr{H}$-class of $S$, or is a union of $\mathscr{H}$-classes.
3. If $J$ is non-regular, then $M = S \setminus J$. Otherwise $J$ is regular.
4. If $M$ doesn't lack $J$ completely, then $M \cap J$ corresponds to a special type of subsemigroup of $J^*$.

# Example: monogenic semigroups (non-group)

$S = \langle a \rangle$. We've done groups.
We've done the infinite monogenic semigroup ($F_{\{a\}} \cong \mathbb{N}$).



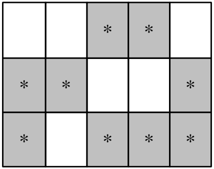
# Maximal subsemigroups of finite zero-simple semigroups (finite regular Rees 0-matrix semigroups over groups)

The theorem tells us to get a maximal subsemigroup we can:

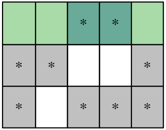
- Remove a whole row of the semigroup.
- Remove a whole column of the semigroup.
- Replace the group by a maximal subgroup.
- Remove the complement of a maximal rectangle of zeroes.

(With certain conditions).

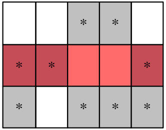
# Removing a row...



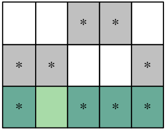
The egg-box diagram of J



Row 1

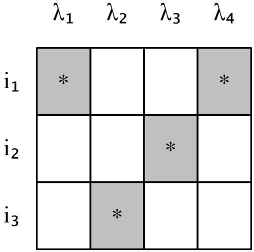


Row 2

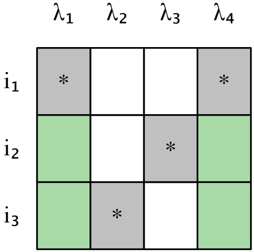


Row 3

# Maximal rectangle of zeroes...

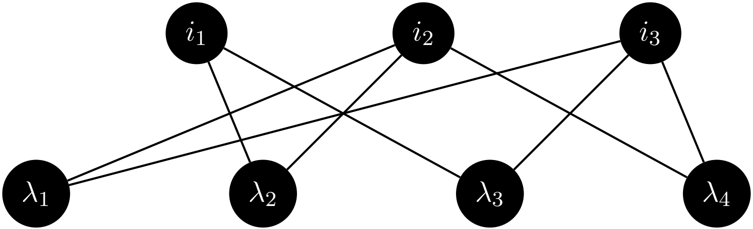



Egg-box diagram

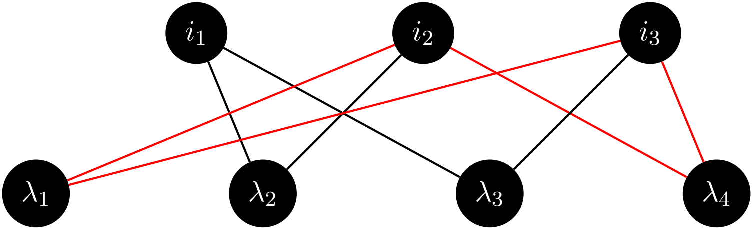



Maximal rectangle

# Create a graph...



# Create a graph...



# How the general MaximalSubsemigroups algorithm works

Suppose $S = \langle X \rangle$, finite, and $X$ is irredundant.

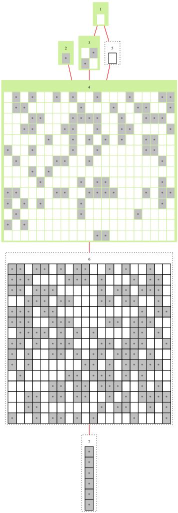
- Every maximal subsemigroup lacks only one $\mathscr{J}$-class, $J$.
- Max. subsemigroups arise from $J \Leftrightarrow J$ contains a generator.
- If $J$ is non-regular, we remove it entirely.
- If $J$ is maximal, then the max. subsemigroups are in one-to-one correspondence with max. subsemigroups of $J^*$.
- If $J$ is non-maximal, it's harder.

# Example: Our semigroup $S$

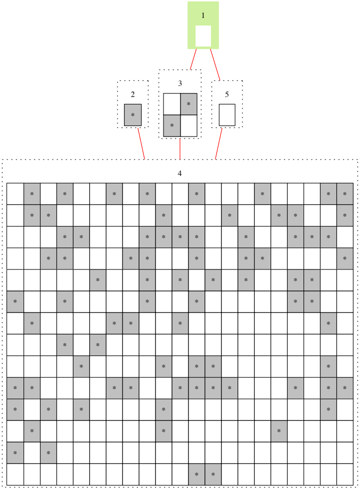
$S$ is the semigroup generated by the following transformations:

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 6 & 5 & 2 & 6 \end{pmatrix}$$
$$\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 5 & 4 & 4 & 3 \end{pmatrix}$$
$$\sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 2 & 2 & 3 & 5 \end{pmatrix}$$
$$\sigma_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 2 & 1 & 3 & 6 \end{pmatrix}$$
$$\sigma_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 1 & 5 & 2 & 1 \end{pmatrix}$$

- $|S| = 2384$
- $S$ is not regular
- $S$ has 7 $\mathscr{J}$-classes.
- 4 $\mathscr{J}$-classes contain generators.



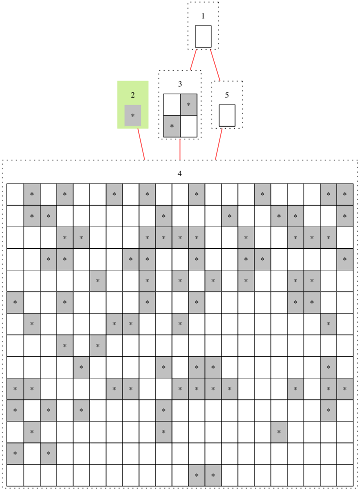
# $\mathscr{J}$-class no. 1: non-regular; maximal




$J_1$ is non-regular.

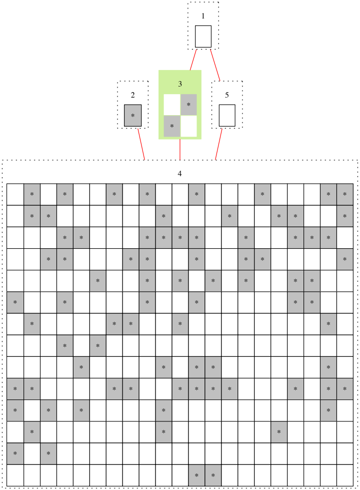
# $\mathscr{J}$-class no. 2: regular; maximal

$J_2^* \cong K_4$ (Klein 4-group).



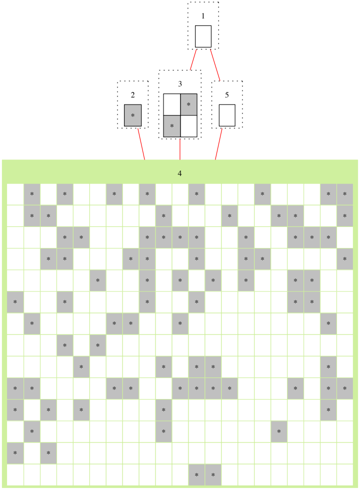
# $\mathscr{J}$-class no. 3: regular; non-maximal

$J_3^* \cong \mathscr{M}^0(C_2; 2, 2; P)$.



# $\mathscr{J}$-class no. 4: regular; non-maximal... and very big

$J_4$ is non-maximal and regular.



End.