Soluble Radicals

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Let G be a finite group and let $\mathrm{sol}(G)$ denote the soluble radical of G, i.e. the largest normal soluble subgroup of G. Paul Flavell conjectured in 2001 that $\mathrm{sol}(G)$ coincides with the set of all elements $x \in G$ such that for any $y \in G$ the subgroup $\langle x, y \rangle$ is soluble. This conjecture has been proved by Guralnick et al. in 2006, using the Classification of Finite Simple Groups ([5]). As a first step towards a proof for this result which does not rely on the Classification, we attempt to show the following:

Theorem A. *Let G be a finite group, let p be a prime and $P \in Syl_p(G)$. Then $P \subseteq sol(G)$ if and only if $\langle P, g \rangle$ is soluble for all $g \in G$.*

In the following let G be a minimal counterexample to Theorem **A**, let p be a prime and let $P \in \mathrm{Syl}_p(G)$ be such that $\langle P, g \rangle$ is soluble for all $g \in G$, but P is not contained in the soluble radical of G. One of the main results so far is

Theorem B. *Suppose that $C_G(P)$ is soluble. Let $\mathcal{L}$ denote the set of maximal P-invariant subgroups M of G such that*
- *$C_G(P) \leq M$,*
- *$[O(F(M)), P] \neq 1$ and*
- *if possible, there exists a prime $q \in \pi(F(M))$ such that $C_{O_q(M)}(P) = 1$.*

If there exists a member $L \in \mathcal{L}$ such that $C_{F(L)}(P)$ is not cyclic, then $\mathcal{L} = \{L\}$.

In [1] it is proved that a group G is p-soluble if and only if for any Sylow p-subgroup P of G, $\langle P, g \rangle$ is p-soluble for all $g \in G$. This result, together with the minimality of G, already implies some restrictions for the structure of G. Let $K := O_{p'}(G)$. Then it turns out that P is cyclic of order p , that $G = PK$ and that K is characteristically simple. Moreover $K = [K, P]$. Whenever $M \in \mathcal{U}_G(P)$ (i.e. M is a P-invariant subgroup of G) is such that $MP < G$, then $[M, P]$ is soluble. So our attention is lead to the maximal P-invariant subgroups of G and we set
$\mathcal{M} := \{M \leq G \mid \text{M is maximally } P\text{-invariant and } MP \neq G\}$.
One of the main ideas is to investigate the structure of the members of $\mathcal{M}$ and how they relate to each other. We first observe that, if $M \in \mathcal{M}$, then $M = P(M \cap K)$. So we have the cyclic p-group P acting on the p'-group $M \cap K$, and coprime action results apply. This yields our first starting point:

Lemma 1. *Let $M \in \mathcal{M}$ be such that $P \nleq Z(M)$. Then there exists a prime q such that $[O_q(M), P] \neq 1$.*

As P is not central in G, we know that $C := C_G(P)$ is contained in a member of $\mathcal{M}$. If moreover C is solube, then whenever $C \leq M \in \mathcal{M}$, it follows that C is properly contained in M and the above lemma is applicable.
In the following, we assume that C is soluble and we focus on the subset $\mathcal{L}$ of $\mathcal{M}$ defined in Theorem **B**, i.e. $\mathcal{L}$ is the set of subgroups $M \in \mathcal{M}$ such that the following hold:

$C_G(P) \leq M$, $[O(F(M)), P] \neq 1$ and if possible, there exists a prime $q \in \pi(F(M))$ such that $C_{O_q(M)}(P) = 1$.
As mentioned above, C being soluble implies that the members of $\mathcal{L}$ contain C properly. So the second hypothesis for $\mathcal{L}$ is basically a statement about the prime 2, avoiding technical difficulties. The last hypothesis also is of a purely technical nature.
When collecting information about the elements in $\mathcal{L}$, then, unsurprisingly, the Bender Method turns out to be very useful. We refer the reader to [4] (p.110 et seq.) where a detailed exposition of it can be found. Very little work has to be done to make sure that the results can be applied in our context (where G is not simple!). The Bender Method can be brought into the picture because of the following result, due to Paul Flavell (Theorem 4.2 in [3]).

Pushing Down Lemma. *Let $M \in \mathcal{M}$. If q is odd and if Q is a C-invariant q-subgroup of G contained in M, then $[Q, P] \leq O_q(M)$.*

The stated version is a special case of Flavell's result, avoiding technical problems related to the prime 2 (and Fermat Primes).
To make sure that two members L_1, L_2 of $\mathcal{L}$ cannot have characteristic q for the same prime q, we apply results from [2]. In fact, this is the only place so far where the solubility of C plays a major role. Then we can successfully apply the Bender Method in order to prove uniqueness results. We start by showing that, for any $M \in \mathcal{L}$, the normaliser of certain C-invariant subgroups of $F(M)$ is contained in a unique member of $\mathcal{M}$.
The penultimate step is

Lemma 2. *Let $M \in \mathcal{L}$, suppose that $|\pi(F(M))| \geq 2$ and that $q \in \pi$ is such that $C_{O_q(M)}(P)$ possesses an elementary abelian subgroup A of order q^2. Then $B := C_{F(M)}(A)$ is contained in a unique member of $\mathcal{M}$. In particular, $C_G(a)$ is contained in a unique member of $\mathcal{M}$ (namely M) for all $a \in A^{\#}$.*

Theorem **B** follows from this by applying the Bender Method. So suppose that $L \in \mathcal{L}$ is such that $C_{F(L)}(P)$ is not cyclic. If $|\pi(F(L))| \geq 2$, then we can apply the previous lemma and obtain the result with tools related to coprime action. If $|\pi(F(L))| = 1$, then the analysis is more difficult and more complicated arguments arise. The main idea is to find a replacement for the previous lemma for this configuration. Theorem **B** can be read in a different way:
If $\mathcal{L}$ possesses more than one element, then for all $L \in \mathcal{L}$ the subgroup $C_{F(L)}(P)$ is cyclic. The next objective is to exclude this case. Then $\mathcal{L}$ has at most one member, and if $\mathcal{L}$ is empty, this gives strong information about the members of $\mathcal{M}$ containing C, hopefully leading to a contradiction.

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