Special primitive pairs in finite groups

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Abstract. This note is concerned with a particular kind of primitive pairs in finite groups. Applications of the results that are proved here play a key role in the author's work towards a new proof of Glauberman's Z*-Theorem.

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1. Introduction

It is well known that if a finite group G admits automorphisms of order coprime to $|G|$, then knowledge about these automorphisms and their fixed points can be used to investigate the structure of G itself. Sometimes, arguments of that type can be mimicked in subgroups of G without any hypothesis on coprime action. For example, in the context of the Z*-project (see [5]), the idea is very often to study proper subgroups of a group and, in there, 2-groups that behave as if they were acting coprimely although they are not. Then it becomes necessary to control certain maximal subgroups of characteristic q, for some odd prime q, when neither Glauberman's ZJ-Theorem nor his K^∞-Theorem nor the recent results by Paul Flavell on primitive pairs could be applied directly. In the Z*-project, this occurs when many maximal subgroups containing involution centralisers have characteristic q, but also in the analysis of different maximal subgroups containing a fixed involution centraliser. In these applications, the results of this paper are used to exclude the existence of primitive pairs with very particular properties. It turned out however that the underlying theorems hold in much more generality and, already, applications for other problems arise.
In order to state the main results, we need the notion of an A-special primitive pair of characteristic q. From now on, G always denotes a finite group.

Definition 1.1. Suppose that H_1, H_2 are distinct proper subgroups of G and that $A \leq H_1 \cap H_2$. Let $\pi := \pi(A)$ and let $q \in \pi'$ be a prime. Then we say that the pair (H_1, H_2) is an **A-special primitive pair of characteristic q of G** if the following hold:

- For all $i \in \{1,2\}$, if $1 \neq X \trianglelefteq H_i$, then $N_G(X) = H_i$;
- for all $i \in \{1,2\}$, we have that $F^*(H_i) = O_q(H_i)$;
- $C_G(A) \leq H_1 \cap H_2$ and
- $A \leq Z^*_\pi(H_1) \cap Z^*_\pi(H_2)$.

Here the subgroup $Z^*_\pi(H_i)$ (with $i \in \{1,2\}$) denotes the full pre-image of $Z(H_i/O_{\pi'}(H_i))$ in H_i. The above definition is inspired by the notion of a primitive pair of characteristic q as, for example, in [4] on page 262. The special requirements on A compensate for the fact that we might not have coprime action and that we do not impose any solubility or stability hypothesis.

Theorem I.
Suppose that A is a subgroup of G and let $\pi := \pi(A)$. Suppose that $q \in \pi'$ is a prime, that $O_q(G) = 1$ and that, whenever $AC_G(A) \leq H < G$, then $\widehat{H} := H/O_{\pi'}(H)$ has a unique maximal $\widehat{AC_G(A)}$-invariant q-subgroup.
If (H_1, H_2) is an A-special primitive pair of characteristic q of G and if $2 \in \pi$ or $q \geq 5$, then $O_q(H_1) \cap H_2 = 1 = O_q(H_2) \cap H_1$.

The proof is short and uses arguments from so-called local group theory. An important special case occurs for centralisers of elements of prime order.

Theorem II.
Suppose that p and q are distinct primes, that $O_q(G) = 1$ and that $x \in G$ is an element of order p. Suppose further that, whenever $C_G(x) \leq H < G$, then $x \in Z^*_p(H)$.
If (H_1, H_2) is an $\langle x \rangle$-special primitive pair of characteristic q of G and if $p = 2$ or $q \geq 5$, then $O_q(H_1) \cap H_2 = 1 = O_q(H_2) \cap H_1$.

This is a consequence of Theorem I. The statements could be extended to include the case $q = 3$ under some additional stability hypothesis that we chose not to invoke. A generalisation of Flavell's result (Theorem 2.6 below) to the prime 3 would also imply that Theorems I and II include the case $q = 3$, but neither such a generalisation nor a counterexample is known to the author at present.
In this note, all groups are meant to be finite and we use the notation that is standard in books such as [1] and [4].

2. Preliminaries

Throughout this section let X be a group, let π be a set of primes and let r be a prime number.

2.1. Notation

– We say that a proper subgroup H of X is **primitive** if, for all non-trivial normal subgroups A of H, we have that $N_X(A) = H$. The standard examples are maximal subgroups of simple groups.

– $O_{\pi'}(X)$ is the largest normal π'-subgroup of X. Moreover $Z^*_\pi(X)$ denotes the full pre-image of $Z(X/O_{\pi'}(X))$ in X and $O_{\pi',\pi}(X)$ denotes the full pre-image of $O_\pi(X/O_{\pi'}(X))$ in X.

– X is **of characteristic** r if $F^*(X) = O_r(X)$. It is equivalent to say that $C_X(O_r(X)) \leq O_r(X)$ because $F^*(X)$ always contains its centraliser (see for example Theorem 6.5.8 in [4]).

- If X is an r-group, then by $K^\infty(X)$ we denote the characteristic subgroup of X introduced by Glauberman in [3]. The exact definition is omitted here because it is extremely technical and does not play any role for the arguments presented.

– Suppose that $U \leq X$. Then by $\text{И}_X(U,\pi)$ we denote the set of U-invariant π-subgroups of X. We write $\text{И}^*_X(U,\pi)$ for the set of maximal members of $\text{И}_X(U,\pi)$ with respect to inclusion and, for simplicity, we write $\text{И}_X(U,r)$ instead of $\text{И}_X(U,\{r\})$.

2.2. General results

Lemma 2.1. *Let P be a π-group that acts on a π'-group Q.*

(1) *We have that $Q = [Q,P]C_Q(P)$ and $[Q,P] = [Q,P,P]$. Moreover if Q is abelian, then $Q = [Q,P] \times C_Q(P)$.*

(2) *Let $r \in \pi'$. Then $\text{И}^*_Q(P,r) \subseteq Syl_r(Q)$ and $C_{QP}(P)$ is transitive on $\text{И}^*_Q(P,r)$.*

Proof. These results are contained in [4], they correspond to 8.2.3, 8.2.7 and 8.4.2. □

Lemma 2.2. *Let P be a π-group that acts on a π'-group Q. Let $X := QP$ and $r \in \pi'$. Let R denote the intersection of all P-invariant Sylow r-subgroups of Q. Then R is the unique maximal $PC_X(P)$-invariant r-subgroup of Q.*

Proof. As $PC_X(P)$ permutes the elements of $\text{И}^*_Q(P,r)$, the subgroup R is $PC_X(P)$-invariant. Let $T_0 \in \text{И}_Q(PC_X(P),r)$ be arbitrary. Then T_0 lies in some P-invariant Sylow r-subgroup T of Q. Let S be an arbitrary P-invariant Sylow r-subgroup of Q. Then by Lemma 2.1 (2) there exists an element $x \in C_X(P)$ such that $T^x = S$. As T_0 is $PC_X(P)$-invariant, we have that $T_0 = T_0^x \leq T^x = S$. It follows that T_0 is contained in every P-invariant Sylow r-subgroup of Q and hence in R. In particular, if $T_0 \in \text{И}^*_Q(PC_X(P),r)$, then we see that $T_0 = R$. □

Lemma 2.3. *Suppose that A is a π-subgroup of $Z^*_\pi(X)$. Then $O_{\pi'}(C_X(A)) \le O_{\pi'}(X)$.*

Proof. As $A \le Z^*_\pi(X)$, we have that $X = C_X(A)O_{\pi'}(X)$. Let $\overline{X} := X/O_{\pi'}(X)$. Then $\overline{A} \le Z(\overline{X})$ and therefore $\overline{X} = C_{\overline{X}}(\overline{A}) = \overline{C_X(A)}$. We conclude that

$$O_{\pi'}(\overline{C_X(A)}) = O_{\pi'}(\overline{X}) = 1$$

and hence $O_{\pi'}(C_X(A)) \le O_{\pi'}(X)$ as stated. □

Lemma 2.4. *Suppose that $r \notin \pi$ and that A is a π-subgroup of $Z^*_\pi(X)$. Then X has a unique maximal $AC_X(A)$-invariant r-subgroup R and $O_r(X)O_r(C_X(A)) \le R \le O_{\pi'}(X)$.*

Proof. Let $Y \in \mathcal{U}_X(AC_X(A), r)$ be arbitrary. The coprime action of A on Y yields that $Y = C_Y(A)[Y, A]$, with Lemma 2.1(1). As $C_Y(A)$ is a $C_X(A)$-invariant r-subgroup of $C_X(A)$ and $r \notin \pi$, Lemma 2.3 gives that

$$C_Y(A) \le O_{\pi'}(C_X(A)) \le O_{\pi'}(X).$$

We also see that $[Y, A] \le Y \cap Z^*_\pi(X) \le O_{\pi'}(X)$ because A is a subgroup of $Z^*_\pi(X)$ and Y is a π'-group. Therefore every member of $\mathcal{U}_X(AC_X(A), r)$ lies in $O_{\pi'}(X)$. Together with the coprime action of A on $O_{\pi'}(X)$ and Lemma 2.2 this implies that the intersection R of all A-invariant Sylow r-subgroups of $O_{\pi'}(X)$ is the unique maximal $AC_X(A)$-invariant r-subgroup of X. As $O_r(X)O_r(C_X(A))$ is, of course, an $AC_X(A)$-invariant r-subgroup of X, it is contained in R as stated. □

Theorem 2.5. *Suppose that X has odd order and let R be an r-subgroup of X containing $O_r(X)$. If X has characteristic r, then $K^\infty(R)$ is normal in X.*

Proof. This is a special case of Theorem A in [3]. □

Theorem 2.6. *Suppose that the group A acts coprimely on X and that X has characteristic r for some prime $r \ge 5$. Let R denote the unique maximal $AC_X(A)$-invariant r-subgroup of X. Then $K^\infty(R)$ is normal in X.*

Proof. This is Theorem **A** in [2]. □

Lemma 2.7. *Suppose that A is an r'-subgroup of X and set $\pi := \pi(A)$. Let $H \le X$ be such that the following hold:*

- *$C_X(A) \le H$;*
- *$A \le Z^*_\pi(H)$;*
- *H is primitive and of characteristic r and*
- *$2 \in \pi$ or $r \ge 5$.*

Then H has a unique maximal $C_X(A)$-invariant r-subgroup R, moreover $H = N_X(K^\infty(R))$ and $R \in \mathcal{U}^*_X(C_X(A), r)$.

Proof. Our hypothesis $A \leq Z^*_\pi(H)$ implies that $\text{И}_H(C_H(A), r) = \text{И}_H(AC_X(A), r)$ has a unique maximal element R, by Lemma 2.4. Now let $R_0 := K^\infty(R)$. Then Theorem 2.5 (if $2 \in \pi$) or Theorem 2.6 (if $r \geq 5$) yield that $R_0 \trianglelefteq RO_{\pi'}(H)$ and that, in particular, $O_{\pi'}(H)$ normalises R_0. But also, R_0 is $C_H(A)$-invariant and therefore $C_X(A)$-invariant. As $A \leq Z^*_\pi(H)$, we have that $H = C_X(A)O_{\pi'}(H)$ whence H normalises R_0. Thus $N_X(R_0) = H$ because $R_0 \neq 1$ and H is primitive. For the last statement let $R \leq R^* \in \text{И}^*_X(C_X(A), r)$. Then $N_{R^*}(R) \leq N_X(R_0) \leq H$ and $N_{R^*}(R)$ is $C_X(A)$-invariant which means that $N_{R^*}(R)$ lies in the unique member R of $\text{И}^*_H(C_X(A), r)$. Therefore $N_{R^*}(R) = R$ and it follows that $R = R^*$. □

3. Proofs of the theorems

On our way to proving Theorem I we work under the following

Hypothesis 3.1.

- G is a finite group and q is a prime such that $O_q(G) = 1$;
- A is a q'-subgroup of G and $\pi := \pi(A)$;
- whenever $AC_G(A) \leq H < G$, then $\widehat{H} := H/O_{\pi'}(H)$ has a unique maximal $\widehat{AC_G(A)}$-invariant q-subgroup ;
- $2 \in \pi$ or $q \geq 5$ and
- $H_1, H_2 \leq G$ are proper subgroups of G such that (H_1, H_2) is an A-special primitive pair of characteristic q of G. (In particular A is abelian.)

Lemma 3.2. *Suppose that Hypothesis* 3.1 *holds. Let* $Q_1, Q_2 \in \text{И}^*_G(C_G(A), q)$ *and suppose that* $Q_1 \cap Q_2 \neq 1$. *Then* $Q_1 = Q_2$.

Proof. Let us assume that this is false and choose Q_1, Q_2 to be distinct members of $\text{И}^*_G(C_G(A), q)$ such that $D := Q_1 \cap Q_2 \neq 1$ is as large as possible. Since $O_q(G) = 1$ by hypothesis, we find a maximal subgroup H of G containing $N_G(D)$. As $D \neq Q_1$, we may choose $R_1 \in \text{И}^*_H(C_G(A), q)$ such that $D < N_{Q_1}(D) \leq R_1$. Then we let $R_1 \leq R_1^* \in \text{И}^*_G(C_G(A), q)$ and see that $D < Q_1 \cap R_1^*$, hence our choice of Q_1 and Q_2 forces $Q_1 = R_1^*$. In particular, this means that $R_1 \leq Q_1$. Arguing similarly for some $R_2 \in \text{И}^*_H(C_G(A), q)$ containing $N_{Q_2}(D)$ and for some $R_2^* \in \text{И}^*_G(C_G(A), q)$ with $R_2 \leq R_2^*$, we also have that $D < Q_2 \cap R_2^*$ whence $Q_2 = R_2^*$ and $R_2 \leq Q_2$.
By hypothesis, H has a unique maximal $C_G(A)$-invariant q-subgroup Q modulo $O_{\pi'}(H)$ and therefore $QO_{\pi'}(H)$ contains $R_1O_{\pi'}(H)$ and $R_2O_{\pi'}(H)$. Now we let $W := QO_{\pi'}(H)C_G(A)$ and we observe that $QO_{\pi'}(H) \leq O_{\pi'}(W)$ and hence $W = O_{\pi'}(W)C_G(A)$. In particular, $A \leq Z^*_\pi(W)$. Now Lemma 2.4 is applicable and yields that W has a unique maximal $C_G(A)$-invariant q-subgroup. But also, we chose $R_1, R_2 \in \text{И}^*_H(C_G(A), q)$ and since $R_1, R_2 \leq W$, this implies that $R_1, R_2 \in \text{И}^*_W(C_G(A), q)$. Then uniqueness forces $R_1 = R_2$. Therefore $N_{Q_1}(D) \leq R_1 = R_1 \cap R_2 \leq Q_1 \cap Q_2 = D$ whence $D = N_{Q_1}(D)$. It follows that $D = Q_1$ and hence $Q_1 = Q_2$, which is a contradiction. □

Proof of Theorem I.
We suppose that Hypothesis 3.1 holds, we assume further that $O_q(H_1) \cap H_2 \neq 1$ or $O_q(H_2) \cap H_1 \neq 1$ and we work towards a contradiction.
By Lemma 2.4 we know that $O_q(H_1)$ lies in the unique maximal $C_G(A)$-invariant q-subgroup Q_1 of H_1 and that $O_q(H_2)$ lies in the unique maximal $C_G(A)$-invariant q-subgroup Q_2 of H_2. Our hypotheses that $2 \in \pi$ or $q \geq 5$ and that H_1 and H_2 are primitive and of characteristic q also yield that $Q_1, Q_2 \in \text{И}^*_G(C_G(A), q)$, with Lemma 2.7. The subgroups $O_q(H_1) \cap H_2$ and $O_q(H_2) \cap H_1$ are $C_G(A)$-invariant q-subgroups of H_1 as well as H_2 and therefore lie in Q_1 and in Q_2. In particular, as one of those intersections is non-trivial by our assumption, we see that $Q_1 \cap Q_2 \neq 1$ and so Lemma 3.2 forces $Q_1 = Q_2$. Then Lemma 2.7 implies that $H_1 = N_G(K^\infty(Q_1)) = H_2$, which is a contradiction. □

Proof of Theorem II.
Suppose that the hypotheses from Theorem II hold and suppose that $C_G(x) \leq H < G$. Then $x \in Z_p^*(H) \leq O_{p',p}(H)$ and Lemma 2.4, applied for $\pi = \{p\}$, yields that H has a unique maximal $C_H(x)$-invariant q-subgroup. As $C_G(x) = C_H(x)$, it follows that $\widehat{H} := H/O_{p'}(H)$ has a unique maximal $\widehat{C_G(x)}$-invariant q-subgroup. This means that the hypotheses from Theorem I are satisfied and we conclude that, if (H_1, H_2) is an $\langle x \rangle$-special primitive pair of characteristic q in G, then $O_q(H_1) \cap H_2 = 1 = O_q(H_2) \cap H_1$ as stated. □

Example.
Let q^k be a prime power. Let $G := SL_3(q^k)$ and let $\lambda \in GF(q^k)$ be an element of prime order p, $p \neq 3$. Let $x \in G$ be represented by the matrix

$$\begin{pmatrix} \lambda^{-2} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Then $C_G(x)$ is isomorphic to $GL_2(q^k)$; it is contained in the maximal parabolic subgroups

$$H_1 := \begin{pmatrix} * & 0 \\ * & GL_2(q^k) \end{pmatrix}$$

and

$$H_2 := \begin{pmatrix} * & * \\ 0 & GL_2(q^k) \end{pmatrix}.$$

Looking at $\overline{G} := PSL_3(q^k)$, the pair $(\overline{H_1}, \overline{H_2})$ exhibits an example to illustrate Theorem II:
the subgroups $\overline{H_1}$ and $\overline{H_2}$ both contain $C_{\overline{G}}(\overline{x})$, they are primitive and have characteristic q and, whenever we see $C_{\overline{G}}(\overline{x})$ in a proper subgroup $\overline{M}$ of $\overline{G}$,

then we have $\overline{x} \in Z_p^*(\overline{M})$. Thus $(\overline{H_1}, \overline{H_2})$ forms an $\overline{\langle x \rangle}$-special primitive pair of characteristic q and $O_q(\overline{H_1}) \cap \overline{H_2} = 1 = O_q(\overline{H_2}) \cap \overline{H_1}$.

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