Finite simple $3'$-groups are cyclic or Suzuki groups

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In this note we prove that all finite simple $3'$-groups are cyclic of prime order or Suzuki groups. This is well known in the sense that it is mentioned frequently in the literature, often referring to unpublished work of Thompson. Recently an explicit proof was given by Aschbacher ([3]), as a corollary of the classification of $\mathcal{S}_3$-free fusion systems. We argue differently, following Glauberman's comment in the preface to the second printing of his booklet [8]. We use a result by Stellmacher [12] and instead of quoting Goldschmidt's result in its full strength, we give explicit arguments along his ideas in [10] for our special case of $3'$-groups.

1. Preliminaries

In this paper, by "group" we always mean a finite group, and we use standard notation as in [11]. We also quote some background results from this monograph and we use without further reference that groups of odd order are soluble (see [5]).

Definition 1.1. *Let p be a prime. We say that a p-subgroup A of G is* **strongly closed in** G *if, whenever $A \leq S \in Syl_p(G)$ and $g \in G$, then $A^g \cap S \leq A$.*

In this note we only consider strongly closed 2-groups, and we begin by collecting and proving a number of well known facts.

Lemma 1.2. *Suppose that A is a strongly closed 2-subgroup of G.*
(1) If $A \leq H \leq G$, then A is strongly closed in H.
(2) If $A \leq T \leq S \in Syl_2(G)$, then $N_G(T) \leq N_G(A)$.
(3) If $N \trianglelefteq G$, then $A \cap N$ is strongly closed in G.
(4) Let $N \trianglelefteq G$ and $\overline{G} := G/N$. Then $\overline{A}$ is strongly closed in $\overline{G}$.
(5) $N_G(A)$ controls fusion of its 2-elements in G.
(6) $G = N_G(A) \cdot \langle A^G \rangle$.

Proof. (1) is clear from the definition. For the remainder of the proof let $A \leq S \in \mathrm{Syl}_2(G)$. In (2) let $A \leq T \leq S$ and $x \in N_G(T)$. Then $A^x = (A \cap T)^x = A^x \cap T \leq A^x \cap S \leq A$ because A is strongly closed. Thus $A^x = A$. For (3) we let $x \in G$. Set $A_0 := A \cap N$ and $T := S \cap N$. Then $A_0 \leq T \in \mathrm{Syl}_2(N)$ and $A_0^x \cap T \leq A^x \cap T \leq A^x \cap S \leq A$ because A is strongly closed in G. Hence $A_0^x \leq A \cap N = A_0$. Now we turn to (4). Let $\overline{a} \in \overline{A}$ and $\overline{g} \in \overline{G}$ be such that $\overline{a}^{\overline{g}} \in \overline{S}$. Then $\overline{a^g} \in \overline{S}$ whence $a^g \in S \cdot N$. By Sylow's Theorem we may suppose that $a^g \in S$, so $a^g \in A^g \cap S \leq A$ because A is strongly closed in G. In particular $\overline{a}^{\overline{g}} \in \overline{A}$ and hence $\overline{A}$ is strongly closed in $\overline{G}$. For (5) let $x, y \in N_G(A)$ be 2-elements and let $g \in G$ be such that $x^g = y$. Then $S \leq N_G(A)$ by (2) and hence $S \in \mathrm{Syl}_2(N_G(A))$, so we may suppose that $x, y \in S$. By Theorem 6.1 in [7] there exist elements $g_1 \in C_G(x)$ and $g_2 \in N_G(A)$ such that $g = g_1 g_2$, so $y = x^g = x^{g_2}$. In (6) we let $N := \langle A^G \rangle$

and $T := S \cap N$. Then $A \leq T$ and a Frattini argument together with (2) yields that $G = N \cdot N_G(T) = N \cdot N_G(A)$. $\square$

Lemma 1.3. *Suppose that p is prime and that P is a p-subgroup of G.*
(1) If X is a p'-subgroup of G and P normalises X, and if moreover P is an elementary abelian, non-cyclic p-group, then $X = \langle C_X(a) \mid a \in P^\# \rangle$.
(2) If N is a normal p'-subgroup of G, then $C_G(P)N/N = C_{G/N}(PN/N)$.

Proof. These are Proposition 11.13 and Lemma 4.2 (i) in [11]. $\square$

Theorem 1.4. *Suppose that, for all sections H^* of G and all $S^* \in Syl_2(H^*)$, there is defined a characteristic subgroup $W(S^*)$ of S^* such that, if $S^* \neq 1$, then $W(S^*) \neq 1$. Suppose further that, whenever H^* is a section of G and $F^*(H^*) = O_2(H^*)$, then $W(S^*) \trianglelefteq H^*$. Then for all $T \in Syl_2(G)$, the subgroup $N_G(W(T))$ controls G-fusion in T.*

Proof. We define a map W that assigns, for all sections H^* of G^*, to each $S^* \in \mathrm{Syl}_2(H^*)$ the subgroup $W(S^*)$ that exists by hypothesis. This is a section conjugacy functor in the sense of [6] (page 15), so the additional hypotheses and Theorem 6.6 and Lemma 5.7 in [6] yield the result. $\square$

Lemma 1.5. *Suppose that $f \in \mathbb{N}$ and that $G \simeq Sz(2^f)$ is simple. Let $S \in Syl_2(G)$. Then $A := \Omega_1(S) = Z(S)$ has order $2^f \geq 8$ with f odd and:*
(1) The only strongly closed abelian 2-subgroups of G are A and its G-conjugates.
(2) $N_G(S)$ is a Frobenius group with cyclic Frobenius complement of order $q-1$ that acts regularly on $A^\#$. Moreover $C_G(a) = C_G(A) = S$ for all $a \in A^\#$.
(3) If E is a quasi-simple group such that $E/Z(E) \simeq G$, then E is simple or $2^f = 8$ and $Z(E)$ is elementary abelian of order 2 or 4.
(4) The group of outer automorphisms of G is of odd order.

Proof. For (3) see the main results of [1]. The other statements can be found in [13], see § 4 and Theorems 9 and 11. $\square$

Lemma 1.6. *Suppose that $a, b \in G$ are commuting involutions and that the only non-abelian composition factors of $C_G(b)$ are Suzuki groups.*
Then $O(C_G(a)) \cap C_G(b) \leq O(C_G(b))$.

Proof. Let $D := O(C_G(a)) \cap C_G(b)$ and assume that $D \nleq O(C_G(b))$. Let $H := C_G(b)$ and let $\overline{H} := H/O(H)$. Then $1 \neq \overline{D} \leq O(C_{\overline{H}}(\overline{a}))$ and hence we may apply (2.6) of [9] to $\overline{H}$ and $\overline{a}$. Let $E \leq H$ be such that $\overline{E}$ is a component of $\overline{H}$ that is normalised, but not centralised by $\overline{D}$ and by $\overline{a}$. By hypothesis and Lemma 1.5 (2),(4) we deduce first that a induces an inner automorphism of order 2 on $\overline{E}$ and then that $C_{\overline{E}}(\overline{a})$ is a 2-group. It follows that $[C_{\overline{E}}(\overline{a}), \overline{D}] \leq C_{\overline{E}}(\overline{a}) \cap O(C_{\overline{H}}(\overline{a})) = 1$. This is impossible because $\overline{D}$ induces field automorphisms or inner automorphisms of odd order on $\overline{E}$. $\square$

2. Strongly closed abelian subgroups in $3'$-groups

Definition 2.1. *We say that a quasi-simple group E* **is of Suzuki type** *if and only if $E/Z(E)$ is isomorphic to a Suzuki group. Moreover, we say that the pair (G, A)* **has property (sc)** *if and only if there exists $S_0 \in Syl_2(\langle A^G \rangle)$ such that the following conditions hold:*

(1) *A is an abelian subgroup of S_0 and A is strongly closed in G.*
(2) *$\langle A^G \rangle / O(\langle A^G \rangle)$ is a central product of an abelian 2-group and quasi-simple subgroups of Suzuki type.*
(3) *If $\overline{\langle A^G \rangle} := \langle A^G \rangle / O(\langle A^G \rangle)$, then $\overline{A} = O_2(\overline{\langle A^G \rangle}) \cdot \Omega_1(\overline{S_0})$.*

Theorem 2.2. *Suppose that G is a $3'$-group, that every proper non-abelian simple section of G is isomorphic to a Suzuki group and that A is a strongly closed abelian 2-subgroup of G. Then (G, A) has property (sc).*

For the remainder of this section, we assume that Theorem 2.2 is false and we work towards a contradiction. We assume that G and A satisfy the hypothesis, but that (G, A) does not have property (sc) and we choose first G and then A of minimal order subject to these constraints. Let $A \le S \in \mathrm{Syl}_2(\langle A^G \rangle)$. Throughout, we follow Goldschmidt's arguments in [10], mainly Sections 4 and 7, and often with simplifications.

Lemma 2.3. *$O(G) = 1$, $G = \langle A^G \rangle$ and $Z(G) = 1$. In particular $S \in Syl_2(G)$.*

Proof. Assume that $O(G) \neq 1$ and let $\overline{G} := G/O(G)$. Then $\overline{A}$ is strongly closed in $\overline{G}$ by Lemma 1.2 (4) and hence the minimal choice of G forces $(\overline{G}, \overline{A})$ to satisfy property (sc). But $O(\langle \overline{A}^{\overline{G}} \rangle) \le O(\overline{G}) = 1$, so (G, A) also has property (sc) contrary to our choice of G. Thus $O(G) = 1$.
Next let $G_0 := \langle A^G \rangle$ and assume that $G_0 \neq G$. Then $A \le G_0$ and $G = G_0 \cdot N_G(A)$ by Lemma 1.2 (6). Thus $G_0 = \langle A^{G_0} \rangle$. The minimal choice of G yields that (G_0, A) satisfies property (sc) and then (G, A) does too. This is impossible.
Finally we assume that $Z(G) \neq 1$. Set $\tilde{G} := G/Z(G)$. We note that $Z(G)$ is a 2-group because $O(G) = 1$. Let $U \trianglelefteq G$ be such that $Z(G) \le U$ and $\tilde{U} = O(\tilde{G})$. Then U has a central Sylow 2-subgroup and therefore U has a normal 2-complement (e.g. Proposition 16.5 in [11]). But $O(U) \le O(G) = 1$ and therefore $U = Z(G)$. This means that $O(\tilde{G}) = 1$ and in particular $O(\langle \tilde{A}^{\tilde{G}} \rangle) = 1$. Now $\tilde{A}$ is strongly closed in $\tilde{G}$ by Lemma 1.2 (4) and hence, by minimality of G, we see that $(\tilde{G}, \tilde{A})$ has property (sc). Then (G, A) has property (sc) as well, which is a contradiction. □

Lemma 2.4. *$E(G) \neq 1$.*

Proof. Assume that $E(G) = 1$. Then $F^*(G) = O_2(G)$ by Lemma 2.3 and in particular $Z(S) \le C_G(O_2(G)) \le O_2(G)$. Now $A \trianglelefteq S$ by Lemma 1.2 (2), so $1 \neq A \cap Z(S) \le A \cap O_2(G) =: N$. Let $g \in G$. Then $N^g = (A \cap O_2(G))^g = A^g \cap O_2(G) \le A^g \cap S \le A$ because A is strongly closed, hence $N^g = N$ and thus $N \trianglelefteq G$. But $N \leq A$ and A is abelian, so $N \leq C_G(A)$. Therefore, by Lemma 2.3, also $N \leq C_G(\langle A^G \rangle) = Z(G) = 1$. This is a contradiction. $\square$

Lemma 2.5. *All components of G are normal in G and simple.*

Proof. Let $A_0 := A \cap E(G)$ and let $L := \langle A_0^{E(G)} \rangle$. Then $L \trianglelefteq E(G)$and hence $L \leq Z(E(G))$ or L is a (non-empty) product of components of G. In the first case L contains no component of G. In the second case L is a non-empty product of components (in particular $A_0 \not\leq Z(L)$) and we let $E \leq L$ be a component of G. Then $E \trianglelefteq L$ and therefore A_0 normalises E. Moreover A_0 is strongly closed in $E(G)$ and hence in L by Lemma 1.2 (3) and (1). By hypothesis E is of Suzuki type, so A_0 induces inner automorphisms on E and centralises a Sylow 2-subgroup of E, by Lemma 1.5. It follows that A_0 lies in the centre of a Sylow 2-subgroup of L and hence A normalises every component of G contained in L. Next let X denote the product of all components of G that are not contained in L. As L is A-invariant, it also follows that X is A-invariant. Moreover $[L, X] = 1$. Let T denote an A-invariant Sylow 2-subgroup of X and let $AT \leq S \in \mathrm{Syl}_2(E(G))$. As $A \trianglelefteq S$ by Lemma 1.2(2), we see that $[A, T] \leq A \cap T \leq A_0 \cap X \leq L \cap X \leq Z(E(G))$ and in particular every component of G that is contained in X is A-invariant. Consequently A normalises every component of G.
As $G = \langle A^G \rangle$ by Lemma 2.3, all components of G are normal in G and, by hypothesis, of Suzuki type. The possible (non-trivial) Schur multipliers have order 2 or 4 (Lemma 1.5), and we also know that $Z(G) = 1$, so $Z(E(G))$ contains no involution that is central in G. Together with the fact that G is a $3'$-group, this implies that all components are simple. $\square$

Lemma 2.6. *G is simple.*

Proof. Assume otherwise and, with Lemma 2.4, let E be a component of $G = \langle A^G \rangle$ such that $A \neq C_A(E)$. Then E is simple and normal in G by Lemma 2.5.
Since E is normal in G the Sylow 2-subgoup S induces inner automorphisms on E by Lemma 1.5 (4). Hence $A \leq S = C_S(E) \cdot (E \cap S) \leq C_G(E) \cdot E \trianglelefteq G$ and thus $G = \langle A^G \rangle = C_G(E) \cdot E$. Let $T := E \cap S$ and $a \in A^\#$. Then $a \in C_S(E) \cdot T$, so there are $c \in C_S(E)$ and $b \in T^\#$ such that $a = c \cdot b$. By Lemma 1.5 (2) there exists $e \in N_E(T) \setminus C_E(b)$. Since A is strongly closed in G, we see that $a^e = c^e b^e = cb^e \in S \cap A^e \leq A$ and $1 \neq b^e b^{-1} = b^e c c^{-1} b^{-1} = a^e a^{-1} \in A \cap E$. Let $B := A \cap E$. Then $\langle B^E \rangle = E$ because $\langle B^E \rangle \trianglelefteq E$. We recall that $G = C_G(E) \cdot E$ and hence $AE = E \cdot C_{AE}(E) = E \cdot O_2(AE)$. Then $T \cdot O_2(AE)$ is a Sylow 2-subgroup of AE containing A. Again using Lemma 1.5 we let X denote a Frobenius complement in $N_E(T)$. As A is strongly closed in $T \cdot O_2(AE)$ by Lemma 1.2 (1), it is X-invariant and hence $[X, A] \leq E \cap A = B$. This implies that $A = B \cdot C_A(X)$. But $C_A(X)$ induces inner automorphisms on E and centralises $\Omega_1(T)X$ by Lemma 1.5 (2) and (4), so it acts trivially on E. We conclude that $A = B \cdot C_A(E)$.
Now we let $H := C_G(E)$ and $C_A(E) \leq P \in \mathrm{Syl}_2(H)$. Then $C_A(E)$ is strongly closed in H by Lemma 1.2 (3) and the minimality of G yields that $(H, C_A(E))$

has property (sc). But $G = E \cdot H$ and so it follows that G is not a counterexample. Thus $G = E$ and G is simple. $\square$

Lemma 2.7. *$N_G(A)$ is 2-perfect and acts irreducibly on A. Moreover A is elementary abelian of order at least* 8.

Proof. We know that $N_G(A)$ controls fusion of its 2-elements by Lemma 1.2 (5). As G is simple by Lemma 2.6, Lemma 15.10 (ii) in [11] implies that $N_G(A)$ is 2-perfect. Assume that $N_G(A)$ does not act irreducibly on A and let B be a proper non-trivial $N_G(A)$-invariant subgroup of A. Then the control of fusion of $N_G(A)$ yields that B is strongly closed in G. Now if we let $H := \langle B^G \rangle$, then the minimal choice of A forces (H, B) to have property (sc). As A normalises H, it follows from Lemma 1.5 first that A induces inner automorphisms on every component and then that $A = B$. This is a contradiction. The irreducible action of $N_G(A)$ on A yields that $A = \Omega_1(A)$ whence A is elementary abelian. Finally assume that $|A| \leq 4$. If $|A| = 2$ and $a \in A^\#$, then a is an isolated involution in G. Together with Glauberman's Z*-Theorem (e.g. Theorem 15.3 in [11]) this contradicts the fact that G is simple (Lemma 2.6). If $|A| = 4$, then the irreducible action of $N_G(A)$ on A yields that 3 divides $|N_G(A)|$, contrary to our hypothesis. $\square$

Lemma 2.8. $G = \langle C_G(a) \mid a \in A^\# \rangle$.

Proof. Let $D := \{a^g \mid a \in A^\#, g \in G\}$. By Lemmas 1.2 (5) and 2.6, Theorem 1 of [2] is applicable and yields that G has a strongly embedded subgroup. Then Bender's main theorem in [4] gives that G is a Suzuki group, which is a contradiction. $\square$

Lemma 2.9. *For all $a \in A^\#$ the group $\overline{E} := \langle A^{C_G(a)} \rangle / O(\langle A^{C_G(a)} \rangle)$ is not quasi-simple.*

Proof. Assume otherwise. Then $\overline{a} \in Z(\overline{E})$ and $\overline{E}/Z(\overline{E}) \simeq \mathrm{Sz}(8)$ by Lemma 1.5 (3). In particular $8 \cdot 7$ divides $|N_G(A)/C_G(A)|$. Bearing in mind that $N_G(A)$ acts irreducibly on A, the lists of maximal subgroups of $\mathrm{GL}_4(2)$ and $\mathrm{GL}_5(2)$ (the possible automorphism groups of A) tell us that $|A| = 16$ and that $N_G(A)/C_G(A)$ is isomorphic to a subgroup of A_7. This is impossible. $\square$

Lemma 2.10. *Let $X := \langle O(C_G(a)) \mid a \in A^\# \rangle$. Then X is an $N_G(A)$-invariant $2'$-subgroup of G. If $B \leq S$ is elementary abelian of order at least* 4, *then $N_G(B) \leq N_G(X)$.*

Proof. For all $a \in A^\#$ we let $\theta(a) := O(C_G(a))$. Then Lemma 1.6 yields that θ defines a soluble A-signalizer functor. This functor is complete by Lemma 2.7 and the Soluble Signalizer Functor Theorem (e.g. Theorem 21.3 in [11]), and its completion is the subgroup X. For the next assertion let $B \leq S$ is non-cyclic elementary abelian. Then Lemmas 1.3 (1) and 1.6 yield that $X = \langle C_X(b) \mid b \in B^\# \rangle \leq \langle O(C_G(b)) \mid b \in B^\# \rangle \leq X$.
Hence $X = \langle O(C_G(b)) \mid b \in B^\# \rangle$ and we see that $N_G(B)$ normalises X. $\square$

Lemma 2.11. *Let $X := \langle O(C_G(a)) \mid a \in A^\# \rangle$ and suppose that $a \in A^\#$ is such that $C_G(a) \nleq N_G(X)$. Then $\langle A^{C_G(a)} \rangle / O(\langle A^{C_G(a)} \rangle)$ is a central product of a cyclic 2-group and a simple Suzuki group.*

Proof. Let $H := C_G(a)$, let $H_0 := \langle A^H \rangle$ and $\overline{H_0} := H_0/O(H_0)$. As $A \leq H$, Lemma 1.2 (6) yields that $H = N_H(A)H_0$ and Lemma 2.10 gives that $N_H(A) \leq N_G(X)$. By definition of X, also $O(H_0) \leq O(H) \leq X \leq N_G(X)$. The minimality of G implies that (H, A) has property (sc), so $O_2(\overline{H_0}) \leq \overline{A}$. Thus, if $O_{2',E}(H_0) \leq N_G(X)$, then it follows that $H_0 \leq N_G(X)$ and hence $H \leq N_G(X)$, contrary to our choice of a. Hence let $E \leq H_0$ be such that $E \nleq N_G(X)$ and that $\overline{E}$ is a component of $\overline{H_0}$. Assume that $O_2(\overline{H_0})$ is not cyclic and let $T \in \mathrm{Syl}_2(O_{2',2}(H_0))$ be such that $T \leq A$. Then Lemma 2.10 yields that $N_G(T) \leq N_G(X)$ and, since $\overline{E}$ and $\overline{T}$ commute, this forces $E \leq N_G(X)$ contrary to our choice of E.
Next let $P \leq H_0$ be a 2-group such that $P \leq S$ and $\overline{P} \in \mathrm{Syl}_2(\overline{E})$. As $\overline{E}$ is of Suzuki type, $\Omega_1(P) \leq A$ is not cyclic and so Lemma 2.10 implies that $N_G(P) \leq N_G(\Omega_1(P)) \leq N_G(X)$. If $\overline{H_0}$ has a component distinct from $\overline{E}$, then it centralises $\overline{P}$ and hence a pre-image of this component is contained in $N_G(X)$. Then, by arguing with a Sylow 2-subgroup of this component instead of $\overline{P}$, we obtain that $E \leq N_G(X)$, which is a contradiction again. Hence $\overline{E} = E(\overline{H_0})$ and it is simple by Lemma 2.9. □

Lemma 2.12. $O(C_G(a)) = 1$ *for all* $a \in A^\#$.

Proof. Assume otherwise. Then $X := \langle O(C_G(a)) \mid a \in A^\# \rangle \neq 1$ and hence $N_G(X) < G$ by Lemma 2.6. We choose a maximal subgroup M of G that contains $N_G(X)$. Applying Lemma 2.8, we choose $a \in A^\#$ such that $C_G(a) \nleq N_G(X)$. Let $C := C_G(a)$ and $C_0 := \langle A^H \rangle$. Then Lemma 2.11 yields that $C_0/O(C_0)$ is a central product of a cyclic 2-group and a simple Suzuki group. Let $E \leq C_0$ be such that $E/O(C_0) = E(C_0/O(C_0))$. Then $EO(C_0)$ is characteristic in C_0 and hence normal in C, so it follows that $N_C(A)$ normalises $EO(C_0)$. In particular $N_C(A)$ normalises $B := A \cap EO(C_0)$ and centralises a. Since $N_E(A)$ acts transitively on $B^\#$ by Lemma 1.5 (2), the orbits on $A^\#$ of $N_C(A)$ are $\{a\}$, $B^\#$ and $Ba \backslash \{a\}$. We let $H := N_G(A)$ and $\bar{H} := H/C_G(A)$.

(1) $N_G(A)$ acts transitively on $A^\#$. In particular $C_G(b) \nleq N_G(X)$ for all $b \in A^\#$ and $N_{C_G(b)}(A)$ has three orbits on $A^\#$, any two of which generate A.

Proof. The irreducible action of $N_G(A)$ on A (see Lemma 2.7) yields that $N_G(A)$ fuses the orbits $\{a\}$ and $B^\#$. As $N_G(A) \leq N_G(X) \leq H$ by Lemma 2.10, this forces for all $b \in B^\#$ that $C_G(b) \nleq C$. Let $C_1 := C_G(b)$. Then Lemma 2.11 and the previous arguments, applied to A and C_1, give a subgroup B_1 of the full pre-image of the unique component of $\langle A^{C_1} \rangle / O(\langle A^{C_1} \rangle)$ in $\langle A^{C_1} \rangle$ such that all involutions in $B_1^\#$ are $N_G(A)$-conjugate to b.
Assume that $N_G(A)$ does not act transitively on $A^\#$ and let $\Delta \subsetneq A^\#$ be a $N_G(A)$-orbit that contains $\{a\}$ and $B^\#$. Then it does not contain any element from $Ba \backslash \{a\}$ and hence $\Delta = \{a\} \cup B^\#$. With the same argument, $\Delta = \{b\} \cup B_1^\#$. In particular $a \in B_1$. Let $c \in B_1^\#$ be such that $c \neq a$. As

$a, c \in B_1 \leq A$, we know that $c \cdot a \in B_1$. Moreover $c \cdot a$ is distinct from 1, a and c. Hence it is contained in $\Delta \backslash \{a, c\}$ and then in $B^{\#}$. We also know that $c \in \Delta \backslash \{a\}$ and therefore $c \in B$. This forces $c \cdot a \in Ba \cap B^{\#}$, which is impossible. □

(2) $\bar{H}$ has odd order.

Proof. Let $t \in H$ be a 2-element such that $t^2 \in C_G(A)$. Let $b \in C_A(t)^{\#}$ and $C_1 := C_G(b)$. Then t normalises $\langle A^{C_1} \rangle / O(\langle A^{C_1} \rangle)$. By (1) and Lemma 2.11 the group $\langle A^{C_1} \rangle / O(\langle A^{C_1} \rangle)$ is a central product of a cyclic 2-group and a simple Suzuki group, so it follows with Lemma 1.5 that t induces an inner automorphism. As t centralises b, it centralises $A \cdot O(\langle A^{C_1} \rangle)/O(\langle A^{C_1} \rangle)$. Then Lemma 1.3 (2) yields that $t \in C_G(A)$, so $\bar{t} = 1$. □

Let $N \leq H$ be such that $\bar{N}$ is a minimal normal subgroup of $\bar{H}$.

(3) Every element $\bar{x} \in \bar{N}^{\#}$ acts fixed point freely on A.

Proof. Assume otherwise. Then by (1) every element from $A^{\#}$ is fixed by some $\bar{x} \in \bar{N}^{\#}$. In particular $C_{\bar{N}}(a)$ is a non-trivial (elementary abelian) normal subgroup of $C_{\bar{H}}(a)$. We note that $C_{\bar{N}}(a) \neq \bar{N}$ by (1) because $\overline{N} \neq 1$. Now $C_{\bar{N}}(a) = C_{\bar{N}}(a^k)$ for all $k \in N$ because $\bar{N}$ is abelian. Hence $C_N(a)$ has another fixed point on $A^{\#}$.
We recall that $N_E(A)$ acts transitively on $B^{\#}$ and we let $Y \leq N_E(A)$ denote a Singer cycle. Then $[Y, C_N(a)] \not\leq C_N(A)$ because $C_H(Y) \leq Y \cdot C_G(A)$. Therefore $[Y, C_N(a)] \leq N_E(A) \cap C_N(a) \leq O(\langle A^C \rangle)$ and $C_N(a) \cap O(\langle A^C \rangle) \trianglelefteq C_H(a)$. Since $C_N(a)$ has another fixed point on $A^{\#}$, the intersection $C_N(a) \cap O(\langle A^C \rangle)$ centralises at least two orbits of $N_C(A)$ on $A^{\#}$. Together with (1) this implies that $[Y, C_N(a)] \leq C_N(a) \cap O(\langle A^C \rangle) \leq C_N(A)$. This is a contradiction. □

Lemma 1.3 (1) and (3) imply that $\bar{N}$ is a cyclic group of prime order, say p. Let $n \in \mathbb{N}$ be such that $|A| = 2^n$. By Lemmas 1.5 and 2.11 the number n is even and E possesses a subgroup Y of order $2^{n-1} - 1$ acting regularly on B. In particular, for any $g \in Y^{\#}$, the only involution of A centralised by g is a, and $|Y| = |\bar{Y}|$. Now let $\bar{h} \in \bar{N}^{\#}$. Then $a \neq a^{\bar{h}}$ by (3). Hence $C_{\bar{Y}}(\bar{N}) = (C_{\bar{Y}}(\bar{N}))^{\bar{h}} = C_{\bar{Y}^{\bar{h}}}(\bar{N})$ centralises a and $a^{\bar{h}}$. Therefore $\bar{Y}$ acts faithfully on $\bar{N}$, i.e. $|\bar{Y}| = |Y| = 2^{n-1} - 1$ divides $|\mathrm{Aut}(\bar{N})| = p - 1$. Since $\bar{N} \leq \bar{L} \leq \mathrm{Aut}(A) \cong \mathrm{GL}_{2^n}(2)$, we see that p divides $\prod_{i=0}^{n-1}(2^n - 2^i)$. Thus there exists an $i \in \{0, .., n-1\}$ such that p divides $2^n - 2^i = 2^i(2^{n-i} - 1)$. Moreover $2^{n-1} - 1$ divides $p - 1$, so $2^{n-1} \leq p \leq 2^{n-i} - 1$. Therefore $i = 0$. As n is even, we also see that p divides $2^{n-i} - 1 = 2^n - 1 = (2^{n/2} + 1)(2^{n/2} - 1)$. But now $2^{n-1} \leq p \leq 2^{n/2} + 1$ whence it follows that $n \leq 3$, contrary to Lemma 2.7. □

Lemma 2.13. *Suppose that $a \in A^{\#}$ and $C_G(a) \leq H \leq G$. Then $O(H) = 1$.*

Proof. We know that $O(H) \cap C_G(a) \leq O(C_G(a)) = 1$ by Lemma 2.12. In particular a acts without fixed points on $O(H)$ and hence a inverts it. Let

$b \in A^{\#}$. Then $O(H) \cap C_G(b) = [O(H) \cap C_G(b), a] \leq \langle A^{C_{O(H)}(b)} \rangle \cap O(H) \leq AE(\langle A^{C_G(b)} \rangle) \cap O(H) = 1$, so Lemma 1.3 (1) yields that
$O(H) = \langle O(H) \cap C_H(b) \mid b \in A^{\#} \rangle = 1$. □

In the remaining steps let $a \in A^{\#}$ and let H be a maximal subgroup of G containing $C_G(a)$. Moreover let K denote a component of $\langle A^H \rangle$.

Step 1 $E(H) = E(KC_G(K))$.

Proof. It is clear that $E(H) \leq KC_G(K)$. As $(A\langle K^H \rangle, A)$ has property (sc), we also see that $A \leq KC_G(K)$. Moreover K is a component of $\langle A^H \rangle$, so it follows from property (sc) (3) that $A \cap K \neq 1$. Now we fix $b \in A \cap K^{\#}$. Then $C_G(K) \subseteq C_G(b) \subseteq N_G(A) \cdot E(\langle A^{C_G(b)} \rangle)$ by Lemma 1.2 (6). Let L be a component of H distinct from K, in particular L centralises K and hence b. Now let E be a component of $C_G(b)$.
Claim: $L = E$ or $[L, E] = 1$.
If $E \leq H$, then E and L are components of $C_H(b)$, so they coincide or they commute. Suppose that $E \not\leq H$, so in particular $[E, A] \neq 1$. This implies that $A \cap E \not\leq Z(E)$. The pair $(A\langle E^{C_G(K)} \rangle, A)$ has property (sc) and therefore A induces inner automorphisms on E by Lemma 1.5. Suppose further that $[A \cap E, L] = 1$. Then L normalises E because $A \cap E \not\leq Z(E)$, hence $C_L(E) \trianglelefteq L$ whence $C_L(E) \leq Z(L)$. But L induces inner automorphisms on E, so $L \leq EC_L(E) \leq EZ(L)$ which forces $L = Z(L)(L \cap E)$. So $L = E$. In the following we suppose that $[A \cap E, L] \neq 1$ and we begin with the case where $L \leq \langle A^H \rangle$. Then L is a component of $\langle A^H \rangle$ and the pair $(\langle A^H \rangle, A)$ has property (sc), so $1 \neq A \cap L$ centralises b and we obtain that $A \cap L \leq AE(C_G(b)) \trianglelefteq C_G(b)$. Now $[A, L] \neq 1$ and therefore $A \cap L \not\leq Z(L)$. This forces $(A \cap L)^L = L$ and, since $L \leq C_G(b)$, we deduce that $L \leq E(C_G(b))$. Then L normalises E, so as above $L = E$. It is left to consider the case where $L \not\leq \langle A^H \rangle$. Then $L \neq (A \cap L)^L$ because $L \leq H$, and consequently $A \cap L \leq Z(L)$. Then $L \leq C_G(A)$ and in particular L centralises $A \cap E$. But we already treated this case. This proves the claim.
Now $L \leq E(C_G(b)) \cdot C_G(E(C_G(b)))$ and therefore $L \leq E(C_G(K))$. Hence $E(H)$ is a product of components of $KC_G(K)$ and so it is normalised by $KE(C_G(K))$. Since $E(H)$ is normal in the maximal subgroup H of G, we have that $H = N_G(E(H)) \geq KE(C_G(K))$. So $E(C_G(K)) \leq E(C_H(K))$. Since $E(H) \subseteq KC_H(K) \trianglelefteq \trianglelefteq H$ it follows that $E(H) = E(KC_H(K)) = KE(C_H(K)) \geq KE(C_G(K)) \geq E(H)$. This concludes the proof. □

Step 2 For all $b \in C_A(K)^{\#}$ the unique maximal subgroup of G containing $C_G(b)$ is H.

Proof. Let $b \in C_A(K)^{\#}$ and let M be a maximal subgroup of G containing $C_G(b)$. Then $K = \langle (A \cap K)^K \rangle \leq \langle A^M \rangle$. Therefore Lemma 2.13 yields that K is a component of $\langle A^M \rangle$ and (using Step 1) that $E(M) = E(KC_G(K)) = E(H)$ is normalised by M and by H. As G is simple (Lemma 2.6), it follows that $M = H$. □

Now we choose a, H and K such that $|K|$ is maximal.
Step 3 $N_G(A) \not\leq H$.

Proof. Assume otherwise. We know from Lemma 2.7 that $O^2(N_G(A)) = N_G(A)$ and that $N_G(A)$ acts irreducibly on A. As $1 \neq A \cap K \neq A$, this implies that $N_G(A)$ does not normalise K. Together with the fact that $N_G(A)$ is not a 2-group and has order coprime to 3, it follows that $\langle A^H \rangle$ has at least five distinct components that are isomorphic to K. Let $b \in A^\#$.
First suppose that $E(\langle A^{C_G(b)} \rangle) = 1$. Then $C_G(b) \leq N_G(A) \leq H$ by Lemma 2.12 and by our assumption.
Next suppose that M is a maximal subgroup of G containing $C_G(b)$ and let E be a component of $\langle A^M \rangle$. From the choice of K it follows that $|E| \leq |K|$, so $|A \cap E| \leq |A \cap K|$. As $(K \cap A)C_A(K) = A$ and as there are at least five conjugates of K, we obtain that $|C_A(E)| \geq |C_A(K)| \geq |K \cap A|^4 \geq \sqrt{|A|}$ and so $C_A(K) \cap C_A(E) \neq 1$. Let $c \in C_A(K) \cap C_A(E)^\#$. Then $E \leq C_G(c) \leq H$ by Step 2, so we conclude that $E(\langle A^M \rangle) \leq H$. Moreover $O(M) = 1$ by Lemma 2.13 and then the fact that (M, A) has property (sc) implies that $C_G(b) \leq M = N_M(A) \cdot E(\langle A^M \rangle) \leq H$, contradicting Lemma 2.8. □

Step 4 $C_A(K) \cap C_A(K)^g = 1$ for all $g \in N_G(A) \backslash N_H(K)$.

Proof. Let $g \in N_G(A)$ be such that $C_A(K) \cap C_A(K)^g \neq 1$ and let $b \in C_A(K)^\# \cap C_A(K)^g$. Since $b^{g^{-1}}, b \in C_A(K)$, Step (2) yields that H is the unique maximal subgroup of G containing $C_G(b)$ and $C_G(b)^{g^{-1}}$. Hence $H^{g^{-1}} = H$ and so $g \in H$. Applying Step 3 we choose $h \in N_G(A)\backslash H$, so that $C_A(K) \cap C_A(K)^h = 1$. This forces $(|C_A(K)|)^2 = |C_K(A)| \cdot |C_K(A)^h| \leq |A|$. This proves also $|K \cap A| \geq \sqrt{|A|}$. Now suppose that $g \notin N_H(K)$. Then $[K, K^g] = 1$ and $\sqrt{|A|} \leq |K \cap A| \leq |C_A(K^g)| = |C_A(K)^g| \leq \sqrt{|A|}$. This forces $K \cap A = C_A(K^g)$ and $K \cap A \cap K^g = (K \cap A) \cap C_A(K) \neq 1$. From all this we deduce $|K \cap A|^2 = |C_A(K)|^2 = |A| = |C_A(K^g)(A \cap K^g)| = |(A \cap K)(A \cap K)^g| = \frac{|A \cap K| \cdot |A \cap K^g|}{|A \cap K \cap K^g|}$. This contradicts $A \cap K \cap K^g \neq 1$. □

Let $X := N_G(A)/C_G(A)$, let Y_0 be a complement of the Sylow 2-subgroup of $N_K(A)$ (a Frobenius group) and $Y = Y_0 C_G(A)/C_G(A)$.

Step 5 X and Y satisfy Hypothesis (2.9) of [10] in their action on A, and $Z(K) \neq 1$.

Proof. $N_G(A)$ acts irreducibly on A and $O^2(N_G(A)) = N_G(A)$ by Lemma 2.7. Hence X acts faithfully and irreducibly on A and $O^2(X) = X$. By Lemma 1.5 (2) the group Y is cyclic of odd order acting transitively on $A \cap K^\# = [A, Y]^\#$. By Step 4 the distinct elements of $\{C_A(Y)^x \mid x \in X\}$ intersect pair-wise trivially. Assume that $Z(K) = 1$. Let $x \in N_X(C_A(K))$ and let $g \in N_G(A)$ be a pre-image of x. Then $C_A(K) \cap C_A(K)^g \neq 1$, so Step 4 implies that $g \in N_H(K) \cap N_G(A)$. Hence $g \in N_H(Y)C_G(A)$ and $x \in N_X(Y)$. Applying (2.11) of [10], we conclude that $|A| = 8$ and $|C_A(K)| = 2$. This contradicts Lemma 1.5 (2). □

For a final contradiction, we want to apply (2.10) of [10] and force $|C_A(K)| = 2$. Then $C_A(K) = Z(K)$ which contradicts Lemma 2.9.

Thus, it remains to show that Y acts semi-regularly on the set $\{C_A(Y)^x \mid x \in X\} \backslash \{C_A(K)\}$.

Assume otherwise. Then there exist $g \in N_G(A) \backslash C_G(A)$ and $y \in Y^\#$ such that $C_A(K)^g \neq C_A(K)$ and $C_A(K)^g$ is fixed by Y. By Lemma 1.5 (2) and (3) Y has order 7 and hence $C_A(K)^g$ is Y_0-invariant. Since $C_A(K)^g \cap C_A(K) = 1$ by Step 4 and since Y_0 acts transitively on $[A, Y_0]^\#$, we conclude that $C_A(K)^g = [A, Y_0]$. Now $|C_A(K)|^2 = |C_A(K)| \cdot |[A, Y_0]| = |A|$ and K is not simple by Step 5, so $K \trianglelefteq H$ and $C_A(K) = O_2(\langle A^H \rangle) \trianglelefteq H$. Since H is maximal in G Lemma 2.6 and the fact that H is a maximal subgroup of G yield that $N_G(C_A(K)) = H$ and that $1 \neq C_A(K) \cap K$ is a proper H-invariant subgroup of $C_A(K)$. But $Y_0^{g^{-1}}$ acts transitively on $[A, Y_0]^{g^{-1}} = C_A(K)$.

This contradiction finishes the proof of Theorem 2.2.

3. Proof of the main theorem

Theorem 3.1. *Suppose that G is a simple $3'$-group. Then G is either cyclic of prime order or G is isomorphic to a Suzuki group.*

Proof. Assume that the theorem is false and let G be a minimal counterexample. In particular all proper simple sections of G are cyclic of prime order or isomorphic to a Suzuki group. For all 2-subgroups T of G, we denote by $W(T)$ the characteristic subgroup of T introduced by Stellmacher in [12].

Now G is non-abelian simple and the fact that 3 does not divide $|G|$ implies that G does not involve $\mathcal{S}_4$. Let H^* denote a non-abelian proper section of G and suppose that $O_2(H^*) = F^*(H^*)$. Moreover let $S^* \in \mathrm{Syl}_2(H^*)$. All simple non-abelian sections of H^* are Suzuki groups by minimality of G, and therefore the main theorem in [12] applies. It yields that S^* has a non-trivial characteristic subgroup $W(S^*)$ such that $\Omega_1(Z(S^*)) \leq W(S^*) \leq ZJ(S^*)$ and that $W(S^*) \trianglelefteq H^*$.

Let $T \in \mathrm{Syl}_2(G)$. Then Theorem 1.4 implies that $N_G(W(T))$ controls G-fusion in T. Now we set $A := \langle \Omega_1(Z(T))^{N_G(W(T))} \rangle$ and we show that A is an abelian strongly closed 2-subgroup of G. We begin by noticing that A is normalised by $N_G(W(T))$.

Let $g \in G$. As $N_G(W(T))$ controls G-fusion in T, we find $h \in N_G(W(T))$ such that $A^g \cap T = A^h \cap T = A \cap T \leq A$. Hence A is in fact strongly closed in G. Moreover $\Omega_1(Z(T)) \leq W(T) \leq ZJ(T)$ by definition of the subgroup $W(T)$ and in particular $W(T)$ is abelian, so A is abelian.

Now G satisfies the hypothesis of Theorem 2.2 and it follows that G is a Suzuki group, contrary to our assumption. $\square$

References

[1] Alperin, J. L. and Gorenstein, D.: The multiplicators of certain simple groups, *Proc. Amer. Math. Soc.* **17** (1966), 515-519.

[2] Aschbacher, M.: A condition for the existence of a strongly embedded subgroup, *Proc. Amer. Math. Soc.* **38** (1973), 509–511.

[3] Aschbacher, M.: $\mathcal{S}_3$-free 2-fusion systems, *Proceedings of the Edinburgh Mathematical Society*, Series 2 **56**, Issue 01 (2013), 27–48.

[4] Bender, H.: Transitive Gruppen gerader Ordnung, in denen jede Involution genau einen Punkt festläßt, *J. Algebra* **17** (1971), 527-554.

[5] Feit, W. and Thompson, J. G.: Solvability of groups of odd order, *Pacific J. Math.* **13** (1963), 775-1029.

[6] Glauberman, G.: Global and local properties of finite groups. *Finite simple groups* 1-64. Academic Press, London, 1971.

[7] Glauberman, G.: A sufficient condition for p-stability, *Proc. London Math. Soc.(3)* **25** (1972), 253-287.

[8] Glauberman, G.: *Factorizations in Local Subgroups of Finite Groups.* Regional Conference Series in Mathematics, No. 33. American Mathematical Society, Providence, R.I., 1977.

[9] Goldschmidt, D. M.: Strongly closed 2-subgroups of finite groups, *Ann. of Math. (2)* **102** (1975), no. 3, 475-489.

[10] Goldschmidt, D. M.: 2-fusion in finite groups, *Ann. of Math. (2)* **99** (1974), 70-117.

[11] Gorenstein, D., Lyons, R. and Solomon, R.: *The classification of the finite simple groups. Number 2.* Mathematical Surveys and Monographs, 40.2. American Mathematical Society, Providence, RI, 1996.

[12] Stellmacher, B.: A characteristic subgroup of Σ_4-free groups, *Israel J. Math.* **94** (1996), 367-379.

[13] Suzuki, M.: On a class of doubly transitive groups, *Ann. of Math. (2)* **75** (1962), 105-145.