Isolated involutions whose centraliser is soluble

Rebecca Waldecker

School of Mathematics, University of Birmingham, Birmingham B15 2TT, UK

Abstract

We analyse a minimal counterexample to Glauberman's Z^*-Theorem from a local group theoretic point of view. One of the main results is a group theoretic proof for the Z^*-Theorem in the special case where the centraliser of an isolated involution is soluble.

Key words: finite groups, isolated involution, involution centraliser, soluble
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1. Introduction

In this paper we prove the following:

The Soluble Z^*-Theorem.
Suppose that G is a finite group and that $z \in G$ is an isolated involution.
If $C_G(z)$ is soluble, then $\langle z\rangle O(G) \trianglelefteq G$.

Here an involution $z \in G$ is **isolated** if the only conjugate of z in G commuting with z is z itself.

The result above is a special case of Glauberman's Z^*-Theorem ([6]) which says that in a finite group G, every isolated involution is central modulo $O(G)$.
We provide a new proof here, for the Soluble Z^*-Theorem, which does not use modular character theory. The deepest results which we apply are the Brauer-Suzuki Theorem (for which Glauberman gave a proof based on ordinary representation theory ([11])) and the Odd Order Theorem ([4]). We would also like to point out that this work is part of the larger project to find a group theoretic proof for Glauberman's Z^*-Theorem in general. We therefore establish our results in the framework of a minimal counterexample to the Z^*-Theorem - most of them are required in subsequent work and do not depend on any solubility assumption. However, the Soluble Z^*-Theorem is a consequence of the general results and might be of independent interest.
After some preliminaries, we look at a group G with an isolated involution z in Section 4. We would like to thank the referee for pointing out the connections between groups with isolated involutions and certain classes of loops of odd order which we briefly refer to in that chapter. It turns out that by defining two binary operations on the set of commutators $K := \{zz^g \mid g \in G\}$ we can show that, as a set, $K = C_K(s)C_K(sz)C_K(s)$ for any involution s which commutes with z (Theorem 4.8). This implies later that G is generated by two involution centralisers and also plays a role in subsequent work when signalizer functors appear. It follows from Theorem 4.8

that G possesses z-invariant Sylow p-subgroups for every prime p (Theorem 4.11). This is one of several places where we see that (unsurprisingly) the isolated involution z behaves as if it acts coprimely on every z-invariant subgroup - not only on those of odd order.
In Section 5 we start investigating a minimal counterexample G to Glauberman's Z^*-Theorem. Thus G is a finite group and $z \in G$ is an isolated involution such that $\langle z \rangle O(G)$ is not normal in G and G is minimal in a particular sense. As $z \notin Z(G)$, we have that $C := C_G(z)$ is contained in a maximal subgroup of G. We point out again that we do *not* suppose that C is soluble. From the minimality we deduce that G is almost simple (Lemma 5.3) and after collecting some additional information about the structure of G we prove our first main result (Theorem 6.3) in Section 6.

Theorem A. *Suppose that M is a maximal subgroup of G containing C. If possible, choose M such that there exists a prime p with $O_p(M) \neq 1 = C_{O_p(M)}(z)$. Then one of the following holds:*
- $M = C$.
- $F^*(M) = O_p(M)$ for some odd prime p.
- $E(M) \neq 1$.

The main ingredients for the proof are the Bender method and applications of coprime action results. We introduce the notation and background results for the Bender method at the beginning of Chapter 6. The Infection Theorem (6.2) is mainly a presentation of results of Bender's for our situation to simplify later quotations. We proceed by contradiction, assuming that M is a maximal subgroup of G which properly contains C. We then find a subgroup U in $O_p(M)$ for some odd prime p such that $1 \neq U = [U, z]$ and either $N_G(U) \leq M$ or $F^*(M) = O_p(M)$ (Lemma 6.4 and Corollary 6.8). This is the basis for further investigation; we gradually work our way up to show that if, in addition, M is not of characteristic p, then for certain subgroups X of $F(M)$, we can force $N_G(X)$ to be contained in a unique maximal subgroup of G, namely in M. The strategy is always the same: We suppose that $N_G(X)$ is contained in a maximal subgroup H of G and then use the Bender Method to show that $H = M$. Our main result there is that if $1 \neq X = [X, z] \leq O_p(M)$, then M is the unique maximal subgroup of G containing $N_G(X)$ (Lemma 6.13). Then we are close to a contradiction. We assume that M is in fact a counterexample to Theorem **A**, i.e. that $C < M$, $F^*(M) \neq O_p(M)$ and $E(M) = 1$. First we show that $O_p(M)$ is a cyclic group which is inverted by z (Lemma 6.16). From there we find a non-trivial normal subgroup of G which is contained in M. This is impossible and finishes the proof of Theorem **A**.

In Section 7 we change perspective and rather than further investigating C only, we look at the centralisers of three involutions at the same time. It turns out that if $O_{2',2}(C)$ contains involutions distinct from z, then this has a strong impact on the structure of the group. The result is (Lemmas 7.4 and 7.6, Theorem 7.21):

Theorem B. *Suppose that $O_{2',2}(C)$ contains an elementary abelian subgroup of order* 4. *Then the following hold:*
- G is simple and of 2*-rank* 2.
- The Sylow 2*-subgroups of G possess precisely three involutions.*
- All involutions of G are isolated and their centralisers are maximal subgroups.

As an immediate consequence of Theorem **B**, under the hypothesis above G has exactly three conjugacy classes of involutions. For the proof we first note that an elementary abelian subgroup

$V = \{1, a, b, z\}$ of order 4 of $O_{2',2}(C)$ is not contained in any larger elementary abelian 2-subgroup (Lemma 7.2). Then a key observation is that whenever a and b are contained in a subgroup H of G, then they are either isolated or conjugate in H. In particular, they are isolated or conjugate in G. In the first case, we show that they "behave" like z and that we can apply all the results for them which we proved for z before. In the second case, we can choose maximal subgroups containing the centralisers of a and b, respectively, to be conjugate. In both cases, we are again in a position where we can appeal to the Bender method. The main objects are a maximal subgroup M containing C, a maximal subgroup L_a containing $C_G(a)$ and a maximal subgroup L_b containing $C_G(b)$. One of the first results is that under the hypothesis of Theorem **B**, G is simple (Lemma 7.4) and the 2-rank of G is precisely 2 (Lemma 7.6). There is a situation which needs special attention - the case where $F^*(L_a)$ and $F^*(L_b)$ (or $F^*(M)$) are q-groups for the same odd prime q. So the next step is to show that this does not occur (Lemmas 7.17 and 7.18). Then in Theorem 7.19 we prove that $C_G(a)$ is a maximal subgroup of G. The observation that a and b are isolated (Lemma 7.20) and a summary of the statements (Theorem 7.21) conclude the section.
We continue investigating the case where $O_{2',2}(C)$ contains an elementary abelian subgroup of order 4 at the beginning of Chapter 8. It turns out that, under this assumption, $C/O(C)$ is perfect (Lemma 8.4). This enables us to prove the next main result (Theorem 8.5):

Theorem C. *$C/O(C)$ possesses at least one component. In particular, C is not soluble.*

We then convince ourselves that Theorems **A**, **B** and **C** hold in a minimal counterexample to the Soluble Z^*-Theorem and obtain a contradiction at the end of Section 8. Throughout, all groups are meant to be finite and we follow standard notation as in [1] and [16].

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2. Preliminaries

We introduce some notation and collect a few well known results about coprime action. We use without further reference that groups of odd order are soluble ([4]).

2.1. Notation

Let X be a group, let p be a prime and let π be a set of primes.

- Let $x \in X$ and $U \leq X$. Then $x^U := \{x^u \mid u \in U\}$. Similarly if $Y \leq X$, then $Y^U := \{Y^u \mid u \in U\}$.
- If $U \leq X$ is such that $C_X(U) \leq U$, then we say that U is **centraliser closed** in X.

- Let $A, B \leq X$ be such that AB is a subgroup of X. We say that AB is a **central product** and write $A * B$ if $[A, B] = 1$. If in addition $A \cap B = 1$, then we say that AB is a **direct product** and we denote this by $A \times B$.

- For any integer $n \geq 1$, we denote by n_p the largest p-power that divides n.

- By $r_p(X)$ we denote the p**-rank of** X. If X is a p-group (and therefore no confusion about the prime is possible), then we write $r(X)$ for the rank of X.

- If $F^*(X) = O_p(X)$, then we say that X has **characteristic** p and we write $\mathrm{char}(X) = p$.

- We abbreviate $O_\pi(F(X))$ by $F_\pi(X)$.

- We define $O_{p',p}(X)$ by $O_{p',p}(X)/O_{p'}(X) = O_p(X/O_{p'}(X))$.

- We define $Z^*(X)$ by $Z^*(X)/O(X) = Z(X/O(X))$.

- If X is a p-group, then by $K^\infty(X)$ we denote the characteristic subgroup of X introduced by Glauberman in [9], and by $ZJ(X)$ we denote the centre of the Thompson subgroup of X (a characteristic subgroup) as defined in [7].

- Suppose that X is a p-group. Then X is **extra-special** if $X' = \Phi(X) = Z(X)$ is cyclic of order p.

- If an involution t acts on a group Q of odd order, then we define $I_Q(t) := \{x \in Q \mid x^t = x^{-1}\}$.

- Let $A \leq X$. We define $\mathcal{V}_X(A, \pi)$ to be the set of all A**-invariant** π**-subgroups of** X. Furthermore, $\mathcal{V}^*_X(A, \pi)$ denotes the set of maximal elements of $\mathcal{V}_X(A, \pi)$ with respect to inclusion. We use the same notation if A is a group which acts on X. If $\pi = \{p\}$, then we abbreviate $\mathcal{V}_X(A, \{p\})$ by $\mathcal{V}_X(A, p)$.

- For convenience, we say that X is **quaternion** if $X \simeq Q_{2^n}$ for some $n \geq 3$ (rather then saying "generalised quaternion"). For all $k \in \mathbb{N}$ let Q_8^k denote the central, non-direct product of k quaternion groups of order 8.

- For all $n \in \mathbb{N}$ we denote by C_n the **cyclic group of order** n and by S_n (A_n) the **symmetric (alternating) group** on n elements.

2.2. Coprime Action

Let P be a π-group acting on a π'-group Q. Then the following hold:

(a) $Q = C_Q(P)[Q, P]$ and $[Q, P, P] = [Q, P]$. If Q is abelian, then $Q = C_Q(P) \times [Q, P]$.
(b) Suppose that P is a non-cyclic elementary abelian p-group. Then $Q = \langle C_Q(x) \mid x \in P^\# \rangle$. If P has order 4, e.g. $P = \{1, x, y, xy\}$, and Q is nilpotent, then $Q = C_Q(x)C_Q(y)C_Q(xy)$. As a corollary, if $C_Q(x) \leq C_Q(y)$, then we have $I_Q(y) \subseteq I_Q(x)$.
(c) If P centralises a centraliser closed subnormal subgroup of Q, then $[Q, P] = 1$.
(d) Let $q \in \pi'$. Then $\mathcal{V}^*_Q(P, q) \subseteq \mathrm{Syl}_q(Q)$ and $C_Q(P)$ is transitive on $\mathcal{V}^*_Q(P, q)$.

Proof. Most of these are contained in [16], they correspond to 8.2.3, 8.2.7, 8.3.4 and 8.4.2 and immediate corollaries thereof. Part (c) is Proposition 1.10 in [3]. $\Box$

Lemma 2.1. *Suppose that an involution $t \in X$ acts on a q-subgroup Q of X where q is odd. If $r(Q) \geq 3$, then Q possesses a t-invariant elementary abelian subgroup of order q^3.*

Proof. This is Lemma 11.18 in [14]. $\Box$

Lemma 2.2. *Let $H \leq X$ be a $2'$-subgroup which is normalised by an involution $t \in X$. Suppose that every t-invariant π-subgroup of H is centralised by t. Then $H = C_H(t)O_{\pi'}(H)$.*

Proof. Without loss of generality, $O_{\pi'}(H) = 1$. Then $F(H)$ is a t-invariant π-subgroup of H and thus, by hypothesis, t centralises $F(H)$. But $F(H) = F^*(H)$ because H is soluble by the Odd Order Theorem. Hence $F(H)$ is centraliser closed and therefore Coprime Action (c) implies that $[H, t] = 1$. $\Box$

3. General results

The following are background results which are needed later and for which we give proofs or references. Again X denotes a group, p a prime and π a set of primes.

Lemma 3.1. *Let X be a p-group. Then X possesses a characteristic subgroup P (a so called **critical subgroup**) such that the following hold:*

- *Every p'-subgroup of $Aut(X)$ acts faithfully on P.*
- *$P' = \Phi(P)$ is elementary abelian and lies in $Z(P)$.*
- *If $X \neq 1$, then $exp(P) = p$ if p is odd and $exp(P) = 4$ if $p = 2$.*

Proof. This is Proposition 11.11 in [14]. $\Box$

Lemma 3.2. *Suppose that X is a 2-group and that $X_0 \trianglelefteq X$. If $r(X_0) \geq 2$, then either X_0 contains a normal elementary abelian subgroup of X of order 4 or X_0 is dihedral or semi-dihedral.*

Proof. This is Lemma 10.11 in [14]. $\Box$

Lemma 3.3. *Let $n \in \mathbb{N}$, $n \geq 2$. Then the following hold:*

- *The number of cyclic subgroups of order 4 in Q_8^n is $\frac{1}{2}(2^{2n} - (-2)^n)$.*
- *The number of cyclic subgroups of order 4 in $Q_8^{n-1} * D_8$ is $\frac{1}{2}(2^{2n} + (-2)^n)$.*
- *If n is even, then $r(Q_8^n) = n + 1$.*
- *If n is odd, then $r(Q_8^n) = r(Q_8^{n-1} * D_8) = n$.*
- *$r(C_4 * Q_8^n) = r(C_4 * Q_8^{n-1} * D_8) = n + 1$.*

Proof. The formulae are derived within the proof of Theorem 5.2 in [13]. Proposition 10.4 and Lemma 10.8 in [14] give the statements about the rank. $\Box$

Lemma 3.4 (Thompson's $P \times Q$-Lemma)**.** *Suppose that X acts on a p-group W and that $X = PQ$ is a central product of a p-group P and a p-perfect group Q (i.e. $O^p(Q) = Q$). If Q centralises $C_W(P)$, then $[W, Q] = 1$.*

Proof. This is on p.112 in [1]. □

Lemma 3.5. *Let Y be a p-subgroup of $O_{p',p}(X)$. Then $O_{p'}(C_X(Y)) \leq O_{p'}(X)$. If X is soluble, then for every p-subgroup P of X we have $O_{p'}(C_X(P)) \leq O_{p'}(X)$.*

Proof. These follow from (31.14) and (31.15) in [1]. □

Lemma 3.6. *Let $t \in O_{2',2}(X)$ be an involution and let $D \leq X$ be a $C_X(t)$-invariant $2'$-subgroup. Then $D \leq O(X)$. If D is nilpotent, then $[t, D] \leq F(X)$.*

Proof. By Coprime Action (a), we have $D = C_D(t)[D, t]$. Now as $t \in O_{2',2}(X)$ and D has odd order, we obtain $[t, D] \leq O_{2',2}(X) \cap D \leq O(X)$. On the other hand $C_D(t)$ is $C_X(t)$-invariant and hence $C_D(t) \leq O(C_X(t)) \leq O(X)$ by Lemma 3.5. Thus $D \leq O(X)$.
For the second assertion, suppose that D is nilpotent. Let $q \in \pi(D)$. We show that $[t, O_q(D)] \leq O_q(X)$. Without loss of generality, $O_q(X) = 1$, which means that we need to show $[t, O_q(D)] = 1$. First we note that $O_q(D)$ is $C_X(t)$-invariant. We set $D_0 := [t, O_q(D)]$ and observe that

$$[C_{F(X)}(t), D_0] \leq F(X) \cap O_q(D) \leq O_q(X) = 1$$

and therefore $C_{F(X)}(t) \leq C_{F(X)}(D_0) =: H$. Now t acts on $N_{F(X)}(H)/H$, and this action is fixed point free. Thus t inverts every element of $N_{F(X)}(H)/H$. Consequently $D_0 = [D_0, t]$ centralises that factor group since it is D_0-invariant, yielding $[N_{F(X)}(H), D_0] = 1$. But this implies that $N_{F(X)}(H) \leq C_{F(X)}(D_0) = H$ and hence H is equal to its normaliser in $F(X)$. It follows that $H = F(X)$ because $F(X)$ is nilpotent. As $O(X)$ is soluble, we deduce that $D_0 \leq C_{O(X)}(F(O(X))) = Z(F(O(X)))$ which gives $D_0 \leq O_q(X) = 1$. □

Lemma 3.7. *Let P be a p-subgroup of $O_{p',p}(X)$. Then $C_X(P)$ is transitive on the set $\mathcal{U}^*_X(P, q)$ for every $q \in p'$.*

Proof. Arguing modulo $O_{p'}(X)$, we may suppose that $O_{p'}(X) = 1$. Let $Q_1, Q_2 \in \mathcal{U}^*_X(P, q)$. Then as $P \leq O_p(X)$, we have $[Q_1, P] \leq Q_1 \cap O_p(X) = 1$ and similarly $[Q_2, P] = 1$. Thus $Q_1, Q_2 \leq C_X(P)$ and in particular $Q_1, Q_2 \in \mathcal{U}^*_{C_X(P)}(P, q)$. But then Q_1 and Q_2 are simply two Sylow q-subgroups of $C_X(P)$ and therefore conjugate in $C_X(P)$ by Sylow's Theorem. □

Lemma 3.8. *Let $P \in Syl_p(X)$ and let $H \leq X$ be such that $P \leq H$. For every p-element $y \in H$ suppose that $y^X \cap H = y^H$. If $O^p(H) \neq H$, then $O^p(X) \neq X$.*

Proof. This follows from Lemma 15.10 in [14]. □

The next results are weakened versions of Theorem A in [9] and Glauberman's ZJ-Theorem (in [7]), respectively.

Theorem 3.9. *Suppose that X has odd order and let P be a p-subgroup of X containing $O_p(X)$. Then if $char(X) = p$, it follows that $K^\infty(P)$ is normal in X.*

Theorem 3.10. *Suppose that X has odd order and that $char(X) = p$. Let $P \in Syl_p(X)$. Then $ZJ(P) \trianglelefteq X$.*

Theorem 3.11. *Suppose that $S \in Syl_2(X)$ is cyclic. Then $X = SO(X)$. In particular the unique involution in S is contained in $Z^*(X)$.*

Proof. This follows from 7.2.2 in [16] and is in fact a corollary of Burnside's Transfer Theorem. □

Theorem 3.12 (Brauer-Suzuki)**.** *Suppose that $S \in Syl_2(X)$ is quaternion and let s be the unique involution in S. Then $s \in Z^*(X)$.*

Proof. For a proof using ordinary character theory see [11]. □

Theorem 3.13. *Suppose that B is an abelian 2-subgroup of X of rank at least 3. Let A be a non-cyclic subgroup of B such that $A \leq O_{2',2}(C_X(b))$ for all involutions $b \in B$. Then $\langle O(C_X(a)) \mid a \in \Omega_1(A)^{\#} \rangle$ is a subgroup of X of odd order.*

Proof. This is Theorem 3.1 in [12]. □

Theorem 3.14. *Suppose the following:*
- *A is a non-trivial proper subgroup of X, $\pi := \pi(A)$ and $r_p(Z(A)) \geq 3$ for some prime $p \in \pi$.*
- *If $A \leq Y < X$, then $\langle \mathcal{V}_Y(A, \pi') \rangle = O_{\pi'}(Y)$.*
- *If $q \in \pi'$ and Q is a non-trivial q-subgroup of X, then $N_X(Q) < X$.*

Then $O_{\pi'}(C_X(A))$ is transitive on $\mathcal{V}^*_X(A, q)$ for every $q \in \pi'$.

Proof. This is Theorem 7.2 in [3]. □

Theorem 3.15. *Suppose that X is a π-group and that A is a π'-group of automorphisms of X. Suppose that t is an automorphism of X of order 2 such that $C_X(t) \leq C_X(A)$. Then $[C_X(A), t]$ and $[X, A]$ are normal subgroups of X and $[X, A]$ is nilpotent of odd order.*

Proof. This is a weakened version of Theorem 1 in [10]. □

The following theorem is needed towards the end of the paper.

Theorem 3.16. *Let T be a finite 2-group with precisely three involutions. Then $Aut(T)$ is soluble.*

Proof. Assume not. Then $\mathrm{Aut}(T)$ possesses a non-trivial perfect subgroup A. We choose A minimal in the sense that every proper subgroup of A is soluble. Let a, b, c be the involutions in T and let $V := \langle a, b \rangle$. Then V is a characteristic subgroup of T. As $Z(T)$ contains at least one involution, there are two cases: $V \leq Z(T)$ or $|V \cap Z(T)| = 2$. In the following, until the last paragraph of the proof, we suppose that $V \leq Z(T)$.

(1) $V \leq C_T(A)$.

Proof. Assume that $C_A(V)$ is a proper subgroup of A. Then $C_A(V)$ is soluble. On the other hand $A/C_A(V)$ is isomorphic to a subgroup of $\mathrm{Aut}(V) \simeq S_3$. But then this factor group is soluble and so is A, a contradiction. Therefore A centralises V. □

(2) Without loss, $\Phi(T) = T'$ is elementary abelian and lies in $Z(T)$.

Proof. By Lemma 3.1, T possesses a critical subgroup T_0. We recall that this means that T_0 is characteristic in T, $O^2(\mathrm{Aut}(T))$ acts faithfully on T_0 and $T_0' = \Phi(T_0)$ is elementary abelian and lies in $Z(T_0)$. If we show that $\mathrm{Aut}(T_0)$ is soluble, then $O^2(\mathrm{Aut}(T)) \leq \mathrm{Aut}(T_0)$ is soluble. As $\mathrm{Aut}(T)/O^2(\mathrm{Aut}(T))$ is a 2-group, this implies that $\mathrm{Aut}(T)$ is soluble, a contradicton. Thus we may replace T by T_0 if T_0 contains exactly three involutions. But assume that T_0 contains a unique involution. Then T_0 is quaternion or cyclic, and in both cases $\mathrm{Aut}(T_0)$ is soluble. We showed above that this implies that $\mathrm{Aut}(T)$ is soluble, a contradiction. Therefore T_0 contains all three involutions and we replace T by T_0. □

We recall that in particular T is now of exponent 4 (by Lemma 3.1) and $T' \leq V$ which implies $|T'| \leq 4$. Moreover we still suppose that $V \leq Z(T)$.

(3) T does not have a series $1 = T_0 \leq T_1 \leq \cdots \leq T_k = T$ of subgroups T_i, with $0 \leq i \leq k \in \mathbb{N}$, such that T_i is A-invariant and $|T_{i+1}/T_i| \leq 4$ for all $i \in \{0, ..., k-1\}$.

Proof. Assume that such a series of subgroups of T exists. Since $C_A(T) = 1$, there is a $j \in \{0, ..., k-1\}$ such that A acts non-trivially on T_{j+1}/T_j. Hence $C_A(T_{j+1}/T_j)$ is soluble. But $A/C_A(T_{j+1}/T_j)$ is isomorphic to a subgroup of S_4 and thus A is soluble, a contradiction. □

(4) Let $v \in V^{\#}$ and $\overline{T} := T/\langle v\rangle$. Then $r(\overline{T}) \leq 3$.

Proof. Let $X \leq T$ be such that $\overline{X}$ is a maximal elementary abelian subgroup of $\overline{T}$. Then $V \leq X$ and there exists a subgroup Y of T such that $\overline{Y}$ is a complement to $\overline{V}$ in $\overline{X}$. In particular $Y \cap V = \langle v\rangle$, so Y contains a unique involution. It follows that Y is cyclic or quaternion, i.e. $\overline{Y}$ is cyclic or dihedral and thus of rank at most 2. As $|\overline{V}| = 2$, this yields $r(\overline{X}) \leq 3$. □

(5) Let $v \in V^{\#}$ and $\overline{T} := T/\langle v\rangle$. Let $Z \leq T$ be such that $\overline{Z} = \Omega_1(Z(\overline{T}))$. Then T has a subgroup X such that $\overline{T} = \overline{Z} * \overline{X}$ and $\overline{X}$ is a central product of a cyclic group of order at most 4 and an extra-special group.

Proof. As T has exponent 4, we see that $\overline{T}/\overline{V}$ is elementary abelian. Choose $V \leq X \leq T$ such that $\overline{X}/\overline{V}$ is a complement to $\overline{Z}/\overline{V}$ in $\overline{T}/\overline{V}$. Then $\overline{X} \cap \overline{Z} \leq \overline{V}$ and $\overline{T} = \overline{Z} * \overline{X}$. We note that $\Phi(\overline{T}) = \overline{V}$ and that $\overline{X}$ has exponent 4. The last assertion follows because $\Phi(\overline{X}) = \overline{X}' \leq Z(\overline{X})$ and $Z(\overline{X})$ is cyclic of order at most 4. □

(6) Let $v \in V^{\#}$ and $\overline{T} := T/\langle v\rangle$. Then one of the following holds:
   (a) $|T| = 2^6$ and $\overline{T} \simeq Q_8 * D_8$.
   (b) $|T| = 2^7$ and $\overline{T} \simeq C_4 * Q_8^2$ or $\overline{T} \simeq C_2 \times (Q_8 * D_8)$.
   (c) $|T| = 2^8$ and $\overline{T} \simeq Q_8^3$.

Proof. Let $Z \leq T$ and $X \leq T$ be such that $\overline{Z} = \Omega_1(Z(\overline{T}))$ and $\overline{T} = \overline{Z} * \overline{X}$ as in (5).
Suppose that $r(\overline{Z}) = 3$. Then (4) yields that $\overline{X}$ contains a unique involution and this forces $\overline{X}$ to be quaternion of order 8 or cyclic of order 4. In both cases T possesses a series of subgroups as in (3), a contradiction.

Next suppose that $r(\overline{Z}) = 2$. Then, again by (4), we have $r(\overline{X}) \leq 2$. The structure of $\overline{X}$ together with Lemma 3.3 leaves the possibilities

$$\overline{X} \simeq C_4,\ Q_8,\ D_8,\ C_4 * Q_8,\ Q_8 * D_8.$$

As $|\overline{Z}| = 4$, the cases $\overline{X} \simeq C_4,\ Q_8$ and D_8 contradict (3). If $\overline{X} \simeq C_4 * Q_8$, then $\overline{X}$ has order 16 with a centre of order 4, so again we find a series of subgroups as in (3), a contradiction. This only leaves the possibility $\overline{X} \simeq Q_8 * D_8$. Then the central involution of $\overline{X}$ lies in $\overline{Z}$ and hence $\overline{T} \simeq C_2 \times (Q_8 * D_8)$ which is the second case in (b).
Finally suppose that $\overline{Z}$ is cyclic. Then $|\overline{Z}| = 2$ and therefore $\overline{Z} = \overline{V}$ and $\overline{T} = \overline{X}$. By (4) and Lemma 3.3, the possibilities for $\overline{T}$ are

$$\overline{T} \simeq C_4,\ Q_8,\ D_8,\ C_4 * Q_8,\ Q_8 * D_8,\ Q_8^2, C_4 * Q_8^2 \simeq C_4 * Q_8 * D_8,\ Q_8^3.$$

By (3), we can exclude the cases $\overline{T} \simeq C_4,\ Q_8,\ D_8$ and $C_4 * Q_8$. The next possibility is $\overline{T} \simeq Q_8 * D_8$ which leads to (a). We observe that the automorphism group of Q_8^2 is soluble which leaves us with $\overline{T} \simeq C_4 * Q_8^2$ or Q_8^3, i.e. the cases (b) and (c).

□

For all $v \in V^\#$ we define $T_v := \{t \in T \mid t^2 = v\}$. Then T is the disjoint union of V, T_a, T_b and T_c and A leaves every member of this partition invariant. Now we set $\widetilde{T} := T/V$ and $\widetilde{T}_v := \{\tilde{t} \mid t \in T_v\}$ for every involution $v \in V$.

(7) $|\widetilde{T}| = 16$ and $|\widetilde{T}_a| = |\widetilde{T}_b| = |\widetilde{T}_c| = 5$.

Proof. The partition of T into V, T_a, T_b and T_c yields that

$$|\widetilde{T}| - 1 = |\widetilde{T}_a| + |\widetilde{T}_b| + |\widetilde{T}_c|.$$

By (6) we have $|\widetilde{T}| \in \{2^4, 2^5, 2^6\}$.
Assume that $|\widetilde{T}| = 2^6$. Then we are in case (c) of (6) and Lemma 3.3 yields that, for all $v \in V^\#$, the factor group $T/\langle v\rangle$ possesses 36 cyclic subgroups of order 4. Thus there are $2^7 - 1 - 72 = 55$ involutions in $T/\langle v\rangle$ which gives $\frac{1}{2}(55-1) = 27$ distinct elements in $\widetilde{T}_v$. It follows that $63 = 2^6 - 1 = |\widetilde{T}| - 1 = 3 \cdot 27$, a contradiction.
Assume that $|\widetilde{T}| = 2^5$. By Lemma 3.3, the group Q_8^2 possesses 6 cyclic subgroups of order 4 and therefore $32 - 1 - 12 = 19$ involutions. Hence referring to the first case in (6)(b), for all $v \in V^\#$, the factor group $T/\langle v\rangle$ possesses $19 + 12 = 31$ involutions, giving $\frac{1}{2}(31-1) = 15$ distinct elements in $\widetilde{T}_v$. It follows that $31 = 2^5 - 1 = |\widetilde{T}| - 1 = 3 \cdot 15$, a contradiction.
Again by Lemma 3.3, the group $Q_8 * D_8$ has 10 cyclic subgroups of order 4 and hence $32 - 1 - 20 = 11$ involutions. Thus for the second case in (6)(b) we obtain that, for all $v \in V^\#$, the factor group $T/\langle v\rangle$ possesses $2 \cdot 11 + 1 = 23$ involutions. This gives $\frac{1}{2}(23-1) = 11$ distinct elements in $\widetilde{T}_v$ and it follows that $31 = 2^5 - 1 = |\widetilde{T}| - 1 = 3 \cdot 11$, again a contradiction.
Therefore we are left with the case that $|\widetilde{T}| = 2^4$ which is (6)(a). Hence $T/\langle v\rangle \simeq Q_8 * D_8$ for all $v \in V^\#$, and we showed above that $T/\langle v\rangle$ has 11 involutions. The 10 non-central involutions yield 5 distinct elements in $\widetilde{T}_v$. Thus $|\widetilde{T}_a| = |\widetilde{T}_b| = |\widetilde{T}_c| = 5$ as stated.

□

It follows from (7) that $A/C_A(\widetilde{T})$ is isomorphic to a (perfect) subgroup of S_5 and has three orbits of length 5 on $\widetilde{T}^{\#}$. This corresponds to the fact that $|T_a| = |T_b| = |T_c| = 20$. Since the image of $A/C_A(\widetilde{T})$ in S_5 contains A_5, it is 2-transitive on every set $\widetilde{T}_v,\ v \in V^{\#}$. Hence so is A.
We choose elements $t_1, ..., t_5 \in T$ such that $\widetilde{T_a} = \{\widetilde{t_1}, ..., \widetilde{t_5}\}$ and we let $\overline{T} := T/\langle a\rangle$. Then $T_a = \{t_i v \mid i \in \{1, ..., 5\},\ v \in V\}$. We recall that $T' \leq V$ and therefore $[t_1, t_2] = w$ for some $w \in V$. But then the 2-transitivity of A yields that $[t_i, t_j] = w$ (and therefore $(t_i t_j)^2 = w$ whenever $i \neq j$) for all $i, j \in \{1, ..., 5\}$. If $w \neq 1$, then the elements $t_i t_j$ and their products with a, b and c give 40 distinct members of T_w, a contradiction. Hence $w = 1$ and it follows that $\langle t_1, ..., t_5\rangle$ is abelian and that all the products $t_i t_j$ $(i \neq j)$ are equal and of order 2. Therefore $\overline{\langle t_1, ..., t_5\rangle}$ is elementary abelian and contains all involutions of $\overline{T}$. This contradicts the fact that $\overline{T} \simeq Q_8 * D_8$.
Thus we have established the theorem in the case where $V \leq Z(T)$.

Now suppose that $Z(T)$ possesses only one involution, say a. Then $T_0 := C_T(b)$ is an A-invariant subgroup of index 2 in T containing V. But $Z(T)$ as well as b lie in $Z(T_0)$. Thus T_0 is a 2-group which has exactly three involutions, and they are all central. Applying the theorem for T_0 yields that $A/C_A(T_0)$ is soluble. On the other hand $[T, A] \leq T_0$ and therefore $C_A(T_0) < A$ is soluble. This is impossible and proves the full statement. □

We would like to mention the work on the classification of 2-groups with precisely three involutions by Janko and others. (See for example [15].) The above theorem is a corollary of this classification, proceeding case by case, but not at all immediate. We therefore decided to give direct arguments.

4. Isolated Involutions

From now on G is a finite group and $z \in G$ is an isolated involution, i.e. an involution z such that the only conjugate of z in G commuting with z is z itself. We set $C := C_G(z)$ and start by collecting some basic facts about isolated involutions. Then we deduce knowledge about the set $K := \{zz^g \mid g \in G\}$ of commutators and use it to make initial statements about the structure of G.

Lemma 4.1. *Let $z \in S \in Syl_2(G)$.*

(1) $z^G \cap S = \{z\}$.
(2) Every z-invariant 2-subgroup of G is centralised by z. In particular $z \in Z(S)$.
(3) zz^g is an element of odd order for all $g \in G$.
(4) Whenever $z \in H \leq G$, then $z^G \cap H = z^H$.
(5) Let $w \in G \backslash z^G$ be an involution. Then the order of zw is even, but not divisible by 4. In particular, the Sylow 2-subgroups of $\langle z, w\rangle$ are elementary abelian of order 4.
(6) If $z \in X \trianglelefteq Y \leq G$, then $Y = XC_Y(z)$.
(7) Suppose that $z \notin N \trianglelefteq G$ and let $\overline{G} := G/N$. Then $C_{\overline{G}}(\overline{z}) = \overline{C}$ and $\overline{z}$ is isolated in $\overline{G}$.
(8) If $C \leq H \leq G$, then H is the only conjugate of H in G which contains z.

Proof. (1)-(3) are straightforward from the definition of "isolated".
(4) Let $g \in G$ be such that $z^g \in H$. We observe that $\langle z, z^g\rangle$ is a dihedral group of twice odd order by (3). Thus z and z^g are conjugate in $\langle z, z^g\rangle$ by Sylow's Theorem.

(5) Set $D := \langle z, w\rangle$ and note that zw has even order because otherwise z and w are conjugate. Let $z \in T \in Syl_2(D)$. Then $z \in Z(T)$ by (2) and on the other hand a power of zw is the unique central involution in D. Therefore T is elementary abelian of order 4.
(6) Let $z \in P \in \mathrm{Syl}_2(X)$. As z is isolated and central in P by (2), we have $N_Y(P) \leq C_Y(z)$. Hence with a Frattini argument, it follows that $Y = XN_Y(P) \leq XC_Y(z)$ as stated.
(7) Let $g \in G$ be such that $(Ng)^z = Ng$. Then $g^z g^{-1} \in N$ and hence $z^{g^{-1}} \in N\langle z\rangle$. By (4) it follows that z and $z^{g^{-1}}$ are conjugate in $N\langle z\rangle$. Choose $x \in N\langle z\rangle$ to be such that $z^x = z^{g^{-1}}$. Then $z^{xg} = z$ which means that $xg \in C$ and therefore $xgz \in C$. On the other hand, since $x \in N\langle z\rangle$, we have $Nxgz = Nzgz = Ng$, so we see that every z-invariant coset of N has a representative in C. Thus $C_{\overline{G}}(\overline{z}) = \overline{C}$ and the second statement follows from there.
(8) Assume that $z \in H^g$ where $g \in G\backslash N_G(H)$. Then $z \in H \cap H^g$ and therefore $z, z^{g^{-1}} \in H$. It follows from (4) that there exists an element $h \in H$ such that $z = z^{hg}$. Hence $hg \in C \leq H$ and thus $g \in H$, a contradiction. □

Lemma 4.2. *Suppose that $z \in H \leq G$. Then $H \cap C$ controls fusion in $H \cap C$ with respect to H.*

Proof. Let $x, y \in H \cap C$ and let $h \in H$ be such that $x^h = y$. We need to show that x and y are conjugate in $H \cap C$. Now as x, x^h are both contained in C, it follows that $z, z^{h^{-1}} \in C_H(x)$. But then Lemma 4.1 (4) yields that z and $z^{h^{-1}}$ are conjugate in $C_H(x)$. Let $a \in C_H(x)$ be such that $z^a = z^{h^{-1}}$. This gives $z = z^{ah}$ which means that $ah \in C \cap H$. As $x^{ah} = x^h = y$, we are done. □

Corollary 4.3. *If $O^2(G) = G$, then $O^2(C) = C$.*

Proof. By Lemma 4.1 (2) we know that C contains a Sylow 2-subgroup of G. Now suppose that $y \in C$ is a 2-element. Then $y^G \cap C = y^C$ by Lemma 4.2. So the result follows from Lemma 3.8. □

Lemma 4.4. *Let $s, t \in z^G$ be distinct. Then $st \notin C$.*

Proof. Assume that $st \in C$ and set $X := C_G(st)$, $Y := X\langle t\rangle$. Then t inverts st and z centralises st and thus $z \in X \trianglelefteq Y$ and $t \notin X$. But z and t are both contained in Y and therefore conjugate in Y by Lemma 4.1 (4). This is impossible. □

Recall that $K = \{zz^g \mid g \in G\}$, and for all $a, b \in K$ let $a \circ b := aba$. In [5], where Fischer proves a special case of the Z^*-Theorem, the operation $\circ$ is introduced more generally in the context of distributive quasi-groups. Glauberman refers to Fischer's result in [6]. In [8] he mentions that the Z^*-Theorem is a group theoretic equivalent to the fact that every finite loop of odd order with certain additional properties - which he refers to as B-loops - is soluble. The reader familiar with these results (and more recent work, e.g. [2]) might therefore recognise the following construction from the context of loop theory.

Lemma 4.5.

(1) K is C-invariant and contains 1.

(2) *Every element in K has odd order and is inverted by z.*
(3) *Let $a \in K$. Then $a^n \in K$ for all $n \in \mathbb{N}$.*
(4) *$\circ$ is a binary operation on K.*

Proof. The first statement is immediate. By Lemma 4.1 (3), the elements of K have odd order. Moreover if $a \in K$, that is $a = zz^g$ for some $g \in G$, then $a^z = zzz^g z = z^g z = a^{-1}$. For (3) we observe that, if $a = zz^g$ with $g \in G$, then $a^n = (zz^g)^n = zh$ where h is conjugate to z. For the last assertion let $a, b \in K$, i.e. let $g, h \in G$ be such that $a = zz^g$ and $b = zz^h$. Then $a \circ b = aba = zz^g zz^h zz^g = zz^{ha} \in K$ and therefore $\circ$ is a binary operation on K. □

Lemma 4.6.

(1) *Let $a, b, d \in K$. If $a \circ b = d$, then $a^{-1} \circ d = b$. Moreover $a^{-1} \circ b^{-1} = (a \circ b)^{-1}$.*
(2) *For all $a \in K$, the maps $k \mapsto k \circ a$ and $k \mapsto a \circ k$ are bijective.*

Proof. For the first result, we recall that $a \circ b = d$ means that $aba = d$. Thus $a^{-1} \circ d = a^{-1} d a^{-1} = b$ as stated. Finally $(a \circ b)^{-1} = (aba)^{-1} = a^{-1} b^{-1} a^{-1} = a^{-1} \circ b^{-1}$.
For the second statement, it suffices to show that both maps are injective. Let $a, b, d \in K$, i.e. let $g, h, k \in G$ be such that $a = zz^g$, $b = zz^h$ and $d = zz^k$. Suppose that $a \circ b = a \circ d$. Then immediately $b = d$. Now if $a \circ b = d \circ b$, then $zz^g zz^h zz^g = zz^k zz^h zz^k$ and it follows that $z^{ha} = z^{hd}$. Hence ha and hd are in the same coset of C in G which means that $z^k z^g \in C$. Then Lemma 4.4 forces $a = d$.

□

Definition. *For all $a, b \in K$, we denote by $a + b$ the (by Lemma 4.6 (2)) unique element d in K with the property that $d \circ a^{-1} = b$. In other words, $(a + b)a^{-1}(a + b) = (a + b) \circ a^{-1} = b$.*

Lemma 4.7. *Let $a, b, d \in K$.*

(1) *$a + b = b + a$.*
(2) *For all $c \in C$ we have $(a + b)^c = a^c + b^c$.*
(3) *$(a + b)^{-1} = a^{-1} + b^{-1}$.*
(4) *$a + b = 1$ if and only if $b = a^{-1}$.*
(5) *$a \circ (b + d) = a \circ b + a \circ d$.*

Proof. By definition, $(a + b) \circ a^{-1} = b$. Applying Lemma 4.6 (1) yields $(a + b)^{-1} \circ b = a^{-1}$ and then $(a + b) \circ b^{-1} = a$. But, again by definition, $a = (b + a) \circ b^{-1}$. This implies $a + b = b + a$.
The second statement is immediate. As z is in C and inverts K (by Lemma 4.5 (2)), the third statement follows. Now $a + b = 1$ means that $1 \circ a^{-1} = b$, but then $a^{-1} = 1a^{-1}1 = b$. Conversely $1 \circ a^{-1} = a^{-1} = (a + a^{-1}) \circ a^{-1}$ by definition of $+$ and therefore $a + a^{-1} = 1$ by Lemma 4.6 (2).
For the last assertion we recall that, by definition, $(b + d) \circ b^{-1} = d$. This gives $a \circ d = a \circ ((b + d) \circ b^{-1})$. On the other hand, by definition of the element $a \circ b + a \circ d$, we have $a \circ d = (a \circ b + a \circ d) \circ (a \circ b)^{-1}$. This yields
$(a \circ b + a \circ d) \circ (a \circ b)^{-1} = a \circ d = a \circ ((b + d) \circ b^{-1}) = a((b + d)b^{-1}(b + d))a$
$= a((b + d)aa^{-1}b^{-1}a^{-1}a(b + d))a = a(b + d)a((a \circ b)^{-1})a(b + d)a,$

by Lemma 4.6 (1). But

$$a(b+d)a((a\circ b)^{-1})a(b+d)a = (a\circ(b+d))(a\circ b)^{-1}(a\circ(b+d)) = (a\circ(b+d))\circ(a\circ b)^{-1}.$$

Therefore

$$(a\circ b + a\circ d)\circ(a\circ b)^{-1} = (a\circ(b+d))\circ(a\circ b)^{-1}$$

and Lemma 4.6 (2) gives the result. □

Theorem 4.8. *Let $a\in K$ and let $s\in C$ be an involution. Then $a = u\circ v$ where $u\in C_K(s)$, $v\in C_K(sz)$, and this representation of a is unique. In particular, $|K| = |C_K(s)||C_K(sz)|$ and $K\subseteq\langle C_K(s), C_K(sz)\rangle$.*

Proof. We have $a+a^s = a^s+a = (a+a^s)^s$ by Lemma 4.7 (1) and (2) and therefore $a+a^s\in C_K(s)$. If for all $b\in K$ we set $\overline{b} := b+b^s$ and if we let $J := \{b\in K \mid \overline{b} = 1\}$, then Lemma 4.7 (4) yields $J = \{b\in K \mid b+b^s = 1\} = I_K(s) = C_K(sz)$.
As $\overline{a}\in K$ is of odd order (Lemma 4.5 (2)), there exists a power y of $\overline{a}$ with the property that $(y^{-1})^2 = \overline{a}$. We pick this element y and observe that, by Lemma 4.5 (3), it is contained in K and thus lies in $C_K(s)$. Furthermore $y\circ\overline{a} = 1$. Lemma 4.7 (5) and the fact that s centralises y imply that

$$y\circ\overline{a} = y\circ(a+a^s) = y\circ a + y\circ a^s = y\circ a + (y\circ a)^s = \overline{y\circ a}.$$

Thus $\overline{y\circ a} = y\circ\overline{a} = 1$ which means that $y\circ a\in J$. Now let $u := y^{-1}$ and $v := y\circ a$. Then

$$a = y^{-1}yayy^{-1} = y^{-1}\circ(y\circ a) = u\circ v\in C_K(s)\circ C_K(sz).$$

This proves the existence of a representation as stated.
For the uniqueness suppose that $a = u'\circ v'\in C_K(s)\circ C_K(sz)$. Then

$$\overline{a} = \overline{u'\circ v'} = (u'\circ v') + (u'\circ v')^s = (u'\circ v') + (u'\circ v'^s) = u'\circ(v'+v'^s)$$

where the last equality comes from Lemma 4.7 (5). Moreover $v'\in J = I_K(s)$ by choice which implies that $\overline{v'} = 1$. We deduce that

$$\overline{a} = u'\circ(v'+v'^s) = u'\circ\overline{v'} = u'\circ 1 = u'^2$$

and therefore $u'^2 = \overline{a} = u^2$. As u and u' are of odd order, we obtain $u = u'$. Finally Lemma 4.6 (2) yields that also $v = v'$. □

Lemma 4.9. *Suppose that $z\in H\le G$. Then $H = C_H(z)(H\cap K)$. More precisely, every coset of $C_H(z)$ in H contains a unique element which is inverted by z .*

Proof. Set $C_0 := C_H(z)$. As K is C-invariant, $H\cap K$ is C_0-invariant and every non-trivial element in $H\cap K$ is inverted and not centralised by z . Therefore $(H\cap K)\cap C_0 = 1$. As $\{zz^h \mid h\in H\}\subseteq H\cap K$, we have $|H\cap K|\ge|H:C_0|$. Now we show that $H\cap K$ contains a unique representative for every coset of C_0 in H. Suppose that $zz^g, zz^h\in H\cap K$ are such that $C_0zz^g = C_0zz^h$. Then $z^gz^h\in C_0\le C$ which by Lemma 4.4 is only possible if $z^g = z^h$. The result follows. □

Corollary 4.10. *$G = CK$ and more precisely, every coset of C in G contains a unique element which is inverted by z . Furthermore $|G| = |C||C_K(sz)||C_K(s)|$ for every involution $s \in C$.*

Proof. This follows from Theorem 4.8 and Lemma 4.9. □

Theorem 4.11. *Let $p \in \pi(G)$. Then $\mathcal{V}^*_G(\langle z\rangle, p) \subseteq Syl_p(G)$.*

Proof. As z lies in a Sylow 2-subgroup of G, we may suppose that p is odd. We proceed by induction on $|G|$ and first show that $\mathcal{V}_G(\langle z\rangle, p) \neq \{1\}$. Suppose that $r_2(G) = 1$. Then the Sylow 2-subgroups of G are cyclic or quaternion. It follows that $z \in Z^*(G)$ by Theorem 3.11 or the Brauer-Suzuki Theorem (3.12), respectively. But then $G = CO(G)$ and at least one of these subgroups has order divisible by p. If p divides $|C|$, then z centralises a non-trivial p-subgroup of G. If p divides $|O(G)|$, then Coprime Action (d) yields that $\{1\} \neq \mathcal{V}_{O(G)}(\langle z\rangle, p) \subseteq \mathcal{V}_G(\langle z\rangle, p)$.
Thus we may suppose that $r_2(G) \geq 2$ and we choose an involution $s \in C$ distinct from z . By Corollary 4.10, p divides one of $|C|$, $|C_K(s)|$ or $|C_K(sz)|$. If p divides $|C|$, then we are done. Suppose therefore that p does not divide $|C|$. Then without loss of generality p divides $|C_K(s)| = |C_G(s) : C_C(s)|$ (by Lemma 4.9). But z is contained in $C_G(s)$, thus by induction $\mathcal{V}_{C_G(s)}(\langle z\rangle, p) \neq \{1\}$ if $C_G(s) < G$. On the other hand, if $C_G(s) = G$, then $s \in Z(G)$. We can therefore argue by induction in the factor group $G/\langle s\rangle$, applying Lemma 4.1 (7). We conclude that $\mathcal{V}_G(\langle z\rangle, p) \neq \{1\}$.
Now let $P_0 \in \mathcal{V}^*_G(\langle z\rangle, p)$ and let $N_0 := N_G(P_0)$. We have $z \in N_0$. If $N_0 < G$, then by induction $\mathcal{V}^*_{N_0}(\langle z\rangle, p) \subseteq \mathrm{Syl}_p(N_0)$. By the maximal choice of P_0, this yields $P_0 \in \mathrm{Syl}_p(N_0)$ and therefore $P_0 \in \mathrm{Syl}_p(G)$. If on the other hand $N_0 = G$, then $P_0 \trianglelefteq G$ and in G/P_0 there exists a z-invariant Sylow p-subgroup by induction. Its preimage in G is a z-invariant Sylow p-subgroup of G and equals P_0 by the maximal choice of P_0. □

Definition. *From now on, for every subgroup H of G and for every prime p, we denote by $Syl_p(H, z)$ the set of all z-invariant Sylow p-subgroups of H. Similarly, if V is a 2-subgroup of G, then we denote by $Syl_p(H, V)$ the set of V-invariant Sylow p-subgroups of H.*

Lemma 4.12. *Let $p \in \pi(G)$. Then C is transitive on $Syl_p(G, z)$.*

Proof. Let $P_1, P_2 \in \mathrm{Syl}_p(G, z)$ and let $g \in G$ be such that $P_1^g = P_2$. Since $z \in N_G(P_2) = (N_G(P_1))^g$, we conclude that z and z^g are both contained in $N_G(P_2)$. They are therefore conjugate in $N_G(P_2)$ by Lemma 4.1 (4). Choose $h \in N_G(P_2)$ such that $z = z^{gh}$. Then $gh \in C$ and $P_1^{gh} = P_2^h = P_2$. □

Remark 4.13. *Let $V \leq G$ be an elementary abelian 2-subgroup generated by (necessarily non-conjugate) isolated involutions. Then the results in 4.11 and 4.12 can easily be generalised to the following:*
For all primes $p \in \pi(G)$ we have $\mathcal{V}^*_G(V, p) \subseteq Syl_p(G)$ and $C_G(V) = N_G(V)$ is transitive on $Syl_p(G, V)$.

Lemma 4.14. *Let $V \leq G$ be elementary abelian of order 4 and such that $z \in V$. Let a, b, z denote the involutions in V. Let p be a prime and suppose that $P \in Syl_p(G, V)$ is such that $P \leq C_G(a)$. Suppose that C does not contain any Sylow p-subgroup of G. Then a and b are not conjugate in G.*

Proof. First we apply Corollary 4.10 to obtain $|G|_p = |C|_p|C_K(a)|_p|C_K(b)|_p$. Our hypothesis gives $|G|_p = |P| = |C_G(a)|_p = |C_C(a)|_p|C_K(a)|_p$ by Lemma 4.9. Therefore $|C|_p|C_K(b)|_p = |C_C(a)|_p$. As $C_C(a) \leq C$, this yields $|C_K(b)|_p = 1$. Now assume that a and b are conjugate in G. Then by Lemma 4.2 there exists an $x \in C$ such that $a^x = b$. In particular $C_C(a)^x = C_C(b)$ and $C_K(a)^x = C_K(b)$. But this implies $|C_K(a)|_p = 1$ and therefore $|G|_p = |C|_p|C_K(a)|_p|C_K(b)|_p = |C|_p$ contrary to our hypothesis that C does not contain any Sylow p-subgroup of G. □

Lemma 4.15. *Let $p \in \pi(G)$ and let $P \in Syl_p(G, z)$. Then $P \cap C \in Syl_p(C)$ and $|K|_p = |I_P(z)| = |P : C_P(z)|$.*

Proof. Let $P \cap C \leq P_0 \in \mathrm{Syl}_p(C)$. Theorem 4.11 yields that $P_0 \leq P_1 \in \mathrm{Syl}_p(G, z)$ and by Lemma 4.12 there exists an $x \in C$ such that $P = P_1^x$. But then $P_0^x \leq C_{P_1}(z)^x = C_P(z) = P \cap C$ and therefore $P \cap C$ is already a Sylow p-subgroup of C. For the second statement, Corollary 4.10 gives $|G| = |C||K|$ and thus $|P| = |G|_p = |C|_p|K|_p$. On the other hand

$$|P| = |C_P(z)||P : C_P(z)| = |C_P(z)||I_P(z)| = |C|_p|I_P(z)|$$

by the previous paragraph. Hence $|K|_p = |I_P(z)|$. □

5. A minimal counterexample to Glauberman's Z^*-Theorem

Throughout this section we assume the following:

Hypothesis 5.1. *Let G be a counterexample to Glauberman's Z^*-Theorem such that in every proper subgroup or factor group of G the Z^*-Theorem holds. Let z be an isolated involution with $z \notin Z^*(G)$ and set $K := \{zz^g \mid g \in G\}$. Let $C := C_G(z)$ and let M be a maximal subgroup of G containing C.*

We point out that Hypothesis 5.1 holds in a minimal counterexample to the Z^*-Theorem with respect to the order of the group.
We begin by collecting some initial observations which we use frequently.

Lemma 5.2.

(1) If $z \in H < G$, then $z \in Z^*(H)$. Hence $H = C_H(z)O(H)$ and in particular $H \cap K \subseteq O(H)$. Moreover, $O_{2',2}(C) \cap H \leq O_{2',2}(H)$, z centralises $E(H)$ and $\mathcal{V}^*_H(\langle z \rangle, p) \subseteq Syl_p(H)$ for all $p \in \pi(H)$.

(2) $r_2(G) \geq 2$.

(3) G possesses at least two conjugacy classes of involutions.

(4) K generates a subgroup of even order.

Proof. The first statement in (1) follows directly from the minimality of G and implies the rest. For the last assertion note that we may suppose that p is odd. Then Coprime Action (d) gives z-invariant Sylow p-subgroups of $O(H)$ and the statement follows. If $r_2(G) = 1$, then the Sylow 2-subgroups of G are cyclic or quaternion and hence Theorem 3.11 or the Brauer-Suzuki Theorem, respectively, yield a contradiction. This proves (2). Then (3) follows from (2) and the fact that G has a conjugacy class of isolated involutions. For (4) note that $\langle K \rangle = [G, z] \trianglelefteq G$. If this group has odd order, then G is not a counterexample. □

Lemma 5.3. *$G = F^*(G)\langle z\rangle$ and $F^*(G)$ is simple.*

Proof. We apply Lemma 4.1 (7) to deduce $O(G) = 1$. Next we show that $G = \langle z^G\rangle$. Assume that $H := \langle z^G\rangle < G$ and note that $H \trianglelefteq G$. Then $z \in Z^*(H) = Z(H)$ by hypothesis and because $O(H) \le O(G) = 1$. Hence z commutes with all its conjugates in G. But z is isolated and therefore this implies $z \in Z(G)$, a contradiction. We note that z centralises $O_2(G)$ by Lemma 4.1 (2). However, if $z \in O_2(G)$, then $z^g \in O_2(G)$ for all $g \in G$ and therefore z commutes with all its conjugates, a contradiction. This forces $z \notin O_2(G)$.
Let N be a minimal normal subgroup of G and first suppose that N is a 2-group. Then $N \le O_2(G)$ is centralised by z (by Lemma 4.1 (2)) and therefore by all conjugates of z . But as $G = \langle z^G\rangle$, this means that $Z(G)$ contains N and therefore an involution t. In the factor group $\overline{G} := G/\langle t\rangle$ we have $\overline{z} \in Z^*(\overline{G})$ by hypothesis and Lemma 4.1 (7). Now let $X \trianglelefteq G$ be such that $\overline{X} = O(\overline{G})$. Then $\langle t\rangle \in \mathrm{Syl}_2(X)$ and finally $X = \langle t\rangle$ by Theorem 3.11 and because $O(X) \le O(G) = 1$. Therefore $\overline{z} \in Z^*(\overline{G}) = Z(\overline{G})$ which means that $z \in O_2(G)$, a contradiction. We conclude that N is of even order, but not a 2-group, and in particular $F(G) = 1$. Assume that $N\langle z\rangle < G$. Then $z \in Z^*(N\langle z\rangle)$ by Lemma 5.2 (1) and therefore $[N, z] \le N \cap O(N\langle z\rangle) = 1$. This implies that N centralises z and all its conjugates which means that $N \le Z(G) = 1$, a contradiction. Hence $N\langle z\rangle = G$.
Finally let Y be a minimal normal subgroup of N and assume that $Y \neq N$. Then Y is not z-invariant because otherwise $Y \trianglelefteq G$, further $Y \cap Y^z = 1$ and Y has even order. It follows by Lemma 4.1 (2) that z centralises a non-trivial 2-subgroup of Y. This subgroup therefore lies in $Y \cap Y^z = 1$, a contradiction. Thus $N = F^*(G)$ is simple. □

Lemma 5.4.

(1) Let $1 \neq N \trianglelefteq G$. Then $G = N\langle z\rangle$.
(2) Let H be a maximal subgroup of G containing a conjugate of z . Then $H = N_G(X)$ for every non-trivial normal subgroup X of H.
(3) $\langle K, z\rangle = G$.

Proof. For (1) we apply Lemma 5.3 to see that $N\langle z\rangle = (N \cap F^*(G))\langle z\rangle$, with Dedekind's Law. But $F^*(G)$ is simple, so $N \cap F^*(G) \trianglelefteq F^*(G)$ forces $N \cap F^*(G) = F^*(G)$ (which gives the assertion) or $N \cap F^*(G) = 1$. The latter means that $N = \langle z\rangle$ is normal in G, a contradiction.
For (2) let $1 \neq X \trianglelefteq H$ and let $z' \in z^G \cap H$. We note that z' is isolated, in fact z' has precisely the same properties as z . In particular Lemma 5.3 (and therefore (1) above) is applicable for z' instead of z . The maximality of H implies that $N_G(X) = H$ or $N_G(X) = G$. In the second case we deduce from (1), applied for z', that $G = X\langle z'\rangle \le H$ which is impossible.
(3) follows from (1) because $\langle K\rangle$ is a non-trivial normal subgroup of G. □

The previous result shows that G behaves almost like a simple group. We therefore refer to property (1) in Lemma 5.4 above by saying that G is **z-simple**. It turns out that a similar statement holds for every isolated involution of G.

Lemma 5.5. *Let $a \in G$ be an isolated involution. Then $a \notin Z^*(G)$, but $a \in Z^*(H)$ for all proper subgroups H of G containing a. Moreover G is a-simple, i.e. $G = N\langle a\rangle$ for every non-trivial normal subgroup N of G.*

Proof. We may suppose that $a \neq z$. By Lemma 5.3 we have $O(G) = 1$ and $Z^*(G) = Z(G) = 1$. In particular $a \notin Z^*(G)$. But as the Z^*-Theorem holds in every proper subgroup of G, we have $a \in Z^*(H)$ whenever $a \in H < G$. Now suppose that N is a non-trivial proper normal subgroup of G. Then N is simple by Lemma 5.3. In particular $Z^*(N) = 1$ which implies that $a \notin N$. But $|G : N| = 2$ with Lemma 5.4 (1) and therefore $N\langle a\rangle = G$. □

Lemma 5.6. *Let $t \in z^G$ and set $n := |M : C|$. Suppose that $t \notin M$. Let $D := M \cap M^t$ and let I denote the set of elements in D which are inverted by t. Then the following hold:*

(1) $D = O(D)C_D(t)$
(2) I, and hence D, is transitive on z^M.
(3) $M = CI$. More precisely, every coset of C in M contains exactly one element of I.
(4) $|I| = |D : C_D(t)| = |D : C_D(z)| = n$.
(5) Let $q \in \pi(G)$ and $Q \in Syl_q(D, t)$. Then $|I_Q(t)| = n_q$.

Proof. As D is t-invariant amd t is isolated, we have $D\langle t\rangle = O(D\langle t\rangle)C_{D\langle t\rangle}(t)$. Hence $[D, t] \leq D \cap O(D\langle t\rangle) \leq O(D)$ which gives the first statement.
Let $u \in z^M$. Then u and z^t are conjugate, by Sylow's Theorem, because uz^t has odd order (Lemma 4.1 (3)). In fact there exists an involution $s \in \langle u, z^t\rangle$ such that $u^s = z^t$. Now $u = z^{ts}$. On the other hand, since $u \in z^M$, Lemma 4.1 (4) yields that z and u are also conjugate in M. Choose $x \in M$ such that $u^x = z$. Then $z = u^x = z^{tsx}$ and therefore $tsx \in C$. This yields $ts \in M$ because $\langle x, C\rangle \leq M$. As ts is inverted by t, it follows that $ts \in M \cap M^t = D$ and thus $ts \in I$. This gives the second statement and implies $M = CI$. To finish the proof of (3), let $x_1, x_2 \in I$ be such that $Cx_1 = Cx_2$. Then $x_1x_2^{-1} \in C$. But x_1t and x_2t are involutions which are conjugate to t and thus to z . Therefore $x_1tx_2t = x_1t(x_2t)^{-1} = x_1x_2^{-1} \in C$. Lemma 4.4 implies that $x_1t = x_2t$ and finally $x_1 = x_2$.
For (4), we apply Lemma 4.9 to the isolated involution t in $D\langle t\rangle$ and it follows that I is a set of representatives for the cosets of $C_D(t)$ in D. To prove (5) we observe that, since t is isolated in $D\langle t\rangle$, we may apply Lemma 4.15. From there we obtain that $C_Q(t) \in \mathrm{Syl}_q(C_D(t))$ and that $n_q = |D : C_D(t)|_q = |Q : C_Q(t)| = |I_Q(t)|$. □

Lemma 5.7. *Suppose that C is a maximal subgroup of G and let $p \in \pi(F(C))$. Then C contains a Sylow p-subgroup of G and every z-invariant p-subgroup of G is centralised by z .*

Proof. Let $P \in \mathrm{Syl}_p(C)$. Then $z \in C_G(P) \leq C_G(O_p(C))$. But $N_G(O_p(C)) = C$ by Lemma 5.4 (2), so it follows that $C_G(P) \leq C$. Now if we set $X := C_G(P)$ and $Y := N_G(P)$, then Lemma 4.1 (6) yields $Y = XC_Y(z)$. But X and $C_Y(z)$ are both contained in C, thus $N_G(P) = Y \leq C$. This implies that $P \in \mathrm{Syl}_p(G)$. The rest follows from Theorem 4.11 and Lemma 4.12. □

Lemma 5.8. *Suppose that C is a maximal subgroup of G and let $\pi := \pi(F(C))$. Let $z \in H < G$. Then $[H, z]$ is a π'-group.*

Proof. Let $H_0 := [H, z]$ and note that H_0 has odd order by Lemma 5.2 (1). Assume that $p \in \pi \cap \pi(H_0)$. Then Coprime Action (d) implies that $\mathcal{U}^*_{H_0}(\langle z\rangle, p) \subseteq \mathrm{Syl}_p(H_0)$. Lemma 5.7 yields that every z-invariant p-subgroup of H_0 is centralised by z . Thus Lemma 2.2 gives

$H_0 = C_{H_0}(z)O_{p'}(H_0)$. But this means that $H_0 = [H_0, z] \leq O_{p'}(H_0)$, contrary to our choice of p. $\square$

Lemma 5.9. *Suppose that q is a prime such that $O_q(M) \nleq C$. Then M does not contain a Sylow q-subgroup of G.*

Proof. First we observe that q is odd by Lemma 4.1 (2). With Lemma 5.2 (1) we choose $Q \in \mathrm{Syl}_q(M, z)$ and assume that $Q \in \mathrm{Syl}_q(G, z)$. As $O_q(M) \nleq C$, we have $1 \neq X := I_{O_q(M)}(z)$. If we set $n := |M : C|$, then Lemma 4.15 implies that $1 \neq |I_Q(z)| = n_q$. Our objective is to show that X lies in every conjugate of M in G.
We see that X is C-invariant and hence Lemma 4.12 gives that X is contained in every z-invariant Sylow q-subgroup of G. For the same reason every z-invariant q-subgroup of G lies in M. Now let $g \in G \backslash M$ and $M_1 := M^g$. We look at $D := M_1 \cap M_1^z$ and see that $D = C_D(z)O(D)$ and $|D : C_D(z)| = n$ by Lemma 5.6 (1) and (4). If we choose $T \in \mathrm{Syl}_q(D, z)$, then part (5) of the same lemma yields $|I_T(z)| = n_q \neq 1$. Moreover $T \leq M$ because T is z-invariant. But then it follows that $I_T(z) = I_{Q^c}(z) = (I_Q(z))^c$ for a suitable element $c \in C$ and finally $X = X^{c^{-1}} \subseteq I_T(z) \subseteq D \leq M_1$. Hence

$$1 \neq X \subseteq N := \bigcap_{g \in G} M^g \trianglelefteq G.$$

By Lemma 5.4 (1) we have $N\langle z \rangle = G$. But $N\langle z \rangle \leq M$, a contradiction. $\square$

The last lemma of this section plays a role as soon as we bring several involutions into the picture at the same time.

Lemma 5.10. *Let $x \in C \backslash \{z\}$ be an involution. Then $C_G(x) \nleq M$.*

Proof. Let $w := zx$. Then we have $1 \neq \langle w^G \rangle \trianglelefteq G$ and Lemma 5.4 (1) implies $G = \langle w^G \rangle \langle z \rangle$. In particular $\langle w^G \rangle \nleq M$ and thus there exists a conjugate u of w which is not contained in M and thus does not centralise z . We note that w and z are distinct and commute. As z is isolated, this implies that w and z are not conjugate and it follows that u and z are not conjugate. Now set $D := \langle u, z \rangle$. By Lemma 4.1 (5) we know that the order of uz is even and not divisible by 4. More precisely the Sylow 2-subgroups of D are elementary abelian of order 4 and contain the unique central involution v of D. As $u \in C_G(v)$ and $u \notin M$, we have $C_G(v) \nleq M$. Let $z \in T \in \mathrm{Syl}_2(D)$ and let $d \in D$ be such that $u^d \in T$. It follows that $T = \langle z, u^d \rangle$ and hence $v = zu^d$. But u^d $(= zv)$ and w $(= zx)$ both centralise z and therefore they are conjugate in C by Lemma 4.2. Thus $v = zu^d$ and $x = zw$ are conjugate in C and $C_G(v) \nleq M$ implies that $C_G(x) \nleq M$. $\square$

6. Maximal subgroups containing C

In this chapter we use the Bender method to investigate under which assumptions C is a maximal subgroup. We obtain even stronger results in the following section by considering a carefully chosen elementary abelian subgroup of order 4 which contains z . Throughout, we assume Hypothesis 5.1. Note that this implies in particular that G is z-simple (Lemma 5.4 (1)).

Definition. *Let H and L be maximal subgroups of G. Then we say that H **infects** L and we write $H \looparrowright L$ if there exists a subgroup A of $F(H)$ such that $AC_{F^*(H)}(A) \leq L$.*

Lemma 6.1. *Suppose that H and L are maximal subgroups of G which both contain a conjugate of z and suppose that H infects L. Let $\sigma := \pi(F(H))$. Then the following hold:*

(1) $Z(F(H))E(H) \leq L$.
(2) $[E(H), O_q(L)] = 1$ *for all* $q \in \sigma$.
(3) If $E(H) \neq 1$ *or* $|\sigma| \geq 2$*, then* $F_\sigma(L) \leq H$.

Proof. By hypothesis, there exists an involution $z' \in z^G \cap H$. Hence if $1 \neq X \trianglelefteq H$, then Lemma 5.4 (2) yields that $H = N_G(X)$. Similarly $L = N_G(Y)$ whenever $1 \neq Y \trianglelefteq L$.

(1) Let $A \leq F(H)$ be such that $AC_{F^*(H)}(A) \leq L$. Then $Z(F(H))$ centralises A and $[E(H), A] \leq [E(H), F(H)] = 1$. Hence it follows that $Z(F(H))E(H) \leq C_{F^*(H)}(A) \leq L$.
(2) Let $q \in \sigma$ and $Q := Z(O_q(H))$. Then $1 \neq Q \trianglelefteq H$ and by (1) we have $Q \leq L$. Moreover $N_G(Q) = H$ and therefore $C_{O_q(L)}(Q) \leq H$ normalises $E(H)$. Conversely $E(H)$, which lies in L by (1), normalises $C_{O_q(L)}(Q)$. Hence $[C_{O_q(L)}(Q), E(H)] = 1$ and then the fact that $O^q(E(H)) = E(H)$ yields that we can apply Thompson's $P \times Q$-Lemma. It gives that $E(H)$ centralises $O_q(L)$ as stated.
(3) If $E(H) \neq 1$ then, by (2), we have $F_\sigma(L) \leq N_G(E(H)) = H$. Now suppose that $|\sigma| \geq 2$ and let $p, q \in \sigma$ be distinct. Again let $Q := Z(O_q(H))$ and set $P := Z(O_p(H))$. Then $1 \neq P$ is q-perfect (i.e. $P = O^q(P)$) and lies in L by (1). On the other hand $C_{O_q(L)}(Q)$ lies in H and therefore $[C_{O_q(L)}(Q), P] \leq O_q(L) \cap P = 1$. Once more we may apply Thompson's $P \times Q$-Lemma and obtain $O_q(L) \leq C_G(P) \leq H$. Repeating this argument for all primes in σ yields the statement.

□

The next theorem is essential for the Bender method. The result is, in fact, due to Bender and is usually stated for maximal subgroups of simple groups. We felt that the fact that G is only z-simple makes the quotation of theorems for simple groups slightly inconvenient - so rather than doing that and dealing with case distinctions every time, we decided to rephrase Bender's results for our purpose and to give a proof.

Theorem 6.2 (Infection Theorem)**.** *Let H and L be maximal subgroups of G which both contain a conjugate of z and suppose that H infects L. Set $\sigma := \pi(F(H))$.*

(1) $F_{\sigma'}(L) \cap H = 1$.
(2) If q is a prime such that $O_q(H) \neq 1$ and $char(L) = q$, then $char(H) = q$.
(3) If $L \looparrowright H$, then $H = L$ unless $char(H) = q = char(L)$ where q is prime.
(4) If $E(L) \leq H$ and $\pi(F(L)) \subseteq \sigma$, then $H = L$ unless $char(H) = q = char(L)$ where q is prime.

(5) *If H and L are conjugate and $E(L) = 1$, then $H = L$ unless char$(H) = q =$ char(L) where q is prime.*

Proof. Let $A \leq F(H)$ be such that $AC_{F^*(H)}(A) \leq L$ and note that, by Lemma 6.1 above, we have $Z(F(H))E(H) \leq L$. In the statements (1)-(4) we use that $H = N_G(X)$ whenever $1 \neq X \trianglelefteq H$ and similarly $L = N_G(Y)$ whenever $1 \neq Y \trianglelefteq L$, by Lemma 5.4 (2).

(1) As $F := F_{\sigma'}(L) \cap H$ acts coprimely on $F(H)$, we can apply Coprime Action (c) to the subnormal centraliser closed subgroup $AC_{F(H)}(A)$ of $F(H)$. Hence from $[F, AC_{F(H)}(A)] \leq F \cap F(H) = 1$ we deduce $[F, F(H)] = 1$. On the other hand, F and $E(H)$ normalise each other and therefore $[F, E(H)] = 1$. Thus $F \leq C_H(F^*(H)) = Z(F(H))$ which yields $F = 1$.

(2) Suppose that char$(L) = q$. By Lemma 6.1 (2) we have that $E(H)$ centralises $O_q(L) = F^*(L)$. But then $E(H) \leq C_L(F^*(L)) = Z(F^*(L))$ and thus $E(H) = 1$. Now let $P_1 := O_{q'}(Z(F(H)))$, let $Q := Z(O_q(H))$ and note that $C_{O_q(L)}(Q) \leq H$ and $P_1 \leq L$ by Lemma 6.1 (1). So we consider the action of $P_1 \times Q$ on $O_q(L) = F^*(L)$ and see that $[C_{O_q(L)}(Q), P_1] \leq O_q(L) \cap P_1 = 1$. Thompson's $P \times Q$-Lemma yields $[O_q(L), P_1] = 1$. Therefore $P_1 \leq C_L(F^*(L)) = Z(F^*(L))$. But then $P_1 = 1$ and thus $F(H) = F^*(H)$ is a q-group.

(3) From Lemma 6.1 (1) we know that $Z(F(H)) \leq L$ and $Z(F(L)) \leq H$. Together with (1) this yields that $\pi(F(L)) = \sigma$. Again by Lemma 6.1 (1) we have $E(H), E(L) \leq H \cap L$, thus each component of H or L is a component of $H \cap L$. If $F(H) = F(L) = 1$, then it follows that $E(H) = E(L)$ which immediately means $H = L$. Therefore we may suppose that $F(H)$ and $F(L)$ are not both trivial. As $\pi(F(L)) = \sigma$, this implies $F(H) \neq 1 \neq F(L)$. We are done if one of $F^*(L)$ or $F^*(H)$ is a q-group for some prime q because then $\pi(F(L)) = \sigma = \{q\}$, by (2). Thus we may suppose that both $F^*(L)$ and $F^*(H)$ are not q-groups. Then Lemma 6.1 (3) implies that $F(H) \leq L$ and also $F(L) \leq H$ because L infects H. So we have $F^*(H) \leq L$ and $F^*(L) \leq H$. Let $p \in \sigma$ and set $P := O_p(H)$ and $R := O_p(L)$. Note that $PF(L) = PR \times F_{p'}(L)$ is nilpotent. By the previous paragraph we have $[P, O^p(F^*(L))] = 1$ and it follows that

$$[P, C_L(R)] \leq C_L(O^p(F^*(L))) \cap C_L(R) \leq C_L(F^*(L)) \leq Z(F(L)).$$

In particular $PF(L)$ is $C_L(R)$-invariant. But then it follows that $PR = O_p(PF(L))$ is normalised by $C_L(R)$ and therefore $[P, O^p(C_L(R))] = 1$. Hence we have $O^p(C_L(R)) \leq C_H(P)$ and symmetry yields $1 \neq O^p(C_H(P)) = O^p(C_L(R))$, implying $H = L$.

(4) By (2) we are done if $F^*(L)$ is a q-group for some prime q. Now suppose that $F^*(H)$ is a q-group. Then by hypothesis, $\pi(F(L)) \subseteq \sigma = \{q\}$, and the result follows if $E(L) = 1$. As $E(L) \leq H$ with Lemma 6.1 (1), the subgroups $E(L)$ and $AC_{F^*(H)}(A)$ normalise and hence centralise each other. Thus Thompson's $P \times Q$-Lemma yields $[F^*(H), E(L)] = 1$. (Note that $F^*(H) = O_q(H)$.) It follows that $E(L) \leq C_H(F^*(H)) = Z(F(H))$ and finally $E(L) = 1$. Thus char$(L) = q$.

    Suppose now that $F^*(H)$ is not a q-group. We have $\pi(F(L)) \subseteq \sigma$ and thus parts (1) and (3) of Lemma 6.1 imply that $F^*(L) \leq H$. Therefore $L \looparrowright H$ and we can apply (3).

(5) By hypothesis, $E(L) = 1$ and $\pi(F(L)) = \pi(F(H))$. Thus part (4) yields the result.

□

There is a technical detail to sort out before we can state the main theorem of this section (a restatement of Theorem **A**). We define a set $\mathcal{M}$ of maximal subgroups of G containing C in the following way: If the set

$$\{H \leq G \mid C \leq H,\ H \text{ is maximal in G and there exists a } p \in \pi(F(H)) \text{ such that } C_{O_p(H)}(z) = 1\}$$

is non-empty, then $\mathcal{M}$ is defined to be this set. Otherwise, $\mathcal{M}$ is just the set of all maximal subgroups of G containing C.

Theorem 6.3. *Assume Hypothesis 5.1 and let $M \in \mathcal{M}$. Then one of the following holds:*
- $M = C$.
- *$char(M) = p$ for some odd prime p.*
- $E(M) \neq 1$.

We proceed by contradiction and begin with a few preparatory results. Then we formulate the main hypothesis for this section.

Definition. *Let $t \in G$ be an involution. Then a t-invariant $2'$-subgroup W of G is called **t-minimal** if W is minimal with respect to being invariant under $C_G(t)$, but not centralised by t.*

Lemma 6.4. *Suppose that $C \neq M$. Then there exists a prime $p \in \pi$ such that $O_p(M)$ contains a z-minimal subgroup U and $U = [U, z]$.*

Proof. With Lemma 5.2 (1) we have $z \in Z^*(M)$, but $z \notin Z(M)$ by hypothesis. Assume that $[F(M), z] = 1$. Then Lemma 5.2 (1) yields $z \in C_M(F^*(M)) \leq Z(F(M))$ and hence $z \in O_2(M)$. But z is isolated and centralises $O_2(M)$ (Lemma 4.1 (2)), so this implies $z \in Z(M)$, a contradiction. Thus we have $[F(M), z] \neq 1$. As z centralises $O_2(M)$, there exists an odd prime p such that $[O_p(M), z] \neq 1$ and $O_p(M)$ then contains a z-minimal subgroup U. The minimality of U implies $U = [U, z]$. □

We collect a few applications of Lemma 3.6 which enable us to bring the Bender method into action.

Lemma 6.5 (Basis Lemma)**.** *Let $z \in L < G$ and let $a \in O_{2',2}(C)$ be an involution.*

1. *If $F \leq L$ is a nilpotent $C_L(z)$-invariant subgroup, then $[F, z] \leq F(L)$.*
2. *$[F(M) \cap L, z] \leq F(L)$.*
3. *Suppose that $C < M$ and that $p \in \pi$ is such that $O_p(M)$ contains a z-minimal subgroup U. If $U \leq L$, then $U \leq O_p(L)$.*
4. *Let $X \leq M$ be a $C_M(a)$-invariant nilpotent $2'$-subgroup. Then $[X, a] \leq F(M)$.*
5. *Suppose that U_a is an a-minimal subgroup of G. If $a \in L$ and $U_a \leq L$, then we have $U_a = [U_a, a] \leq F(L)$.*

Proof. By hypothesis and Lemma 5.2 (1) we have $z \in Z^*(L)$ and therefore $[F, z] \leq [L, z] \leq O(L)$. Hence $[F, z]$ is a nilpotent $C_L(z)$-invariant $2'$-subgroup of L . By Lemma 3.6 and Coprime Action (a) we obtain $[F, z] = [F, z, z] \leq F(L)$ which is the first result.

The second part follows from the first because $F(M) \cap L$ is a nilpotent $C_L(z)$-invariant subgroup of L. For (3) we note that $U = [U, z] \leq [F(M) \cap L, z] \leq F(L)$ by (2). To obtain the fourth assertion we may apply Lemma 3.6 for a and X (with Lemma 5.2 (1)) which yields $[X, a] \leq F(M)$.
For (5) we recall that U_a has odd order and is $C_G(a)$-invariant. By Lemma 5.2 (1), we have $a \in O_{2',2}(L)$ and hence another application of Lemma 3.6 gives the result. □

Lemma 6.6. *Suppose that $C < M$ and let $p \in \pi$ be such that $O_p(M)$ contains a z-minimal subgroup U. Then $[O_p(C), U] = 1$ and in particular $[C_{F(M)}(z), U] = 1$.*

Proof. Assume that $[O_p(C), U] \neq 1$ and let $U_0 := C_U(O_p(C))$. Then U_0 is centralised by z because of the minimal choice of U. Thus Thompson's $P \times Q$ Lemma yields $[U, z] = 1$, a contradiction. The assertion follows since $[O_{p'}(M), U] \leq [O_{p'}(M), O_p(M)] = 1$. □

Lemma 6.7. *Suppose that $C < M \in \mathcal{M}$. Let H be a maximal subgroup of G containing C and suppose that $M \looparrowright H$. Then $M = H$ or $char(M) = char(H) = p$.*

Proof. By Lemma 5.2 (1) we have $E(H) \leq M$. Now we show that $\pi(F(H)) \subseteq \pi$ in order to apply the Infection Theorem (6.2), part (4). Assume that $F := F_{\pi'}(H)$ is non-trivial. By part (1) of the Infection Theorem, $F \cap M \leq F_{\pi'}(H) \cap M = 1$. In particular, $F \cap C = 1$ which means that F is inverted by z. Hence, as $M \in \mathcal{M}$, there exists a prime q such that $Q := O_q(M)$ is inverted by z. In particular Q is abelian, so $Q \leq Z(F(M)) \leq H$ by Lemma 6.1 (1). Now F is inverted by z and normalised by Q which implies that $Q = [Q, z]$ centralises F. Therefore $F \leq C_G(Q) \leq M$ and it follows $F = F \cap M = 1$. Applying part (4) of the Infection Theorem yields that $M = H$ unless $\mathrm{char}(M) = \mathrm{char}(H) = p$. □

Corollary 6.8. *Suppose that $C < M \in \mathcal{M}$. If $|\pi(F(M))| \geq 2$, then M is the unique maximal subgroup of G containing $N_G(U)$.*

Proof. Suppose that $N_G(U)$ is contained in a maximal subgroup H of G. Then as U lies in $F(M)$ and $UC_{F^*(M)}(U) \leq N_G(U) \leq H$, we have $M \looparrowright H$. On the other hand, U is C-invariant which implies $C \leq H$. Thus Lemma 6.7 and the hypothesis $|\pi(F(M))| \geq 2$ force $H = M$. □

Lemma 6.9. *Suppose that $C < M$ and that $1 \neq X \leq F(M)$ is a $\langle z \rangle U$-invariant subgroup. Let H be a maximal subgroup of G containing $N_G(X)$. If $N_G(U) \leq M$, then $M = H$ or $char(M) = char(H) = p$. In particular if $M \in \mathcal{M}$ and $|\pi(F(M))| \geq 2$, then $N_G(X) \leq M$.*

Proof. By hypothesis, $X \leq F(M)$ and hence $N_G(X) \leq H$ means that $M \looparrowright H$. On the other hand we have $\langle z \rangle U \leq H$ and thus the Basis Lemma (3) yields $U \leq O_p(H)$. If $N_G(U) \leq M$, then H infects M and the Infection Theorem (3) yields the statement. If $M \in \mathcal{M}$ and $|\pi| \geq 2$, then $N_G(U) \leq M$ by Corollary 6.8 which means that again $H \looparrowright M$. But this time part (3) of the Infection Theorem only leaves the possibility $M = H$. □

Hypothesis 6.10. *Assume Hypothesis 5.1 and let $M \in \mathcal{M}$.*
In addition, assume that $C < M$, let $\pi := \pi(F(M))$ and suppose that $|\pi| \geq 2$. Let $p \in \pi$ be such that $O_p(M)$ contains a z-minimal subgroup U.

From now on, until the proof of Theorem **A**, we assume Hypothesis 6.10. We recall that, by Lemma 5.2 (1), we have $\mathcal{U}^*_M(\langle z\rangle, p) \subseteq \mathrm{Syl}_p(M)$. For the remainder of this section we let $P \in \mathrm{Syl}_p(M, z)$ and $Z := \Omega_1(Z(P))$.

Lemma 6.11. *$Z \nleq O_p(M)$. In particular Z is not cyclic.*

Proof. By Hypothesis 6.10 and Lemma 5.9 we have $P \notin \mathrm{Syl}_p(G, z)$ and therefore $N_G(P)$ is not contained in M. Assume that $Z \leq O_p(M)$. As $[U, Z] = 1$ and Z is z-invariant, we can apply Lemma 6.9 to Z and see that $N_G(Z) \leq M$ since M is not of characteristic p. But then $N_G(P) \leq N_G(Z) \leq M$, a contradiction. The second assertion follows because, if Z is cyclic, then $|Z| = p$ and $1 \neq Z \cap O_p(M)$ forces $Z \leq O_p(M)$. □

Lemma 6.12. *Suppose that $1 \neq X = [X, z] \leq O_p(M)$ and that $r_p(C_C(O_p(M))) \geq 2$. Then M is the unique maximal subgroup of G containing $N_G(X)$.*

Proof. Let $N_G(X)$ be contained in a maximal subgroup H of G. As $X \leq O_p(M)$, we have $M \looparrowright H$. By hypothesis, there exists an elementary abelian p-subgroup $W \leq C_C(O_p(M))$ of order at least p^2. We note that W and z both lie in H. Our objective is to apply the Infection Theorem (4) and so we show first that $F := F_{\pi'}(H)$ is trivial, i.e. $F(H)$ is a π-group. We apply the Infection Theorem, part (1), to see that $F \cap M \leq F_{\pi'}(H) \cap M = 1$. In particular $F \cap C = 1$, so F is inverted by z .
Let $w \in W^\#$ and let L be a maximal subgroup of G containing $C_G(w)$. By hypothesis, $z \in C_G(W)$ and since $W \leq C_C(O_p(M)) \leq C_C(U)$, it follows that $U\langle z\rangle \leq L$. The Basis Lemma (3) implies that $U \leq O_p(L)$. Hence the fact that M is not of characteristic p and Corollary 6.8 give that L infects M. As we observed above, F is inverted by z, in particular $C_F(w) \leq L$ is inverted by z . It follows that $C_F(w) = [C_F(w), z] \leq [L, z] \leq O(L)$ because $z \in Z^*(L)$, with Lemma 5.2 (1). Since $X \leq C_G(W) \leq L$, we also have $X = [X, z] \leq O(L)$.
By the Basis Lemma (2) we have $X = [X, z] \leq [O_p(M) \cap H, z] \leq O_p(H)$ and this forces $[F, X] = 1$. Thus Lemma 3.5 gives

$$C_F(w) = C_{C_F(w)}(X) \leq O_{p'}(C_{O(L)}(X)) \leq O_{p'}(O(L)) \leq O_{p'}(L).$$

As $U \leq O_p(L)$, it follows that $[U, C_F(w)] = 1$. By Coprime Action (b) and since W is not cyclic, we have $F = \langle C_F(w) \mid v \in W^\# \rangle$ and thus $[U, F] = 1$. Now Corollary 6.8 and the hypothesis that M is not of characteristic p imply that $F \leq N_G(U) \leq M$. This forces $F = F \cap M = 1$ and hence $\pi(F(H)) \subseteq \pi$. Moreover z centralises $E(H)$ by Lemma 5.2 (1). Therefore $E(H) \leq C \leq M$ and the Infection Theorem (4) yields $M = H$. □

Lemma 6.13. *Suppose that $1 \neq X = [X, z]$ is a subgroup of $O_p(M)$. Then M is the unique maximal subgroup of G containing $N_G(X)$.*

Proof. Let H be a maximal subgroup of G which contains $N_G(X)$. Then $M \looparrowright H$. We apply the Infection Theorem (4) once more and argue as in the proof of Lemma 6.12.
First we observe that by Lemma 5.2 (1), we have $E(H) \leq C_H(z) \leq M$. Thus it only remains to show that $F(H)$ is a π-group. Let $F := F_{\pi'}(H)$. Since $z \in H$, an application of the Basis

Lemma (2) yields that $X = [X, z] \leq [O_p(M) \cap H, z] \leq O_p(H)$ and thus $[F, X] = 1$. We know that Z is elementary abelian of order at least p^2, by Lemma 6.11. As $[Z, O_p(M)] = 1$, we are done by Lemma 6.12 if $[Z, z] = 1$. Thus we may suppose that $[Z, z] \neq 1$, i.e. Z possesses an element $w \neq 1$ which is inverted by z. The subgroup F is also inverted by z by part (1) of the Infection Theorem. On the other hand $w \in Z \leq C_G(X) \leq H$ which implies that F is w-invariant. We conclude that F is centralised by $\langle w \rangle = [\langle w \rangle, z]$. Now let L be a maximal subgroup of G containing $N_G(\langle w \rangle)$. Then z, X, U, Z and – as we have just seen – F are contained in L. The Basis Lemma (parts (2) and (3)) yields that X and U are both contained in $O_p(L)$ and hence in $O_p(C_G(w))$. By Lemma 5.2 (1) we have $z \in Z^*(L)$ and therefore F, which is inverted by z, lies in $O(L)$. Lemma 3.5 gives

$$F \leq O_{p'}(C_G(X)) \cap C_{O(L)}(w) \leq O_{p'}(C_{O(C_G(w))}(X)) \leq O_{p'}(C_G(w)).$$

As $U \leq O_p(C_G(w))$, it follows that $[U, F] = 1$ and therefore F is contained in $C_G(U) \leq M$, with Corollary 6.8 and our hypothesis that M is not of characteristic p. But then $F = F \cap M = 1$ and the Infection Theorem (4) gives the statement. □

From now on, until the proof of Theorem 6.3, we assume that $E(M) = 1$, i.e. M is a counterexample to Theorem 6.3.

Lemma 6.14. *Suppose that W is an elementary abelian subgroup of M of order p^2 which is centralised or inverted by z. Then z inverts W and $[C_{O_p(M)}(W), z] = 1$. In particular $C_P(z)$ is cyclic.*

Proof. Assume that W is a counterexample, so $[W, z] = 1$ or $[C_{O_p(M)}(W), z] \neq 1$. If $[W, z] = 1$, then Thompson's $P \times Q$-Lemma yields $[C_{O_p(M)}(W), z] \neq 1$ because, by hypothesis, $O_p(M)$ is not centralised by z. We conclude that $C_{O_p(M)}(W)$ possesses an element x of order p which is inverted by z. By Lemma 5.2 (1), W lies in a z-invariant Sylow p-subgroup of M, so we may suppose that $W \leq P$. Lemma 5.9 implies that $P \notin \mathrm{Syl}_p(G)$ and in particular $N_G(P) \nleq M$. Thus we find an involution $t \in N_G(P)$ such that t is conjugate to z and $M \neq M^t$ (Lemma 4.9). Now $P = P^t \leq M \cap M^t$ and therefore $W\langle x \rangle \leq M \cap M^t$. In particular $W\langle x \rangle$ acts on $Q := O_{p'}(M^t)$. As W is elementary abelian and not cyclic, Coprime Action (b) yields that $Q = \langle C_Q(w) | w \in W^\# \rangle$. Let $w \in W^\#$ and set $L := N_G(\langle w \rangle)$. Then $z \in L$ and thus the Basis Lemma (2) gives $\langle x \rangle = [\langle x \rangle, z] \leq [O_p(M) \cap L, z] \leq O_p(L)$. It follows that $[x, C_Q(w)] \leq O_p(L) \cap Q = 1$ and therefore $C_Q(w) \leq C_G(x)$. So we have $Q \leq C_G(x)$. But as $1 \neq \langle x \rangle = [\langle x \rangle, z] \leq O_p(M)$, Lemma 6.13 implies $C_G(x) \leq M$. Thus $Q \leq M$ and also $O_p(M^t) \leq P \leq M$. This gives $F^*(M^t) = F(M^t) \leq M$ ($E(M^t) = 1$ because by hypothesis $E(M) = 1$). But this means that M^t infects M. As both M and M^t contain a conjugate of z and are not of characteristic p, part (5) of the Infection Theorem forces $M = M^t$. This is impossible. In particular $r(C_P(z)) = 1$. □

Lemma 6.15. *Z is elementary abelian of order p^2 and $|C_Z(z)| = |I_Z(z)| = p$. In fact, $C_Z(z)$ is the unique subgroup of P of order p which is centralised by z and $I_Z(z)$ is the unique subgroup of P of order p which is inverted by z.*

Proof. First we observe that $O_p(M) \cap Z \neq 1$ and that $|Z| \geq p^2$ by Lemma 6.11. With Coprime Action (a) we have $Z = C_Z(z) \times [Z, z] = C_Z(z) \times I_Z(z)$. Assume that $I_Z(z)$ possesses a subgroup V of order p^2. Then $[C_{O_p(M)}(V), z] = 1$ by Lemma 6.14, but on the other hand $V \leq Z(P)$ is centralised by $O_p(M)$. So $[O_p(M), z] = 1$, a contradiction. In particular, as $|Z| \geq p^2$, it follows that Z is not inverted by z. Lemma 6.14 yields that, on the other hand, Z cannot be centralised by z. Hence $C_Z(z)$ also has order p and is, by the same lemma, the unique subgroup of P of order p which is centralised by z .
It remains to show that $I_Z(z)$ is the only subgroup of P of order p which is inverted by z . First assume that $Y \leq O_p(M)$ is distinct from $I_Z(z)$, has order p and is inverted by z . Then $W := YI_Z(z)$ is elementary abelian of order p^2 and we may apply Lemma 6.14. Thus $[Y, z] \leq [C_{O_p(M)}(W), z] = 1$, a contradiction. Next assume that $Y_1 \leq P$ is distinct from $I_Z(z)$, has order p and is inverted by z . Then $W_1 := Y_1 I_Z(z)$ is elementary abelian of order p^2, and Lemma 6.14 yields $[I_Z(z), z] \leq [C_{O_p(M)}(W_1), z] = 1$, again a contradiction. □

Lemma 6.16. *$O_p(M)$ is a cyclic group which is inverted by z .*

Proof. We know from Lemma 6.15 that $I_Z(z)$ is the unique subgroup of order p in $O_p(M)$ which is inverted by z . If there is any subgroup of order p centralised by z in $O_p(M)$, then, again with Lemma 6.15, it can only be $C_Z(z)$. But $Z \nleq O_p(M)$ by Lemma 6.11. Therefore $C_{O_p(M)}(z) = 1$ and it follows that $O_p(M)$ is abelian and contains a unique subgroup of order p. This forces $O_p(M)$ to be cyclic. □

Lemma 6.17. $Z = \Omega_1(P)$.

Proof. We have $Z \leq \Omega_1(P)$. Assume that there exists a subgroup of P of order p which is not contained in Z. Then as $Z \leq Z(P)$, it follows that $r(P) \geq 3$. But P is z-invariant, so by Lemma 2.1 there exists a z-invariant elementary abelian subgroup X of P of order p^3. By Coprime Action (a), we have $X = C_X(z) \times [X, z]$. But by Lemma 6.15, every element in X which is centralised or inverted by z lies in Z. This yields $X \leq Z$, a contradiction. Hence $\Omega_1(P) \leq Z$. □

Lemma 6.18. *$I_Z(z)$ is contained in every conjugate of M in G.*

Proof. Let $g \in G\backslash M$, let $t \in z^G \cap M^g$ and set $D := M \cap M^t$. We note that, by Lemma 4.1 (8), the involution t is not contained in M. As z does not centralise P, Lemmas 4.15 and 5.6 (4) yield that p divides $|M : C| = |D : C_D(t)|$ and therefore $|[D, t]| \,/\, |[D, t] \cap C_D(t)|$ is divisible by p. Thus there exists a subgroup X of D of order p which is inverted by t. We show that X is conjugate to $I := I_Z(z)$.
By Lemma 6.17 we have $Z = \Omega_1(P)$. Now let $P_1 \in \mathrm{Syl}_p(N_G(Z), z)$ (with Lemma 5.2 (1)). Lemmas 6.16 and 6.17 imply that $I = Z \cap O_p(M) \trianglelefteq M$, thus $N_G(I) = M$ with Lemma 5.4 (2) and $P = N_{P_1}(I)$. Let $P_2 := N_{P_1}(P)$. By Lemma 5.9 we know that P_1 is not contained in M and therefore $|I^{P_2}| = p$. Now z leaves $C_Z(P_2)$ invariant and $C_Z(P_2) \neq I$ because $P < P_2$ (i.e. $P_2 \nleq M$) and $C_G(I) \leq M$. As I and $C_Z(z)$ are precisely the z-invariant subgroups of Z of order p, we conclude that $C_Z(P_2) = C_Z(z)$.

We know that $X \leq M$. Therefore, by Lemma 6.17 and Sylow's Theorem, X is conjugate in M to a subgroup of order p in Z, i.e. to a member of I^{P_2} or to $C_Z(z)$. If X is conjugate to $C_Z(z)$ then we may replace X by $C_Z(z)$. But then z and t are both contained in $N_G(X)$ and hence conjugate in $N_G(X)$ by Lemma 4.1 (4). This is impossible because z centralises X $(= C_Z(z))$ whereas t inverts it. Thus X is conjugate to I.
Now let $y \in G$ be such that $X = I^y$. Then $t \in N_G(X) = N_G(I^y) = M^y$. As every conjugate of z is contained in a unique conjugate of M, by Lemma 4.1 (8), this yields $M^y = M^g$. Now we see that $X \leq D \leq M$ normalises $I \trianglelefteq M$ and therefore $[X, I] = 1$. So we have $I \leq N_G(X) = M^g$. As $g \in G \backslash M$ was arbitrary, it follows that I lies in every conjugate of M in G, as stated. □

Proof of Theorem **A**.
Assume that Theorem **A** (i.e. Theorem 6.3 above) fails. Then Hypothesis 5.1 holds and C is properly contained in $M \in \mathcal{M}$. Hence Lemma 6.4 implies that there exists an odd prime p such that $O_p(M)$ contains a z-minimal subgroup U. Again by the failure of Theorem A, we have $F^*(M) \neq O_p(M)$ and $E(M) = 1$. It follows that Hypothesis 6.10 is also satisfied and thus the results which have been proved earlier in this section are applicable. In particular Lemma 6.18 says that M contains a non-trivial subgroup I which lies in every conjugate of M in G. This means that

$$1 \neq I \leq N := \bigcap_{g \in G} M^g \trianglelefteq G.$$

With Lemma 5.4 (1) we deduce that $G = N\langle z \rangle \leq M$, a contradiction. □

We conclude with a theorem that is applied in the next chapter, but holds more generally than under the main hypothesis there. It uses arguments from [3].

Theorem 6.19. *Suppose that C is a maximal subgroup of G, that $p \in \pi := \pi(F(C))$ is an odd prime and that $Y \leq O_p(C)$ is elementary abelian of order p^3. Then, for all $y \in Y^\#$, C is the unique maximal subgroup of G containing $C_G(y)$.*

Proof. We first observe that, by hypothesis, $|\pi| \geq 2$ since $2 \in \pi$. Let $A_0 := C_{F(C)}(Y)$, let $A := A_0 E(C)$ and note that $z \in Z(C) \leq A$. Moreover A_0 is centraliser closed in $F(C)$ and contains $Z(F^*(C))$. Suppose that A lies in a maximal subgroup H of G. Then we have $C \looparrowright H$.

- $C_G(A) \leq C$ and $C_G(O_p(A)) \leq C$ for all $p \in \pi$.

  This follows from Lemma 5.4 (2) because $Z(F(C)) \leq A$ and $O_p(Z(F(C))) \leq O_p(A)$.

- $C_G(A)$ is a π-group and $\mathcal{U}_C(A, \pi') = \{1\}$.

  Suppose that $x \in C_G(A)$ is a π'-element. Since x centralises A_0, a centraliser closed subgroup of $F(C)$, we have $[F(C), x] = 1$ by Coprime Action (c). Moreover x centralises $E(C)$ which lies in A. But $C_G(A)$ is contained in C and therefore it follows that $x \in C_C(F^*(C)) = Z(F(C))$. Thus $x = 1$.

  For the second assertion let $X \in \mathcal{U}_C(A, \pi')$ and note that X has odd order because $2 \in \pi$. First we have $[X, A_0] \leq X \cap F(C) = 1$ and therefore, again by Coprime Action (c), it follows that $[X, F(C)] = 1$. On the other hand $[X, E(C)] \leq X \cap E(C) \trianglelefteq E(C)$. By the Odd Order Theorem, $E(C)$ has even order and thus $X \cap E(C) \leq Z(E(C))$. This yields $[X, E(C)] = 1$ with the 3-Subgroups-Lemma. We conclude that $X \leq C_C(F^*(C)) = Z(F(C))$ and hence $X = 1$.

- $\langle \text{И}_H(A, \pi') \rangle \leq O_{\pi'}(H)$.

  To prove this, let $Q \in \text{И}_H(A, \pi')$ and let $r \in \pi$ be odd. We showed above that $\text{И}_C(A, \pi') = \{1\}$ and this implies that $Q \cap C = 1$, i.e. that Q is inverted by z . As $z \in H$ and therefore $z \in Z^*(H)$ by Lemma 5.2 (1), it follows that $Q = [Q, z] \leq [H, z] \leq O(H)$.

  Set $A_r := O_r(A)$. We recall that r is odd and thus $A_r \leq O(C)$. Hence

  $$A_r \leq O(C) \cap H \leq O(C_H(z)) \leq O(H)$$

  by Lemma 3.5 and because $z \in O_{2',2}(H)$. Let $W := O(H)\langle z \rangle$. We have $z \in O_2(C) \leq O_{r'}(C)$ and therefore, since $C_G(A_r) \leq C$, it follows that $z \in O_{r'}(C_W(A_r))$. As W is soluble and contains A_r, we may apply Lemma 3.5 once more to obtain $z \in O_{r'}(W)$. We showed above that $Q \leq O(H)$ and thus $Q = [Q, z] \leq O_{r'}(W)$. Repeating this argument for all odd primes in π, it follows that $Q \leq O_{\pi'}(W)$. But π' consists of odd primes and thus $O_{\pi'}(W) = O_{\pi'}(O(W)) = O_{\pi'}(O(H)) = O_{\pi'}(H)$. Hence $Q \leq O_{\pi'}(H)$.

- Let $q \in \pi'$. Then $\text{И}^*_G(A, q)$ possesses a unique element Q^* and $Q^* \trianglelefteq C$.

  We recall that $F^*(G)$ is simple by Lemma 5.3, so that in particular G does not normalise any non-trivial q-subgroup. Hence we may apply Theorem 3.14 which yields that $O_{\pi'}(C_G(A))$ is transitive on $\text{И}^*_G(A, q)$. But $O_{\pi'}(C_G(A)) = 1$ and therefore $\text{И}^*_G(A, q)$ has a unique element Q^*. Let $A_1 := N_{F^*(C)}(A)$. Then A_1 leaves Q^* invariant which means that $Q^* \in \text{И}_G(A_1, q)$. Let $Q^* \leq Q_1 \in \text{И}^*_G(A_1, q)$. As $\text{И}_G(A_1, q) \subseteq \text{И}_G(A, q)$, it follows that $Q_1 \leq Q^*$ and so Q^* is also the unique member of $\text{И}^*_G(A_1, q)$. But A is subnormal in $F^*(C)$ and thus the previous argument implies that $\text{И}^*_G(F^*(C), q) = \{Q^*\}$. Hence C normalises Q^*. The fact that $F^*(G)$ is simple (Lemma 5.3) forces $N_G(Q^*) < G$. As C lies in $N_G(Q^*)$ and is a maximal subgroup of G, we obtain $Q^* \leq N_G(Q^*) = C$. Thus $Q^* \trianglelefteq C$.

- $H = C$.

  As $z \in A \leq H$ we see that $E(H)$ lies in C (Lemma 5.2 (1)). Moreover $|\pi| \geq 2$ by hypothesis. Thus the Infection Theorem, part (4), yields the statement if $F_{\pi'}(H) = 1$. Assume, therefore, that there exists a prime $q \in \pi'$ such that $F := O_q(H) \neq 1$. Then $F \in \text{И}_H(A, q)$. We showed in the previous paragraph that $\text{И}^*_G(A, q)$ has a unique element Q^* and that Q^* lies in C. Now $F \leq Q^* \leq C$ whereas by the Infection Theorem (1) we have $F \cap C \leq F_{\pi'}(H) \cap C = 1$. This is a contradiction and hence $H = C$.

Finally A (and then $C_G(y)$ for all $y \in Y^\#$) is contained in a unique maximal subgroup of G, namely in C. □

7. Maximal subgroups containing the centraliser of an involution in $O_{2',2}(C)$

Throughout this chapter, we assume:

Hypothesis 7.1. *Assume Hypothesis 5.1 as well as the following:*

- *Let $V = \{1, a, b, z\} \le O_{2',2}(C)$ be an elementary abelian subgroup of order 4 and let $V \le S \in Syl_2(G)$.*
- *For every $v \in \{a, b, z\}$ let L_v be a maximal subgroup containing $C_G(v)$ such that, if possible, there exists a prime $p \in \pi(F(L_v))$ with $C_{O_p(L_v)}(v) = 1$. (In particular we choose $L_z = M \in \mathcal{M}$ as in the previous section.)*
- *Let $\pi := \pi(F(M))$. Whenever $C_G(v) \neq L_v$, let U_v denote a v-minimal subgroup of G which is contained in $F(L_v)$. If $C \neq M$, then let U be a z-minimal subgroup of G contained in $F(M)$.*
- *If a and b are conjugate, then assume L_a and L_b to be conjugate.*

In this section, the idea is to look at maximal subgroups containing the centralisers of z, a and b, respectively, at the same time. It turns out that a and b are *either* conjugate *or* isolated in G. In both cases, the implied information helps us to bring the Bender method into the picture once more. The main objective is to show that the centralisers of z, a and b are maximal subgroups of G.

Lemma 7.2. $r_2(C_G(V)) = 2$.

Proof. Assume that $r_2(C_G(V)) > 2$ and let $V \le B$ where B is elementary abelian of order 8. For all involutions $t \in B$ we have $z \in C_G(t)$ and $V \le O_{2',2}(C) \cap C_G(t) \le O_{2',2}(C_G(t))$ by Lemma 5.2 (1). Now Theorem 3.13 forces $W := \langle O(C_G(v)) \mid v \in V^\# \rangle$ to be a group of odd order. On the other hand we know that $z \in Z^*(C_G(a))$ and therefore $C_K(a)$ is contained in $O(C_G(a))$, again by Lemma 5.2 (1). Similarly $C_K(b)$ is contained in $O(C_G(b))$. But by Theorem 4.8 this means $K \subseteq \langle C_K(a), C_K(b) \rangle \le W$. In particular it implies that $\langle K \rangle$ is a normal subgroup of G of odd order, contradicting Lemma 5.2 (4). $\square$

Lemma 7.3. *Suppose that $a, b \in H \le G$. Then a and b are either conjugate or isolated in H.*

Proof. Let $V \le T \in \mathrm{Syl}_2(H)$. By Lemma 4.1 (2) we have $z \in Z(T)$. Now if we suppose that a and b are not conjugate in H, then we have $N_T(V) = C_T(V)$. But $V = \Omega_1(Z(C_T(V)))$ by Lemma 7.2 and $N_T(N_T(V))$ therefore centralises V. This implies that $N_T(V)$ is equal to its normaliser in T and forces $T = N_T(V) = C_T(V)$. In particular a, b and z are the only involutions in T which gives the assertion. $\square$

Lemma 7.4. *G is simple.*

Proof. By Lemma 5.3 we have $G = F^*(G)\langle z\rangle$ and $F^*(G)$ is simple. We note that $N := F^*(G)$ intersects non-trivially with V because $|G : N| = 2$. So without loss of generality we may suppose that $a \in N$. If a and b are conjugate, then let $g \in G$ be such that $a^g = b$. It follows that $z = ab = aa^g = [a, g] \in [N, g] \leq N$ and thus $G = N$. On the other hand if a and b are isolated in G, then $G = N\langle a\rangle = N$ with Lemma 5.5. By Lemma 7.3 there are no more cases to consider. □

Corollary 7.5. *$G = \langle C_K(a), C_K(b)\rangle$ and $O^2(C) = C$.*

Proof. By Lemma 7.4 above, G is simple. This yields $G = \langle K\rangle = \langle C_K(a), C_K(b)\rangle$ with Theorem 4.8 and moreover $O^2(C) = C$ with Corollary 4.3. □

Lemma 7.6. *$r_2(G) = 2$.*

Proof. Set $S_0 := S \cap O_{2',2}(C)$. Then $V \leq S_0 \trianglelefteq S$ and in particular $r(S_0) \geq 2$. Assume that S_0 does not contain any normal elementary abelian subgroup of S of order 4. Then Lemma 3.2 implies that S_0 is dihedral or semidihedral, in particular $|S_0| \geq 8$. It follows that $\mathrm{Aut}(S_0)$ is a 2-group. Now in $\overline{C} := C/O(C)$ we have $\overline{S_0} = O_2(\overline{C})$ and therefore $[O^2(\overline{C}), \overline{S_0}] = 1$. But $O^2(\overline{C}) = \overline{C}$ by Corollary 7.5 and therefore $[\overline{C}, \overline{S_0}] = 1$. This is impossible because $\overline{S_0}$ is not abelian. Hence we may suppose that V is normal in S.
Assume that $T \leq S$ is elementary abelian of order 8. Then $C_T(V) \leq V$ by Lemma 7.2. On the other hand, as V is normal in S and hence T-invariant, we have $|T : C_T(V)| \leq 2$. Thus $C_T(V) = V$. In particular $V \leq T$ which is not possible by Lemma 7.2. Therefore $r_2(G) = 2$. □

Lemma 7.7. *Suppose that $V \leq H < G$ and that $C_K(a) \subseteq H$. Then a and b are isolated in H.*

Proof. Assume that a and b are not isolated in H. Then Lemma 7.3 yields that they are conjugate in H. So by Lemma 4.2 they are conjugate in $C_H(z)$. But if $x \in C_H(z)$ is such that $b = a^x$, then $C_K(b) = C_K(a^x) = (C_K(a))^x \subseteq H$. So Theorem 4.8 forces K to be contained in H. This contradicts Corollary 7.5. □

Lemma 7.8. *$[V, L_a] \leq O(L_a)$. If all involutions in V are isolated, then $[V, M] \leq O(M)$.*

Proof. First we observe that a and b are isolated in L_a by Lemma 7.7. As the Z^*-Theorem holds in L_a, this yields $[V, L_a] \leq O(L_a)$. Now suppose that a, b and z are isolated in G. Then $[V, M] \leq O(M)$ because the Z^*-Theorem holds in M. □

Corollary 7.9. *Let $V \leq H < G$ and suppose that a and b are isolated in H. Then $\mathcal{U}^*_H(V, p) \subseteq$ $Syl_p(H)$ for all $p \in \pi(H)$.*

Proof. As the Z^*-Theorem holds in H, we have $[H, V] \leq O(H)$. Coprime Action (d) yields that $\mathcal{U}^*_{O(H)}(V, p) \subseteq \mathrm{Syl}_p(O(H))$ for all $p \in \pi(O(H))$ and then the statement follows. □

Lemma 7.10. *Let $v \in \{a, b\}$ and suppose that $C_G(v) < L_v$. Then $E(L_v) = 1 = O_2(L_v)$. If $C < M$, then $E(M) = 1 = O_2(M)$.*

Proof. We may suppose that $v = a$. Lemma 7.8 yields that $[V, O_2(L_a)] \leq O(L_a) \cap O_2(L_a) = 1$. On the other hand z and a are not contained in $O_2(L_a)$ by hypothesis and Lemma 5.10. Hence if $O_2(L_a) \neq 1$, then Lemma 7.2 implies that b is the unique involution in $O_2(L_a)$. But since a and b are isolated in L_a by Lemma 7.7, this forces $b \in Z(L_a)$. It follows that $C_G(a) \leq L_a = C_G(b)$ and thus $G \leq L_a$ with Corollary 7.5. This is impossible and hence $O_2(L_a) = 1$. Moreover, $V \leq Z^*(L_a)$ which is a soluble subgroup of L_a and thus $[V, E(L_a)] = 1$. As $V \cap E(L_a) \leq O_2(L_a) = 1$, this implies $E(L_a) = 1$ by Lemma 7.2 because components have even order.
Let us finally consider the case where $C < M$. Let $p \in \pi$ be such that $U \leq O_p(M)$. In particular p is odd and we note that our choice of M implies that $M \in \mathcal{M}$. Assume that $t \in O_2(M)^\#$. Then $|\pi| \geq 2$, so Hypothesis 6.10 from the previous chapter is satisfied. Now t is centralised by z (by Lemma 4.1 (2)) and by U. Thus we may apply Lemma 6.9 which yields that $C_G(t) \leq M$, contradicting Lemma 5.10. This forces $O_2(M) = 1$. Since $[V, E(M)] = 1$ and $V \cap E(M) \leq O_2(M) = 1$, we also have $E(M) = 1$, again by Lemma 7.2. □

Later in this section, some arguments require that $N_G(U_a) \leq L_a$ in the case where $C_G(a) < L_a$. The situation where $F^*(L_a)$ is a p-group for some prime p is of particular interest and needs a bit more work.

Lemma 7.11. *Suppose that $a \in H < G$, that $a \in Z^*(H)$ and that q is an odd prime. Then $Q := O_q(H)O_q(C_H(a))$ is the unique maximal $C_H(a)$-invariant q-subgroup of H. Furthermore if char$(H) = q$, then $K^\infty(Q)$ is normal in H.*

Proof. By hypothesis we have $H = C_H(a)O(H)$. Next we observe that Q is in fact a $C_H(a)$-invariant q-subgroup of H. Now suppose that $Y \in \mathcal{U}_H(C_H(a), q)$ is arbitrary. As $C_H(a)$ normalises Y and q is odd, Lemma 3.6 yields $[Y, a] \leq O_q(H)$. Moreover $C_Y(a)$ is normal in $C_H(a)$ and therefore lies in $O_q(C_H(a))$. But then Coprime Action (a) yields $Y = [Y, a]C_Y(a) \leq O_q(H)O_q(C_H(a)) = Q$. So every element of $\mathcal{U}_H(C_H(a), q)$ is contained in Q which means that $\mathcal{U}^*_H(C_H(a), q) = \{Q\}$.
For the second assertion, suppose that char$(H) = q$ and let $W := O(H)Q$. By Dedekind's Law we have $O_q(W)Q = (O(H) \cap O_q(W))Q \leq Q$ and thus $O_q(W) \leq Q$. We note that $F^*(H) = O_q(H) \leq Q \leq W$ by hypothesis which implies $F^*(H) \leq F^*(W)$. Therefore

$$O^q(F^*(W)) \leq C_H(O_q(W)) \leq C_H(O_q(H)) = C_H(F^*(H)) = Z(F^*(H))$$

which yields that $F^*(W)$ also is a q-group. Now we can apply Theorem 3.9 and obtain $K^\infty(Q) \trianglelefteq W$. As $C_H(a)$ normalises Q, the subgroup $K^\infty(Q)$ is invariant under both $C_H(a)$ and $O(H)$. Thus $K^\infty(Q)$ is normal in H. □

Lemma 7.12. *Let q be an odd prime and suppose that $P, Q \in \mathcal{U}^*_G(C_G(a), q)$ are such that $P \cap Q \neq 1$. Then $P = Q$.*

Proof. Assume that this is not true and choose $P, Q \in \mathcal{U}^*_G(C_G(a), q)$ to be distinct and such that their intersection $D \neq 1$ is as large as possible. By Lemma 7.4 we have that G is simple and therefore $H := N_G(D)$ is a proper subgroup of G. Next we note that, as P and Q are $C_G(a)$-invariant by hypothesis, $C_G(a)$ is contained in H. Hence $W := O_q(H)O_q(C_G(a)) \in \mathcal{U}_G(C_G(a), q)$. As $V \leq C_G(a) \leq H$, we may apply Lemma 7.7 whence it follows that $a \in Z^*(H)$. Thus Lemma

7.11 above yields that W is the unique maximal element of $\text{И}_H(C_H(a), q)$. But $N_P(D)$ is a $C_G(a)$-invariant q-subgroup of H, so it follows that $N_P(D) \leq W$. Now we choose $W^* \in \text{И}^*_G(C_G(a), q)$ to be such that W^* contains W. Then $D < N_P(D) \leq P \cap W \leq P \cap W^*$, so the choice of P and Q implies $P = W^*$. Similarly $Q = W^*$ and hence $P = Q$, a contradiction. □

Theorem 7.13. *Suppose that $C_G(a) < L_a$. Then L_a is the unique maximal subgroup of G containing $N_G(U_a)$.*

Proof. Let H be a maximal subgroup of G containing $N_G(U_a)$. Then we have $C_G(a) \leq H$ and $L_a \looparrowright H$ because $U_a \leq F(L_a)$. Lemma 7.7 yields that a is isolated in H and therefore $a \in Z^*(H)$. This implies that $[a, E(H)] \leq [Z^*(H), E(H)] = 1$ and hence $E(H) \leq C_G(a) \leq L_a$. We let $\sigma := \pi(F(L_a))$ and show that $F(H)$ is a σ-group in order to apply the Infection Theorem (4). Let $F := F_{\sigma'}(H)$. Then by the Infection Theorem (1), we have $F \cap L_a \leq F_{\sigma'}(H) \cap L_a = 1$ which means that F is inverted by a. Thus the choice of L_a implies that there exists a prime $q \in \sigma$ such that $C_{O_q(L_a)}(a) = 1$. In particular $T := O_q(L_a)$ is abelian and we have $[T, a] = T \leq N_G(U_a) \leq H$. As F is T-invariant and inverted by a, it follows that $T = [T, a]$ centralises F. Therefore $F \leq C_G(T) \leq L_a$. Now we have $F = F \cap L_a = 1$ and the Infection Theorem (4) forces $L_a = H$ or $\mathrm{char}(L_a) = \mathrm{char}(H) = q$.
Suppose that $F^*(L_a)$ and $F^*(H)$ are both q-groups. We recall that, as $C_G(a)$ is contained in L_a and in H, Lemma 7.7 yields that a is isolated in L_a as well as in H. Thus $a \in Z^*(L_a)$ and $a \in Z^*(H)$. Now we apply Lemma 7.11 to see that $\text{И}_{L_a}(C_G(a), q)$ has a unique maximal element P and similarly $\text{И}_H(C_G(a), q)$ has a unique maximal element Q. We even have $P, Q \in \text{И}_G(C_G(a), q)$ because $C_G(a)$ is contained in L_a and H. Our hypothesis and Lemma 7.11 imply that $K^\infty(P)$ is normal in L_a and $K^\infty(Q)$ is normal in H. Now choose $P \leq P^* \in \text{И}^*_G(C_G(a), q)$. Then $N_{P^*}(P) \leq N_G(K^\infty(P)) = L_a$, by Lemma 5.4 (2), whence $P = P^*$ because P^* is a q-group. Similarly $Q \in \text{И}^*_G(C_G(a), q)$. Now let $X := Z(O_q(L_a))$. We note that $1 \neq X$ centralises U_a and therefore lies in H. Furthermore X is $C_G(a)$-invariant and thus contained in P and in Q. Finally Lemma 7.12 forces $P = Q$. But then we have $L_a = N_G(K^\infty(P)) = N_G(K^\infty(Q)) = H$ also in this case, as stated. □

Lemma 7.14. *If a is isolated in G, then $L_a = C_G(a)$ or $\mathrm{char}(L_a) = q$ where q is an odd prime.*

Proof. By Lemma 5.5, the involution a behaves like z. So we can apply Theorem 6.3 for a instead of z. Moreover, if $C_G(a) < L_a$, then $E(L_a) = 1$ by Lemma 7.10. This yields the statement. □

The following lemmas help us later when the Bender method returns to the scene.

Lemma 7.15. *Suppose that q is a prime such that $\mathrm{char}(L_a) = q = \mathrm{char}(L_b)$. Then a and b are conjugate.*

Proof. Assume that a and b are not conjugate. Then Lemma 7.3 implies that a and b are both isolated in G. It follows from Lemma 4.1 (2) that V centralises $O_2(L_a)$. Thus if $q = 2$, then $V \leq C_{L_a}(F^*(L_a)) \leq F^*(L_a)$ and therefore $L_a = C_G(a) = C_G(z) \leq M$. This contradicts Lemma

5.10. Hence q is odd and a does not lie in $Z(L_a)$ which in particular forces $C_G(a) < L_a$. Moreover we note that $U_a \leq O_q(L_a)$.
Choose Q_a to be a V-invariant q-subgroup of G, containing $O_q(L_a)$, such that $ZJ(Q_a)$ is invariant under $C_K(a)$ and such that Q_a is maximal subject to these constraints. Subgroups satisfying these conditions exist – $O_q(L_a)$ is an example. By Lemma 7.4 we have that G is simple and thus $N_G(ZJ(Q_a))$ is a proper subgroup of G containing V, $C_K(a)$ and Q_a. Let $N_G(ZJ(Q_a))$ be contained in a maximal subgroup H_a of G. Then $C_K(a) \subseteq O(H_a)$ because $z \in Z^*(H_a)$, by Lemma 5.2 (1). Now we have $F^*(L_a) = O_q(L_a) \leq Q_a \leq H_a$ and therefore $L_a \looparrowright H_a$. On the other hand the Basis Lemma (5) yields $U_a \leq O_q(H_a)$. So, as $C_G(a) < L_a$, we appeal to Lemma 7.13 to obtain $H_a \looparrowright L_a$. By the Infection Theorem (3) it follows that $L_a = H_a$ or $\mathrm{char}(H_a) = q$. But as $\mathrm{char}(L_a) = q$ by hypothesis, we have $\mathrm{char}(H_a) = q$ in both cases.
Now we deduce that Q_a is a Sylow q-subgroup of G. By Corollary 7.9 we may choose $Q_a \leq Q \in \mathrm{Syl}_q(H_a, V)$. We have $\mathrm{char}(H_a) = q$ and therefore $\mathrm{char}(QO(H_a)) = q$, so we can apply Theorem 3.10 and obtain $ZJ(Q) \trianglelefteq QO(H_a)V$. Hence $ZJ(Q)$ is $C_K(a)$-invariant and the choice of Q_a yields that $Q_a = Q \in \mathrm{Syl}_q(H_a)$. But $N_G(Q_a) \leq N_G(ZJ(Q_a)) \leq H_a$ and hence $Q_a \in \mathrm{Syl}_q(N_G(Q_a))$. It follows that Q_a is in fact a V-invariant Sylow q-subgroup of G.
We find $Q_b \in \mathrm{Syl}_q(G, V)$ and H_b with similar properties, arguing in the same way. By Remark 4.13 and since all involutions in V are isolated, we may choose $x \in C_G(V)$ such that $Q_a^x = Q_b$. Then $ZJ(Q_a)^x = ZJ(Q_b)$ and thus we suppose that $H_a^x = H_b$. But $C_K(a) \subseteq H_a$ and it follows that $C_K(a) = C_K(a)^x \subseteq H_a^x = H_b$. This, by Corollary 7.5, forces $G = \langle C_K(a), C_K(b)\rangle$ to be contained in H_b, a contradiction. $\square$

Lemma 7.16. *Let q be an odd prime and let $Q_1,\ Q_2 \in \mathcal{U}^*_G(V, q)$ be such that $Q_1 \cap Q_2 \neq 1$. Then Q_1 and Q_2 are conjugate under $C_G(V)$.*

Proof. Assume that this is not the case and choose Q_1 and Q_2 such that they are not conjugate under $C_G(V)$ and moreover such that $D := Q_1 \cap Q_2 \neq 1$ is maximal. Let $H := N_G(D)$ and note that, by Lemma 7.4, we have $H < G$. Then D, $N_{Q_1}(D)$ and $N_{Q_2}(D)$ are V-invariant subgroups of H. We choose $N_{Q_i}(D) \leq \widehat{Q_i} \in \mathcal{U}^*_H(V, q)$. As q is odd and $V \leq O_{2',2}(H)$ by Lemma 5.2 (1), we may apply Lemma 3.7 which yields an element $h \in C_H(V)$ such that $\widehat{Q_1}^h = \widehat{Q_2}$. Now let $\widehat{Q_1} \leq Q_1^* \in \mathcal{U}^*_G(V, q)$. Then $\widehat{Q_2} \leq Q_1^{*h} \in \mathcal{U}^*_G(V, q)$. Therefore we have $D < N_{Q_1}(D) \leq Q_1 \cap Q_1^*$ and $D < N_{Q_2}(D) \leq Q_2 \cap Q_1^{*h}$. By our choice of Q_1 and Q_2, it follows that Q_1 and Q_1^* as well as Q_2 and Q_1^{*h} are conjugate under $C_G(V)$, respectively. On the other hand $h \in C_H(V)$ which yields a contradiction. $\square$

Lemma 7.17. *There does not exist a prime q such that $\mathrm{char}(L_a) = q = \mathrm{char}(L_b)$.*

Proof. Assume that there is such a prime q. If $q = 2$, then z centralises $O_2(L_a) = F^*(L_a)$ by Lemma 4.1 (2) and this implies that $z \in O_2(L_a)$. Hence $z \in Z(L_a)$ contradicting Lemma 5.10. We deduce that q is odd. By Lemma 7.15 we know that a and b are conjugate in G, but on the other hand a is isolated in L_a by Lemma 7.7. Now the basic idea is to argue as in Lemma 7.15. In order to do that, we show that there exist $Q \in \mathrm{Syl}_q(L_a, V)$ and $Q_1 \in \mathrm{Syl}_q(L_b, V)$ such that their intersection is non-trivial. Then we can apply Lemma 7.16 and use the same arguments as before.

Assume that such subgroups Q and Q_1 do not exist and, with Corollary 7.9, let $Q \in \mathrm{Syl}_q(L_a, V)$ be arbitrary. From the same result it follows that $Q \cap L_b = 1$. Hence Q is inverted by b, in particular Q is abelian. So $Q \leq C_{L_a}(O_q(L_a)) = Z(O_q(L_a))$ because $\mathrm{char}(L_a) = q$. This implies $Q = O_q(L_a) = F^*(L_a)$ and forces $N_G(Q)$ to be contained in L_a (by Lemma 5.4 (2)). Therefore Q is a Sylow q-subgroup of G. In particular $O_q(M)$ is abelian, with Sylow's Theorem. As q is odd, we have $a \notin Z(L_a)$. Thus $C_G(a) < L_a$ and in particular $U_a \leq Q$. Since $U_a = [U_a, a]$ is abelian now, it is inverted by a. So U_a is inverted by a and b and therefore centralised by z. The Basis Lemma (5) gives $U_a \leq O_q(M)$. But $O_q(M)$ is abelian and therefore $O_q(M) \leq C_G(U_a) \leq L_a$ by Theorem 7.13. Hence M infects L_a. By part (2) of the Infection Theorem it follows that $\mathrm{char}(M) = q$. Hence $Q \leq C_M(F^*(M)) \leq O_q(M)$ and thus $F^*(L_a) = F^*(M)$, a contradiction because M and L_a are distinct (Lemma 5.10).

Now we choose $Q \in \mathrm{Syl}_q(L_a, V)$ and $Q_1 \in \mathrm{Syl}_q(L_b, V)$ such that $Q \cap Q_1 \neq 1$. Let Q_a be a V-invariant q-subgroup of G containing $O_q(L_a)$ and such that $ZJ(Q_a)$ is $C_K(a)$-invariant. Choose Q_a to be maximal subject to these constraints and let H_a be a maximal subgroup of G containing $N_G(ZJ(Q_a))$. Then V lies in H_a and Lemma 7.7 yields that a and b are isolated in H_a. Applying Corollary 7.9 we let $Q_a \leq Q_a^* \in \mathrm{Syl}_q(H_a, V)$. In particular, $F^*(L_a) = O_q(L_a) \leq Q_a \leq H_a$ and hence $L_a \looparrowright H_a$. The Basis Lemma (5) yields that $U_a \leq O_p(H_a)$ and hence $H_a \looparrowright L_a$ with Theorem 7.13. By the Infection Theorem (3), we have $L_a = H_a$ or $\mathrm{char}(H_a) = q$ whence in both cases it follows that $\mathrm{char}(H_a) = q$. In particular $\mathrm{char}(Q_a^* O(H_a)) = q$, so we may appeal to Theorem 3.10 and obtain $ZJ(Q_q^*) \trianglelefteq Q_a^* O(H_a) V$. Thus $ZJ(Q_a^*)$ is $C_K(a)$-invariant and the choice of Q_a yields that $Q_a = Q_a^* \in \mathrm{Syl}_q(H_a)$. But $N_G(Q_a) \leq N_G(ZJ(Q_a)) \leq H_a$ and hence $Q_a \in \mathrm{Syl}_q(N_G(Q_a))$. It follows that Q_a is in fact a V-invariant Sylow q-subgroup of G and so we may suppose that $Q \leq Q_a$. Since a and b are conjugate, we have $Q_b \in \mathcal{U}^*_G(V, q)$ and a maximal subgroup H_b of G with the corresponding properties and we may suppose that $Q_1 \leq Q_b$.

As $1 \neq Q \cap Q_1 \leq Q_a \cap Q_b$, Lemma 7.16 yields an $x \in C_G(V)$ such that $Q_a^x = Q_b$. Then $ZJ(Q_a)^x = ZJ(Q_b)$ and thus we suppose that $H_a^x = H_b$. But $C_K(a) \subseteq H_a$ and therefore $C_K(a) = C_K(a)^x \subseteq H_a^x = H_b$. Then Corollary 7.5 implies that $G \leq H_b$ which is impossible. □

Lemma 7.18. *There does not exist a prime q such that $\mathrm{char}(M) = q = \mathrm{char}(L_a)$.*

Proof. Assume that there is such a prime q. If a and b are conjugate in G, then also $\mathrm{char}(L_b) = q$ by our choice of L_a and L_b, and then Lemma 7.17 yields a contradiction. If a and b are isolated in G, then we can interchange the roles of a, b and z as we like and apply Lemma 7.17. This gives a contradiction again. By Lemma 7.3 there are no more cases to consider. □

Theorem 7.19. *$C_G(a)$ is a maximal subgroup of G.*

Proof. Assume that this is false. Let $F := F(L_a)$ (which equals $F_{2'}(L_a) = F^*(L_a)$ by Lemma 7.10).

(1) $[C_F(z), a] \neq 1$.

Proof. Assume that $C_F(z) \leq C_F(a)$. Then from Coprime Action (b) we deduce that $[F, a] \leq [F, z] \cap C_G(b)$ and thus $U_a \leq L_b$. By the Basis Lemma (5) we have that $U_a \leq$

$O_p(L_b)$ for some prime p and by Lemma 7.13 it follows that $L_b \looparrowright L_a$. Corollary 7.5 and Lemma 7.17 imply that L_a and L_b are neither equal nor both of characteristic p. So parts (3) and (5) of the Infection Theorem imply that L_b is not infected by L_a and that L_a and L_b are not conjugate. By our hypothesis this means that a and b are not conjugate. Lemma 7.3 gives that a and b are isolated in G and then Lemma 7.14 forces $\mathrm{char}(L_a) = p$. Now the Infection Theorem (2) and the fact that $1 \neq U_a \leq O_p(L_b)$ yield that also $\mathrm{char}(L_b) = p$, a contradiction. $\square$

Let $p \in \pi(F(L_a))$ and let $P := O_p(L_a)$ be such that $X := [C_P(z), a] \neq 1$. In particular $[P, a] \neq 1$ so that we may choose $U_a \leq P$. Applying the Basis Lemma (4) we obtain that $X \leq [P \cap M, a] \leq O_p(M)$ and therefore $X \leq C_{F(M)}(z)$.

(2) $C = M$.

Proof. Assume that $C < M$. We choose M to be of characteristic p and such that $N_G(U) \leq M$, in the following way: By hypothesis, $M \in \mathcal{M}$. So by Lemma 7.10 we have $E(M) = 1 = O_2(M)$ and thus Theorem 6.3 and the fact that $1 \neq X \leq O_p(M)$ imply that $\mathrm{char}(M) = p$. If $N_G(U) \leq M$, then we are done. Otherwise let H be a maximal subgroup of G containing $N_G(U)$. Then $C < H$ and M infects H. If it is possible to choose $H \in \mathcal{M}$, then we do this and then replace M by H. Otherwise, with parts (1) and (4) of the Infection Theorem and since $M \neq H$, we deduce that $\mathrm{char}(H) = p$. So we found a maximal subgroup of characteristic p containing $N_G(U)$ (but not necessarily in $\mathcal{M}$) in either case.

Now as $X \leq C_{F(M)}(z)$, Lemma 6.6 yields $[X, U] = 1$ and therefore X is a $U\langle z\rangle$-invariant subgroup of $O_p(M)$. As $C < M$ and $N_G(U) \leq M$, we may apply Lemma 6.9 and obtain that $N_G(X) \leq M$ or that $N_G(X)$ lies in a maximal subgroup of characteristic p. In both cases, the fact that $X \leq F(L_a)$ implies that L_a infects a maximal subgroup of G of characteristic p. But then, applying the Infection Theorem (2), we obtain $\mathrm{char}(L_a) = p$ contradicting Lemma 7.18. $\square$

Now (2) and Lemma 5.7 imply that every z-invariant π-subgroup of G is contained in $C = M$. In particular we have $[F_\pi(L_a), z] = 1$. We already know that $X \leq O_p(M)$ and therefore $p \in \pi$, furthermore $P \leq C$ since P is z-invariant. This yields that $U_a \leq X = [P, a] \trianglelefteq F$, so X is normalised by $U_a\langle a\rangle$.

(3) M infects L_a and (therefore) $F_{\pi'}(L_a) \neq 1$ is inverted by z .

Proof. As $U_a \leq X \leq O_p(M)$, the first statement follows from Theorem 7.13. Then the fact that M and L_a can neither be equal nor both of characteristic p (by Lemmas 5.10 and 7.18) gives $F_{\pi'}(L_a) \neq 1$, because $E(L_a) = 1$. Moreover $F_{\pi'}(L_a)$ is inverted by z by part (1) of the Infection Theorem. $\square$

(4) $N_G(X) \leq L_a$.

Proof. Suppose that $N_G(X)$ lies in a maximal subgroup H of G. Then as $X \leq F(L_a)$, we have that L_a infects H. Moreover $M \looparrowright H$ because $X \leq O_p(M)$, and $M \looparrowright L_a$ by (3). The Basis Lemma (5) implies that $U_a \leq F(H)$ and therefore Lemma 7.13 yields that we also have $H \looparrowright L_a$. Now the Infection Theorem (3) forces H and L_a to be equal or both of characteristic p. But if $\mathrm{char}(L_a) = p$, then the Infection Theorem (2) implies that $\mathrm{char}(M) = p$, contradicting Lemma 7.18. Thus $H = L_a$. □

(5) a and b are conjugate in G.

Proof. Assume that a and b are isolated. Then $C_a < L_a$ implies that $\mathrm{char}(L_a) = p$ by Lemma 7.14. But now, as $F^*(L_a) = P \leq M$, it follows that $L_a \looparrowright M$ and then $\mathrm{char}(L_a) = p = \mathrm{char}(M)$ or $M = L_a$ by (3) above and the Infection Theorem (4). This contradicts Lemmas 5.10 and 7.18 and then Lemma 7.3 yields the result. □

By our choice of L_a and L_b it follows that $C_G(b) < L_b$ and that $Y := [O_p(L_b), b] \neq 1$. We also recall that $F_\pi(L_a)$ is centralised by z by Lemma 5.7 and that $F_{\pi'}(L_a)$ is inverted by z by (3). In particular $F_{\pi'}(L_a)$ is abelian. Now we have $[L_a, z] \leq C_{L_a}(F) \leq F$ and together with Lemma 2.2 this implies $L_a = C_{L_a}(z)F_{\pi'}(L_a)$.

(6) $Y \leq O_p(M)$ and $N_G(Y) \leq L_b$. In particular M infects L_b.

Proof. This follows because a and b are conjugate in C by Lemma 4.2 and $X \leq O_p(M)$. Again by conjugacy, $N_G(Y) \leq L_b$ and then $M \looparrowright L_b$. □

(7) $F_{\pi'}(L_a)$ is inverted by b and by z and (therefore) centralised by a.

Proof. Let $D := F_{\pi'}(L_a) \cap C_G(b)$. Then $D \trianglelefteq F_{\pi'}(L_a)$ because $F_{\pi'}(L_a)$ is abelian, moreover D is invariant under $C_G(b) \cap C_G(a) = C_C(b) = C_C(a)$. Thus $C_G(a) \leq C_C(a)F_{\pi'}(L_a)$ normalises D. But z inverts $D \leq C_G(b)$, therefore D is contained in $[C_G(b), z] \leq F_{\pi'}(L_b)$. As a and b are conjugate, also $F_{\pi'}(L_b)$ is abelian. In particular $D \trianglelefteq F_{\pi'}(L_b)$. We deduce that D is $C_G(b)$-invariant, because $C_G(b) = C_C(b)F_{\pi'}(L_b)$, which means that by Corollary 7.5 we have $G \leq \langle C_G(a), C_G(b)\rangle \leq N_G(D)$. This is impossible. □

(8) $O_p(M) \nleq L_a$.

Proof. Otherwise $Y \leq O_p(M) \leq L_a$ and it follows that $Y \leq O_p(L_a)$ (by the Basis Lemma (4)) and then $L_a \looparrowright L_b$ by (6). Hence (5) above and Part (5) of the Infection Theorem force L_a and L_b to be equal or both of characteristic p. But this is contradicted by Lemma 7.17 and the fact that L_a and L_b are distinct (by Corollary 7.5). □

(9) $\pi \cap \pi(F(L_a)) = \{p\}$.

Proof. Assume that there are two distinct primes p and q in $\pi \cap \pi(F(L_a))$. Then $P \times O_q(L_a)$ acts on $O_p(M)$ and we have $[C_{O_p(M)}(P), O_q(L_a)] \le O_p(M) \cap O_q(L_a) = 1$. By Thompson's $P \times Q$-Lemma it follows that $[O_p(M), O_q(L_a)] = 1$ and therefore $O_p(M) \le L_a$, contradicting (8). $\square$

(10) Let $P_0 := P \cap O_p(M)$. Then $X = P_0 \trianglelefteq L_a$ is a cyclic group which is inverted by a and b.

Proof. First assume that $\Omega := \Omega_1(Z(O_p(M))) \le P_0$. Then as $\Omega \le P$, we have $F_{\pi'}(L_a) \le C_G(\Omega) \le M$ which is a contradiction. Thus $\Omega \not\le P_0$.

Now assume that P_0 is not cyclic. Then let $W_1 \le P_0$ be elementary abelian of order p^2 and let $W_1 \le W \le W_1\Omega$ be such that W is elementary abelian of order p^3. As $M = C$, we may apply Theorem 6.19 and obtain $C_G(w) \le M$ for all $w \in W^\#$. On the other hand $W_1 \le P$ and thus $O_{p'}(L_a) \le C_G(W_1)$. This implies $F_{\pi'}(L_a) \le O_{p'}(L_a) \le M$, a contradiction. We conclude that P_0 is cyclic and that, therefore, a and b invert it. It follows that $X \le P_0 = [P_0, a] \le [P, a] = X$. By (9) we have $F_{\pi'}(L_a) = F_{p'}(L_a)$. Since $X = P \cap O_p(M)$ is $C_{L_a}(z)$-invariant, the fact that $L_a = C_{L_a}(z)F_{p'}(L_a)$ yields that X is normal in L_a. $\square$

(11) For all primes $q \neq p$, a Sylow q-subgroup of L_a is centralised by a.

Proof. As X is inverted by a (by (10)) and $P = C_P(a)X$ by Coprime Action (a), we have that $[L_a, a]$ centralises X and (since $X \trianglelefteq L_a$) also P/X. Moreover (7) and (9) yield that $[F_{p'}(L_a), a] = 1$ and therefore $[L_a, a]$ centralises $F_{p'}(L_a)$. This implies that $[L_a, a] \le O_p(L_a)C_{L_a}(F^*(L_a)) = O_p(L_a) = P$.

Let $q \neq p$ be prime and let $Q \in \mathrm{Syl}_q(L_a)$. As a is isolated in L_a, we may suppose, with Corollary 7.9, that Q is a-invariant. But then $[Q, a] \le [L_a, a] \cap Q \le P \cap Q = 1$. $\square$

Let $q \in \pi' \cap \pi(F)$ and, with Corollary 7.9, let $Q \in \mathrm{Syl}_q(L_a, V)$. Then Q is centralised by a, but not by z because z inverts $F_{p'}(L_a)$ by (7) and (9). Therefore if $a \in T \in \mathrm{Syl}_2(C_G(Q))$, then a is the unique involution in T (by Lemma 7.6) and therefore $N_G(T) \le C_G(a)$. Using the Frattini argument we obtain $N_G(Q) = C_G(Q)N_{N_G(Q)}(T) \le C_G(Q)C_G(a) \le L_a$ which implies $Q \in \mathrm{Syl}_q(G)$. But then by Lemmas 4.12 and 4.14 and since $Q \not\le C$, it follows that a and b are not conjugate. This contradicts (5) and finishes the proof of the theorem. $\square$

Lemma 7.20. *a and b are isolated.*

Proof. We have $z \notin Z(L_a)$ by Lemma 5.10. Therefore and by Lemma 5.2 (1) there exists a prime $q \in \pi(F(L_a))$ such that $O_q(L_a)$ is not centralised by z. Let $Q \in \mathrm{Syl}_q(L_a, V)$ with Corollary 7.9. Then $C_G(Q) \le C_G(O_q(L_a)) \le L_a$ and Q is centralised by a (because $L_a = C_G(a)$ by Theorem 7.19), but not by z. Now we let $a \in T \in \mathrm{Syl}_2(C_G(Q))$ and argue as in the last paragraph in the proof of the previous theorem. As $r_2(G) = 2$ by Lemma 7.6, it follows that a is the unique involution in T and we have $[a, N_G(T)] = 1$. Using a Frattini argument, we obtain $N_G(Q) = C_G(Q)N_{N_G(Q)}(T) \le C_G(Q)C_G(a) \le L_a$. Thus $Q \in \mathrm{Syl}_q(G)$. But Q is contained in $C_G(a)$ and no Sylow q-subgroup of G lies in C by Lemma 4.12 and because Q is not centralised by z.

Therefore we can apply Lemma 4.14 which says that a and b are not conjugate in G. Finally Lemma 7.3 forces a and b to be isolated in G. $\square$

Theorem 7.21. *S contains precisely three involutions, they are all isolated and their centralisers are maximal subgroups of G.*

Proof. By Lemma 7.6 we have $r(S) = 2$. Also Lemmas 7.20 and 4.1 (2) imply that $V \leq Z(S)$. Together this yields that $V = \Omega_1(S)$ and hence that $\{a, b, z\}$ is precisely the set of involutions in S. Again by Lemma 7.20 we may interchange the roles of z, a and b as we like, so Theorem 7.19 yields that the centralisers C, $C_G(a)$ and $C_G(b)$ are maximal subgroups. $\square$

Theorem 7.21 and Lemmas 7.4 and 7.6 establish Theorem **B**.

8. The Soluble Z^*-Theorem

After a few preparatory results, we show that $C/O(C)$ possesses at least one component (and therefore C is not soluble) if Hypothesis 5.1 holds. Then we can prove the Soluble Z^*-Theorem.

Lemma 8.1. *Assume Hypothesis 5.1. Suppose that $M = C$ and let $\pi := \pi(F(M))$. Let $a \in M \backslash \{z\}$ be an involution, let L_a be a maximal subgroup of G containing $C_G(a)$ and suppose that $O(F(M)) \cap L_a \neq 1$. Then $[L_a, z]$ is contained in $F_{\pi'}(L_a)$.*

Proof. Define $L_0 := [L_a, z]$. Then by Lemma 5.8 we have that L_0 is a π'-group. We set $D := O(F(M)) \cap L_a$ and see that $D \times \langle z \rangle$ acts coprimely on L_0. As $[C_{L_0}(z), D] \leq C_{L_0}(z) \cap O(F(M)) = 1$, it follows that $C_{L_0}(z) \leq C_{L_0}(D)$. This means that we can apply Theorem 3.15 which yields that $[C_{L_0}(D), z]$ is normal in L_0 and that $[L_0, D]$ is a nilpotent normal subgroup of L_0. Let $C_G(D)D$ be contained in a maximal subgroup H of G. Then $z \in H$ and Lemma 5.2 (1) implies that $H_0 := [H, z] \leq O(H)$. So H_0 is a soluble π'-group, again by Lemma 5.8. Moreover $M \looparrowright H$ because $D \leq F(M)$. With part (1) of the Infection Theorem we deduce that $M \cap H_0 \leq M \cap F_{\pi'}(H) = 1$ and that, therefore, H_0 is inverted by z. But this means that H_0 is an abelian normal subgroup of H and in particular $H_0 \leq F(H)$. Now we have

$$[C_{L_0}(D), z] \leq [C_G(D), z] \cap C_{L_0}(D) \leq H_0 \cap C_{L_0}(D) \leq F(H) \cap C_{L_0}(D) \leq F(C_{L_0}(D)).$$

Hence $[C_{L_0}(D), z]$ is nilpotent and normal in L_0 by the previous paragraph, i.e. $[C_{L_0}(D), z] \leq F(L_0)$. By Coprime Action (a), it follows that $L_0 = C_{L_0}(D)[L_0, D]$ and finally

$$L_0 = [L_0, z] \leq [C_{L_0}(D), z][L_0, D, z] \leq F(L_0) \leq F_{\pi'}(L_a)$$

as stated. $\square$

Lemma 8.2. *Assume Hypothesis 7.1 and further that $[M, a] \nleq F(M)$. Then $[L_a, z] \leq F(L_a)$ and $[L_b, z] \leq F(L_b)$.*

Proof. Theorem 7.21 yields that $C_G(v) = L_v$ for all $v \in V^\#$ and that we can interchange the roles of a, b and z . As $[M, a] \not\leq F(M)$ by hypothesis, we can conclude, applying Lemma 8.1 with a and z interchanged, that $O(F(L_a)) \cap M = 1$. Thus $O(F(L_a))$ is inverted by z . On the other hand z centralises $O_2(L_a)$ as well as $E(L_a)$ (by Lemmas 4.1 (2) and 5.2 (1)) and thus $F^*(L_a)$ is centralised by $[L_a, z]$. Therefore $[L_a, z] \leq C_{L_a}(F^*(L_a)) \leq F(L_a)$ as stated. Similarly $[L_b, z] \leq F(L_b)$. □

Lemma 8.3. *Assume Hypothesis 7.1, let p be a prime and let $P \in Syl_p(G, V)$. Then there exists an involution $v \in V$ such that $[P, v] = 1$.*

Proof. By Theorem 7.21, a and b are isolated and the centralisers of a, b and z are maximal subgroups. Therefore the result follows from Lemma 5.7 if p is contained in one of the sets π, $\pi(F(L_a))$ or $\pi(F(L_b))$. Now suppose that $[P, z] \neq 1$. Then in particular $P \not\leq M = C$ and $p \notin \pi$ by Lemma 5.7. Hence Corollary 4.10 implies that without loss of generality p divides $|C_K(a)|$ and therefore $|[L_a, z]|$, by Lemma 5.2 (1). If $[L_a, z] \not\leq F(L_a)$, then Lemma 8.2 with a in the role of z yields $[M, a] \leq F(M)$ and $[L_b, a] \leq F(L_b)$. We are done if $p \in \pi(F(L_b))$, so let us assume that this is not the case. Then it follows that a centralises every V-invariant p-subgroup of M and of L_b. In particular $[C_P(z), a] = 1 = [C_P(b), a]$ because P is V-invariant. But by Coprime Action (b) this forces $[P, a] = 1$. On the other hand if $[L_a, z] \leq F(L_a)$, then $p \in \pi(F(L_a))$ and we are done. □

Lemma 8.4. *Assume Hypothesis 7.1. Then $C/O(C)$ is perfect.*

Proof. Assume not and let $\overline{C} := C/O(C)$. Then $\overline{C}' < \overline{C}$, i.e. $\overline{C}$ possesses a non-trivial abelian factor group. As $O(C_G(V)) \leq O(C)$ by Lemma 3.5, we have $O(C_G(V)) = O(C) \cap C_G(V)$ and therefore $\overline{C} \simeq C_G(V)/O(C_G(V))$. Now $C_G(V)/O(C_G(V))$ possesses a non-trivial abelian factor group. We note that v is isolated and $C_G(v) = L_v$ is a maximal subgroup for all $v \in V^\#$ by Theorem 7.21. So, arguing as for C in the previous paragraph, we see that $L_v/O(L_v) \simeq C_G(V)/O(C_G(V))$ for all $v \in V^\#$. This means that $\overline{C}$, $L_a/O(L_a)$ and $L_b/O(L_b)$ have a non-trivial p-factor group, respectively, for some prime p. On the other hand at least one of these maximal subgroups contains a Sylow p-subgroup of G by Lemma 4.13 (recall that a, b and z are isolated) and Lemma 8.3. Now Lemmas 4.2 and 3.8 and the fact that G is simple (Lemma 7.4) yield a contradiction. □

Theorem 8.5. *Assume Hypothesis 5.1. Then $C/O(C)$ possesses at least one component.*

Proof. Let $\overline{C} := C/O(C)$ and assume that $E(\overline{C}) = 1$. Let $T \in \mathrm{Syl}_2(O_{2',2}(C))$.
First assume that Hypothesis 7.1 holds, so $r(T) \geq 2$. Then Theorem 7.21 implies that T contains precisely three involutions. By hypothesis we have $F^*(\overline{C}) = O_2(\overline{C}) = \overline{T}$ and therefore $C_{\overline{C}}(\overline{T}) \leq \overline{T}$. Now we apply Theorem 3.16 to deduce that $\overline{C}$ is soluble. On the other hand $\overline{C}$ is perfect by Lemma 8.4, i.e. $\overline{C}$ is trivial, a contradiction.
Thus $r_2(O_{2',2}(C)) = 1$ and T is cyclic or quaternion. In the first case, $\mathrm{Aut}(T)$ is a cyclic 2-group and we deduce $[O^2(\overline{C}), \overline{T}] = 1$ and thus $[\overline{C}, \overline{T}] = 1$. This is impossible because $\overline{T} = F^*(\overline{C})$. In the second case, for similar reasons, $T \simeq Q_8$ since the automorphism groups of larger quaternion groups are 2-groups. We note that $\mathrm{Aut}(Q_8) \simeq S_4$.

By Lemma 5.3 we have $G = \langle z \rangle F^*(G)$ where $N := F^*(G)$ is simple. But T is quaternion with central involution z, so $z \in T' \leq G' = N$ which forces $G = N$ to be simple. Corollary 4.3 yields $O^2(\overline{C}) = \overline{C}$. In particular, $\overline{C}/C_{\overline{C}}(\overline{T})$ is isomorphic to a subgroup of A_4. Let $T \leq S \in \mathrm{Syl}_2(G)$. Then by the previous paragraph, $\overline{S}$ induces inner automorphisms on $\overline{T}$ and it follows that $S = T \simeq Q_8$, contradicting Lemma 5.2 (2). □

Proof of the Soluble Z^*-Theorem.
Assume that G is a minimal counterexample to the Soluble Z^*-Theorem. Let $z \in G$ be an isolated involution such that $C := C_G(z)$ is soluble, and assume that $z \notin Z^*(G)$. If $z \in H < G$, then $H = C_H(z)O(H)$ and thus H is soluble.
Let $t \in G$ be an involution. Lemma 4.1 (2) and Sylow's Theorem imply that $C_G(t)$ contains a conjugate of z . So $C_G(t)$ is soluble by the previous paragraph. From the minimality of G and the fact that every involution centraliser is soluble, it follows that the Z^*-Theorem holds in every proper subgroup and every proper section of G. This means that if we let M be a maximal subgroup of G containing C, then Hypothesis 5.1 is satisfied.
In particular, Theorem 8.5 is applicable and yields that $C/O(C)$ has at least one component. This is impossible because, by hypothesis, C is soluble. □

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