manuscripta math. 99, 73 – 91 (1999)

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Recognizing unstable equidimensional maps, and the number of stable projections of algebraic hypersurfaces

Received: 23 July 1998

Abstract. We study the recognition of $\mathcal{A}$-classes of multi-germs in families of corank-1 maps from n-space into n-space. From these recognition conditions we deduce certain geometric properties of bifurcation sets of such families of maps. As applications we give a formula for the number of $\mathcal{A}_e$-codimension-1 classes of corank-1 multi-germs from $\mathbb{C}^n$ to $\mathbb{C}^n$ and an upper bound for the number of stable projections of algebraic hypersurfaces in $\mathbb{R}^{n+1}$ into hyperplanes.

Introduction and notation

A smooth map (where smooth means either C^∞ or analytic) is unstable if it has positive $\mathcal{A}_e$-codimension as an s-germ for some set of source points $x_1, \ldots, x_s$. We study the recognition of unstable maps in families F of equidimensional corank-1 maps, both in the local situation where F is an unfolding germ and in the global situation where F is the restriction of the family of all (central or parallel) projections into hyperplanes to a smooth hypersurface given as the zero-set of some smooth function. Using these recognition conditions, we deduce certain local and global properties of the bifurcation set $\mathcal{B}$ in the parameter space of F.

Let $F = (u, f_u(x))$ be a family of smooth maps $f_u : \mathbb{F}^n \to \mathbb{F}^p$ (where $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$). In Section 1 we give an upper bound $s(n, p)$ for the number of source points (when $n < p$) or non-submersive source points (when $n \geq p$) in $f_u^{-1}(y)$ for a "generic" point $u \in \mathcal{B}$ (i.e. for a point $u \in \mathcal{B}$ in the complement of strata of $\mathcal{B}$ that correspond to multi-germs of $\mathcal{A}_e$-codimension ≥ 2). In Sections 2.1 and 2.2 we study the recognition of open $\mathcal{A}$-orbits within $\mathcal{K}$-orbits of type $A_{k_1}|\ldots|A_{k_s}$ for families of projections of hypersurfaces and for general families of equidimensional corank-1 maps, respectively. Using these conditions one shows that, for versal corank-1 families F, the closures of the $A_{k_1}|\ldots|A_{k_s}$ strata are smooth submanifolds of the source space of F. Section 2.3 describes the recognition conditions for s-germs of positive $\mathcal{A}_e$-codimension, which define closed subsets $\tilde{\mathcal{B}}(s)$

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Mathematics Subject Classification (1991): 58C27 (primary), 14J70 (secondary)

in the source space of F. The union of the projections of the $\tilde{\mathcal{B}}(s)$, $s = 1, \dots, s(n, n) = n+1$, onto the parameter space of F is the bifurcation set $\mathcal{B}$. The sets $\tilde{\mathcal{B}}(s)$, for $s \leq n$, can be singular, but $\tilde{\mathcal{B}}(n+1)$ is always smooth. For s-germs from $\mathbb{F}^n \to \mathbb{F}^p$, where $n > p$, the same conditions are valid for $p = 1$ and 2; for $p \geq 3$ there are additional unstable s-germs that are not recognized by these conditions (see Remark 1 at the beginning of Section 2). Sections 3 and 4 contain applications of the recognition conditions in Section 2. In Section 3 it is shown that, for complex-analytic equidimensional s-germs, there is exactly one connected orbit of $\mathcal{A}_e$-codimension 1 in each $\mathcal{K}$-orbit of type $A_{k_1}|\dots|A_{k_s}$, $2 \leq \sum k_i \leq n+1$. From this we deduce that there are $\sum_{i=1}^{n} p(i+1)$ (where $p(m)$ denotes the number of partitions of m) $\mathcal{A}_e$-classes of corank-1 s-germs from $\mathbb{C}^n$ to $\mathbb{C}^n$ of $\mathcal{A}_e$-codimension equal to one. Finally, in Section 4, we consider the special case of projections of algebraic hypersurfaces $M \subset \mathbb{F}^{n+1}$ into hyperplanes, and give bounds for the degree of $\mathcal{B}$ and, in the case $\mathbb{F} = \mathbb{R}$, for the number of distinct stable projections of M in terms of n and $d := \deg M$.

For the standard definitions of the (pseudo) groups of equivalences $\mathcal{A}_e$ and $\mathcal{K}_e$ of mono-germs and their tangent spaces, see, for example, the books [GG] and [M] and the survey article on determinacy by Wall [Wa]. For multi-germs $f = \{f_1, \dots, f_s\} : \mathbb{F}^n, S \to \mathbb{F}^p, f(S)$, we set $\theta_f := \bigoplus_{i=1}^{s} \theta_{f_i}$ where the θ_{f_i} are, as usual, sections of $f_i^* T\mathbb{F}^p$. Let C_{n_i}, $1 \leq i \leq s$ denote the local rings of smooth function germs at the ith source point and C_p the local ring of smooth function germs at the target point, and m_{n_i} and m_p the corresponding maximal ideals. Let $T\mathcal{R}_e \cdot f := (tf_1(\theta_{n_1})|\dots|tf_s(\theta_{n_s}))$, where $\theta_{n_1}, \dots, \theta_{n_s}$ are C_{n_i}-modules of germs of (independent) source vector fields, denote the extended right tangent space and $T\mathcal{L}_e \cdot f := wf(\theta_p)$ the extended left tangent space (here θ_p is the C_p-module of germs of target vector fields). The $\mathcal{A}_e$-tangent space and codimension are then given by $T\mathcal{A}_e \cdot f := T\mathcal{R}_e \cdot f + T\mathcal{L}_e \cdot f$ and $\mathrm{cod}(\mathcal{A}_e, f) := \dim_{\mathbb{F}} \theta_f / T\mathcal{A}_e \cdot f$. For the (restricted) groups of source- and target-preserving equivalences $\mathcal{A}$, $\mathcal{R}$ etc. one obtains analogous definitions of the tangent spaces and codimension by multiplying by the appropriate maximal ideals m_{n_i} and m_p. Given a s-germ f, there is an inclusion $\mathcal{A} \cdot f \subset \mathcal{K} \cdot f$ of orbits that does not hold for the orbits of the extended (pseudo) groups $\mathcal{A}_e$ and $\mathcal{K}_e$. We shall frequently refer to the open $\mathcal{A}$-orbit in a $\mathcal{K}$-orbit of $\mathcal{A}_e$-codimension 1, meaning that the s-germs in this $\mathcal{A}$-orbit have $\mathcal{A}_e$-codimension 1 (because we cannot refer to the open $\mathcal{A}_e$-orbit in a $\mathcal{K}_e$-orbit).

1. A bound for the number of source points for a generic point of $\mathcal{B}$

The "complexity" of the bifurcation set $\mathcal{B}$ of a family F of maps $f : \mathbb{F}^n \to \mathbb{F}^p$ depends on the number of unfolding parameters, on n and on the number

$s(n, p)$ which is defined as follows. (Here "complexity" refers, say, to the Betti numbers of $\mathcal{B}$ or, for real semi-algebraic bifurcation sets, to the number of connected components in the complement of $\mathcal{B}$.) For $n < p$, the number $s(n, p)$ is the maximal s amongst the s-germs $f = \{f_1, \ldots, f_s\} : \mathbb{F}^n, S \to \mathbb{F}^p, f(S)$ of $\mathcal{A}_e$-codimension no greater than one. For $n \geq p$, it is easy to see that we can add submersion germs f_i to a given s-germ (and hence increase s) without changing the $\mathcal{A}_e$-codimension. We therefore define $s(n, p)$ as above, with the restriction that the component germs of f be non-submersive.

The bound for $s(n, p)$ below is a corollary to the following formula for the $\mathcal{A}_e$-codimension of an s-germ. Analogous formulas for mono-germs ($s = 1$) for several groups of equivalences are given in Theorem 4.5.1 and Proposition 4.5.2 of [Wa], and the proofs of these formulas (including the one below) closely follow Mather's proof of Theorem 2.5 in [MaIV]. (After writing-up the proof below I found a reference to unpublished notes by L. C. Wilson [Wi] which also contain a proof of this formula, but I do not know whether his proof is different.) In [Ri96] there is also a related formula for multi-germs having "mixed" source dimensions, but this is not needed here.

Proposition 1. *Let*

$$f = \{f_1, \ldots, f_s\} : \mathbb{F}^n, S \to \mathbb{F}^p, f(S)$$

be an s-germ of finite $\mathcal{A}_e$-codimension. Then

$$\mathrm{cod}(\mathcal{A}_e, f) = \max[0,\ \mathrm{cod}(\mathcal{A}, f) + p(s-1) - ns].$$

Proof. For stable f, $\mathrm{cod}(\mathcal{A}_e, f) = 0$. Hence suppose f unstable. In this case the formula is equivalent to:

$$\dim_{\mathbb{F}} \frac{T\mathcal{A}_e \cdot f}{T\mathcal{A} \cdot f} = ns + p.$$

This, in turn, is equivalent to the following: if $\xi_i \in \theta_{n_i}$, $1 \leq i \leq s$, and $X \in \theta_p$ are such that

$$(tf_1(\xi_1)| \ldots |tf_s(\xi_s)) + wf(X) \in T\mathcal{A} \cdot f := T\mathcal{R} \cdot f + T\mathcal{L} \cdot f$$

then $\xi_i \in m_{n_i} \cdot \theta_{n_i}$, $1 \leq i \leq s$, and $X \in m_p \cdot \theta_p$. This condition fails if there exist $\bar{\xi}_i \in m_{n_i} \cdot \theta_{n_i}$, $1 \leq i \leq s$, such that

$$(tf_1(\xi_1 - \bar{\xi}_1)| \ldots |tf_s(\xi_s - \bar{\xi}_s)) \in T\mathcal{L}_e \cdot f.$$

Since $\xi_i - \bar{\xi}_i \notin m_{n_i} \cdot \theta_{n_i}$ we can, after a change of coordinates at the source points, assume that for some i

$$\xi_i - \bar{\xi}_i = \partial/\partial x_i^1,$$

where $x_i = (x_i^1, \dots, x_i^n)$ are the coordinates of the ith source point. This means that the s-germs f_t at

$$x_1, \dots, x_{i-1}, x_i + t \cdot \partial/\partial x_i^1, x_{i+1}, \dots, x_s$$

are $\mathcal{A}_e$-equivalent for all t. But $f = f_0$ is unstable, hence all the f_t are unstable: f has therefore infinite $\mathcal{A}_e$-codimension (by the Mather-Gaffney criterion) which contradicts the hypothesis of the proposition. □

Corollary 1. *Let*

$$s(n, p) := \sup\{s := |S| : \exists f : \mathbb{F}^n, S \to \mathbb{F}^p, f(S) : \operatorname{cod}(\mathcal{A}_e, f) \leq 1\},$$

where for $n \geq p$ all the component germs f_i of f are non-submersive. Then $s(n, p) = p + 1$ (for $n \geq p$) and $s(n, p) = \lfloor \frac{p+1}{p-n} \rfloor$ (for $n < p$).

Proof. For $n < p$ this follows directly from the formula for the $\mathcal{A}_e$-codimension. For $n \geq p$, all component germs f_i of f are non-submersive: hence, by the corank product formula, the $\mathcal{A}$-codimension of f is at least $s(n - p + 1)$. □

2. Recognizing unstable maps

Let $f = \{f_1, \dots, f_s\} : \mathbb{F}^n, S \to \mathbb{F}^n, f(S), S = \{x_1, \dots x_s\}$, be an s-germ. The $\mathcal{K}$-class of f is $A_{k_1}|\dots|A_{k_s}$ if the ith component germ f_i of f has an A_{k_i} singularity at x_i (i.e. a corank-1 singularity of multiplicity $m_i = k_i + 1$) and $f_1(x_1) = \dots = f_s(x_s)$. In the following two sections we describe recognition conditions for such $A_{k_1}|\dots|A_{k_s}$ singularities that are well-behaved on the diagonal, where two or more source points coalesce. In Section 2.1 we consider the slightly more complicated case where f is the restriction of the projection $\mathbb{F}^{n+1} \to H$, where H is some hyperplane, to some smooth hypersurface M. Section 2.2 contains the analogous recognition conditions for general equidimensional corank-1 maps. Finally, in Section 2.3, we supplement the conditions for an $A_{k_1}|\dots|A_{k_s}$ singularity by additional conditions – the resulting set of conditions detects s-germs of positive $\mathcal{A}_e$-codimension. Using the conditions in Sections 2.2 and 2.3 we deduce some properties of bifurcation sets and of the closures of the $A_{k_1}|\dots|A_{k_s}$ strata for versal families of corank-1 maps.

Remark 1. The conditions in Sections 2.2 and 2.3 are also valid for s-germs $f : \mathbb{F}^n \to \mathbb{F}^p$, $n > p$, of $\mathcal{K}$-type $A_{k_1}|\dots|A_{k_s}$. Using a "splitting lemma" for maps, one checks that the component germs $f_i : \mathbb{F}^p \times \mathbb{F}^{n-p} \to \mathbb{F}^p$ of such an f are equivalent to

$$(x_1, \dots, x_{n-1}, g\left(x_1, \dots, x_n) + \sum_{j=1}^{n-p} \pm y_j^2\right), \quad g(0, \dots, 0, x_n) = x_n^{k_i+1}.$$

Setting $\tilde{f}_i := f_i(x_1, \ldots, x_n, 0, \ldots, 0)$, we see that $\Sigma_{f_i} = \Sigma_{\tilde{f}_i} \times \{0\}$, $\Delta_{f_i} = \Delta_{\tilde{f}_i}$ (where Σ and Δ denote the critical set and the discriminant, respectively) and $\mathrm{cod}(\mathcal{A}_e, f_i) = \mathrm{cod}(\mathcal{A}_e, \tilde{f}_i)$. However, for $p \geq 3$ there exist unstable corank-1 s-germs of $\mathcal{K}$-type different from $A_{k_1}|\ldots|A_{k_s}$ that are not detected by the conditions described below. The first such unstable germ $f : \mathbb{F}^4 \to \mathbb{F}^3$ has $\mathcal{K}$-type D_4.

2.1. Families of projections of hypersurfaces

Let $M := g^{-1}(0) \subset \mathbb{F}^{n+1}$ be a hypersurface, and consider parallel (or central) projections along the direction (or from the centre) ω into hyperplanes. This yields a family of corank 1 maps from $\mathbb{F}^n$ into $\mathbb{F}^n$ with parameter ω. The kernels of this family of projections are the families of rays $L(t) = p + t\cdot\omega$, where $p \in \mathbb{F}^{n+1}$ and $\omega \in \mathbb{FP}^n$ (or, for central projection with centre $\omega \in \mathbb{F}^{n+1} \setminus M$, $L(t) = p + t\cdot(\omega - p)$). All $\mathcal{A}$-classes of s-germs of this family lie in some $\mathcal{K}$-orbit $A_{k_1}|\ldots|A_{k_s}$, and the $\mathcal{K}$-orbit membership is determined by the contact-orders of M and $L(t)$ at the points $L(\lambda_i)$, $1 \leq i \leq s$. The straightforward conditions for contact order $\geq m_1, \ldots, \geq m_s$

$$K^{(i)}(\lambda_j) = 0, \quad 0 \leq i \leq m_j - 1, \quad 1 \leq j \leq s, \quad \lambda_1 \equiv 0, \qquad (+)$$

where $K(t) := g \circ L(t)$, are not well-behaved on the diagonal, where $L(\lambda_i) = L(\lambda_j)$.

We now define "modified conditions" $K_j^{(i)}$, which define the same zero-set away from the diagonal, by iteration. Let $\epsilon_{j+1} := \lambda_{j+1} - \lambda_j$ and $K_1^{(i)} := \partial^i K/\partial t^i$, then we set for $j = 1, \ldots, s-1$:

$$K_{j+1}^{(0)} := \sum_{\alpha \geq m_j} K_j^{(\alpha)} \epsilon_{j+1}^{\alpha - m_j} / \alpha!,$$

where, for $j \geq 2$, $K_j^{(i)} := \partial^i K_j / \partial \epsilon_j^i$. The modified set of conditions

$$K_j^{(i)} = 0, \quad 0 \leq i \leq m_j - 1, \quad 1 \leq j \leq s, \qquad (*)$$

defines a variety in $\mathbb{F}^{s-1} \times \mathbb{F}^{n+1} \times \mathcal{V}$, where $\mathcal{V} = \mathbb{FP}^n$ or $\mathbb{F}^{n+1}$ and where $\epsilon_2, \ldots, \epsilon_s$ are coordinates in $\mathbb{F}^{s-1}$. Away from the "diagonal", where one or more consecutive ϵ_js vanish, this variety coincides with the zero-set of the original set of equations obtained by substituting $\lambda_j = \sum_{i=2}^{j} \epsilon_i$, $2 \leq j \leq s$ into $(+)$. This is so because the modified equations $K_j^{(i)}$, multiplied by some suitable power of ϵ_j, and the original equations generate the same ideal.

Further, notice that

$$K_j^{(i)} = c \cdot K_1^{(i + \sum_{l=1}^{j-1} m_l)} + R(\epsilon_2, \ldots, \epsilon_j, K_1^{(m)})$$

where $c \neq 0$ and $m > i + \sum_{l=1}^{j-1} m_l$. Also note that $\lambda_i = \lambda_j$, $i < j$, if and only if $\sum_{k=i+1}^{j} \epsilon_k = 0$, and in this case the required contact order at $L(\lambda_i) = L(\lambda_{i+1}) = \ldots = L(\lambda_j)$ is at least $\sum_{k=i}^{j} m_k$. The modified conditions are therefore "additive" with respect to contact-order. The boundaries of the s-local bifurcation sets made up of strata of type

$$A_{k_1}|\ldots|A_{k_i}|\ldots|A_{k_j}|\ldots|A_{k_s}$$

are therefore closed subsets of $(s - j + i)$-local bifurcation sets made up of strata of type

$$A_{k_1}|\ldots|A_{(\sum_{r=i}^{j} k_r)+j-i}|\ldots|A_{k_s}.$$

(the strange index in the middle stems from the fact that an A_k singularity has contact-order, or multiplicity, $k + 1$).

Note that the conditions above are already sufficient to detect the open $\mathcal{A}$-orbits within a given $\mathcal{K}$-orbit. In order to detect unstable s-germs contained in $\mathcal{A}$-orbits that are closed in their respective $\mathcal{K}$-orbit the conditions have to be supplemented by additional conditions (see Section 2.3). The number of additional conditions is equal to the codimension of the $\mathcal{A}$-orbit within a given $\mathcal{K}$-orbit.

2.2. General families of corank 1 maps $\mathbb{F}^n \to \mathbb{F}^n$

Consider an unfolding $F = (u, \bar{f}(u, z))$ of a corank 1 equidimensional map $f(z) = f(0, z)$. We can assume that $\bar{f}$ is of the form $(x_1, \ldots, x_{n-1}, g(u, x, y))$, where $z = (x, y)$ are coordinates in $\mathbb{F}^n$. In order to recognize an $A_{k_1}|\ldots|A_{k_s}$ singularity at $(x, y_1), \ldots, (x, y_s)$ we, again, define in an iterative fashion $g_1^{(i)} := \partial^i g/\partial y_1^i$ and for $j = 1, \ldots, s-1$:

$$g_{j+1}^{(0)} := \sum_{\alpha \geq k_j+1} g_j^{(\alpha)} \epsilon_{j+1}^{\alpha-k_j-1}/\alpha!,$$

where $\epsilon_{j+1} = y_{j+1} - y_j$ and $g_{j+1}^{(i)} := \partial^i g_{j+1}/\partial\epsilon_{j+1}^i$. The conditions

$$g_j^{(b_j)} = \ldots = g_j^{(k_j)} = 0,\ 1 \leq j \leq s, b_1 = 1, b_{\geq 2} = 0 \qquad (**)$$

then define the desired s-local stratum and are again "additive" (w.r.t. the multiplicities of the component germs) on the diagonal. In fact, all the properties stated in the previous section hold with $g_j^{(i)}$ in place of $K_j^{(i)}$.

For future reference we also state the corresponding "naive" conditions (that have excess dimension on the diagonal):

$$g\left(x, y_1 + \sum_{i=2}^{r} \epsilon_i\right) = g(x, y_1);\ \ g^{(\alpha)}\left(x, y_1 + \sum_{i=1}^{j} \epsilon_i\right) = 0, \epsilon_1 \equiv 0,\ \ (++)$$

where $g^{(\alpha)} := \partial^\alpha g/\partial y^\alpha$ and with the index ranges $2 \leq r \leq s$, $1 \leq \alpha \leq k_j$ and $1 \leq j \leq s$.

Using the conditions $(**)$, it is straightforward to show the following.

Proposition 2. *Let $F : \mathbb{F}^d \times \mathbb{F}^n \to \mathbb{F}^d \times \mathbb{F}^n$ be an $\mathcal{A}_e$-versal unfolding of an s-germ f of corank 1. Then the strata in $\mathbb{F}^d \times \mathbb{F}^{sn}$ corresponding to the closure of the $A_{k_1}|\ldots|A_{k_s}$-stratum are smooth submanifolds.*

Proof. Set $k := \sum_{j=1}^s (k_j + 1)$ and let $W \subset J^k(n+s-1, n)$ denote the $A_{k_1}|\ldots|A_{k_s}$-stratum. The conditions $(**)$ above define the closure $\bar{W}$ of W and are all linear in some coordinate of the jet-space, and these coordinates are pairwise distinct. The closure $\bar{W}$ of the $A_{k_1}|\ldots|A_{k_s}$-stratum in $J^k(n+s-1, n)$ is therefore a smooth submanifold of codimension $(\sum_{i=1}^s k_i)+s-1$.

Now note that $J^k(n+s-1, n)$ and $\Sigma^1[{}_sJ^k(n,n)]$ are isomorphic, and the coordinate change

$$(x_1, \ldots, x_{n-1}, y_1, \epsilon_2, \ldots, \epsilon_s) \mapsto \left(x_1, \ldots, x_{n-1}, y_1, \ldots, y_1 + \sum_{j=2}^s \epsilon_j\right)$$

maps the submanifold $\bar{W}$ in the former jet-space diffeomorphically to a submanifold $\bar{W}'$ in the latter jet-space. Since F is versal, we can pull-back $\bar{W}'$ to a submanifold in $\mathbb{F}^d \times \mathbb{F}^{sn}$. $\square$

Remark 2. The smoothness of the closure of the $A_{k_1}|\ldots|A_{k_s}$ stratum simplifies certain arguments in [MMR], where formulas are given for the number of isolated stable singularities appearing in a deformation of a weighted homogeneous, complex corank-1 singularity.

2.3. The bifurcation set

A multi-germ of a corank-1 map $f : \mathbb{F}^n \to \mathbb{F}^n$ is stable if and only if its component germs are Morin singularities and it satisfies the normal crossings condition (NC), see e.g. Theorem 6.4, p. 192, of [GG]. The stable s-germs are precisely the open $\mathcal{A}$-orbits in the $\mathcal{K}$-orbits of type $A_{k_1}|\ldots|A_{k_s}$, for $\sum k_i \leq n+1$ and $s \leq n+1$. The unstable s-germs can therefore be characterized by the property that their jet-extensions (of the appropriate order) fail to be transverse to some submanifold defined by the recognition conditions for the closure of one of the $\mathcal{K}$-classes $A_{k_1}|\ldots|A_{k_s}$, where $\sum k_i \leq n+1$ and $s \leq n+1$. Recall that the recognition conditions for an $A_{k_1}|\ldots|A_{k_s}$ singularity in Sections 2.1 and 2.2 are conditions on the k-jet, $k = \sum_{i=1}^s (k_i + 1)$, of a function $K : \mathbb{F}^{n+s} \to \mathbb{F}$ (with source coordinates $x_1, \ldots, x_{n+1}, \epsilon_2, \ldots, \epsilon_s$) or of a map $f : \mathbb{F}^{n+s-1} \to \mathbb{F}^n$ (with source coordinates $x_1, \dots, x_{n-1}, y_1, \epsilon_2, \dots, \epsilon_s$), respectively. Hence we will consider transversality to submanifolds in $J^k(n+s, 1)$ or $J^k(n+s-1, n)$, respectively.

The conditions for the failure of transversality require some extra notation. Let

$$\mathbf{k}(s, m) := (k_1, \dots, k_s), \text{ where } k_i \geq k_{i+1}, k_s \geq 1, \sum k_i = m$$

denote a partition of m involving s non-zero summands, and let $\mathcal{P}(s, m)$ be the set of all such partitions. Let $A_{\mathbf{k}(s,m)} := A_{k_1}|\dots|A_{k_s}$ be the $\mathcal{K}$-class associated with such a partition, and let $Q_{\mathbf{k}(s,m)} : \mathbb{F}^{n+s} \to \mathbb{F}^{m+s}$ and $G_{\mathbf{k}(s,m)} : \mathbb{F}^{n+s-1} \to \mathbb{F}^{m+s-1}$ denote the maps with component functions the recognition conditions $(*)$ and $(**)$ for the closure of the $A_{\mathbf{k}(s,m)}$-stratum of Sections 2.1 and 2.2, respectively.

Notice that the isolated stable singularities of an s-germ f from $\mathbb{F}^n$ to $\mathbb{F}^n$ are the open $\mathcal{A}$-orbits within the $\mathcal{K}$-classes $A_{\mathbf{k}(s,n)}$. All s-germs of type $A_{\mathbf{k}(s,n+1)}$ are therefore unstable. Furthermore, the orbit through the stable mono-germ $(x_1, \dots, x_{n-1}, y^2)$ is the only $\mathcal{A}_e$-orbit in $A_{\mathbf{k}(1,1)}$. Hence it is sufficient to find the conditions for the failure of transversality to the submanifolds $A_{\mathbf{k}(s,m)}$, where $2 \leq m \leq n$. We first consider the case of parametrized corank-1 maps and then indicate the necessary changes in the more complicated global case of projections of hypersurfaces.

For parametrized corank-1 maps the closure of the $A_{\mathbf{k}(s,m)}$ stratum is a submanifold in $J^k(n+s-1, n)$ of codimension $m+s-1$ which is given as the zero-set of a regular map $\varphi : J^k(n+s-1, n) \to \mathbb{F}^{m+s-1}$. Let $G_{\mathbf{k}(m,s)} = (G_1, \dots, G_{m+s-1}) : \mathbb{F}^{n+s-1} \to \mathbb{F}^{m+s-1}$ be the map whose component functions are the recognition conditions $(**)$ of Section 2.2, and let $H_{\mathbf{k}(m,s)}$ be the corresponding map with the "naive" conditions $(++)$ as components. The map $H_{\mathbf{k}(s,m)}$ is the composition of the jet-extension $j^k f$ with φ. Now, $j^k f$ fails to be transverse to $\varphi^{-1}(0)$ at q if and only if $H_{\mathbf{k}(s,m)}$ fails to be a submersion at q. It is easy to see that $H_{\mathbf{k}(s,m)}$ fails to be a submersion at source points belonging to the closure of $A_{\mathbf{k}(s,m+1)}$, but we are only interested in the failure of transversality to the proper $A_{\mathbf{k}(s,m)}$ stratum. Letting $\hat{H}_{\mathbf{k}(s,m)}$ denote the map defined by omitting the s maximal derivative conditions $g^{(k_j)}(p_j) = 0$, $1 \leq j \leq s$, from $(++)$ and $d_x\hat{H}_{\mathbf{k}(s,m)}$ its differential with respect to $x_1, \dots, x_{n-1}$, and restricting to the $A_{\mathbf{k}(s,m)}$ stratum, we see that $d_x\hat{H}_{\mathbf{k}(s,m)}$ has maximal rank if and only if $dH_{\mathbf{k}(s,m)}$ has.

However, $d_x\hat{H}_{\mathbf{k}(s,m)}$ is not well-behaved on the diagonal, where some $\epsilon_j = 0$: we have to add to certain columns appropriate linear combinations of others and divide by powers of ϵ_j. The resulting matrix is the differential, $d_x\bar{H}_{\mathbf{k}(s,m)}$, of a map $\bar{H}_{\mathbf{k}(s,m)}$, whose component functions are again defined by iteration: set $g_1^{(i)} := \partial^i g/\partial y_1^i$, for $0 \leq i < k_1$, and for $j = 2, \ldots, s$ set

$$g_j^{(0)} := \sum_{\alpha \geq k_j} g_{j-1}^{(\alpha)} \epsilon_j^{\alpha - k_j}/\alpha!; \quad g_j^{(i)} := \partial^i g_j^{(0)}/\partial \epsilon_j^i, 1 \leq i < k_j.$$

Notice that, away form the diagonal, $\bar{H}_{\mathbf{k}(s,m)} := (\bar{H}_1, \ldots, \bar{H}_{m-1})$ and $\hat{H}_{\mathbf{k}(s,m)}$ define the same ideal. Set $\rho := \sum_{i=1}^{m-1} v_i \bar{H}_i$, where $(v_1 : \ldots : v_{m-1}) \in \mathbb{FP}^{m-2}$, then the component functions of the map

$$\bar{G}_{\mathbf{k}(s,m)} := (\bar{G}_1, \ldots, \bar{G}_{n-m+1}) : \mathbb{F}^{n+s-1} \to \mathbb{F}^{n-m+1},$$

which are defined by eliminating the v_i between the functions $\partial \rho/\partial x_j$ $(1 \leq j \leq n-1)$, vanish if and only if $\bar{H}$ (and hence $G_{\mathbf{k}(s,m)}$) fails to be a submersion. Hence $\bar{G}_{\mathbf{k}(s,m)}$ is the desired condition for the non-transversality to $A_{\mathbf{k}(s,m)}$ in the case of parametrized corank-1 maps.

For projections of hypersurfaces, the closure of the $A_{\mathbf{k}(s,m)}$ stratum is a submanifold in $J^k(n+s, 1)$ of codimension $m+s$. The recognition conditions $(*)$ and $(+)$ of Section 2.1 define maps $Q_{\mathbf{k}(s,m)} = (Q_1, \ldots, Q_{m+s})$ and $K_{\mathbf{k}(s,m)} = (K_1, \ldots, K_{m+s})$ in the variables x_i $(1 \leq i \leq n+1)$, ϵ_j $(2 \leq j \leq s)$, recall that $\epsilon_{j+1} := \lambda_{j+1} - \lambda_j$ and $\lambda_1 \equiv 0$. We now follow the same procedure as in the case of parametrized corank-1 maps, with $K_{\mathbf{k}(s,m)}$ in place of $H_{\mathbf{k}(s,m)}$. Remove again the highest derivative conditions at the s source points and let $\bar{K}_{\mathbf{k}(s,m)}$ be the map, whose m component functions are defined as follows. Set $\bar{K}_1^{(i)} := \partial^i K/\partial t^i$, for $0 \leq i < k_1$, and for $j = 2, \ldots, s$ set

$$\bar{K}_j^{(0)} := \sum_{\alpha \geq k_{j-1}} \bar{K}_{j-1}^{(\alpha)} \epsilon_j^{\alpha - k_{j-1}}/\alpha!; \quad \bar{K}_j^{(i)} := \partial^i \bar{K}_j^{(0)}/\partial \epsilon_j^i, 1 \leq i < k_j.$$

Let $\ell := \omega$ (for parallel projection) or $\ell := \omega - x$ (for central projection). If ℓ is the kernel direction of the projection then, at an $A_{\mathbf{k}(s,m)}$ singularity, $d_x \bar{K}_j^{(i)}(\ell) = 0$ for $0 \leq i < k_j$, $1 \leq j \leq s$ but $d_x \bar{K}_1^{(k_1)}(\ell) \neq 0$. Let $e_1, \ldots, e_n$ be a basis for $\{x \in \mathbb{F}^{n+1} : \langle x, \ell \rangle = 0\}$ and set $\rho := \sum_{i=1}^{m} v_i \bar{K}_i$, where $(v_1 : \ldots : v_m) \in \mathbb{FP}^{m-1}$. The component functions of the map

$$\bar{Q}_{\mathbf{k}(s,m)} := (\bar{Q}_1, \ldots, \bar{Q}_{n-m+1}) : \mathbb{F}^{n+s} \to \mathbb{F}^{n-m+1},$$

which are defined by eliminating the v_i between the functions $d_x \rho(e_j)$ $(1 \leq j \leq n)$, vanish if and only if the restriction of $Q_{\mathbf{k}(s,m)}$ to the $A_{\mathbf{k}(s,m)}$ stratum fails to be submersive – they therefore represent the desired non-transversality conditions to $A_{\mathbf{k}(s,m)}$ for projections of hypersurfaces.

The unstable s-germs in families of projections of hypersurfaces, where the parameter space $\mathcal{V}$ is either $\mathbb{F}^{n+1}$ (for central projection) or $\mathbb{FP}^n$ (for parallel projection), or in general d-parameter families of corank-1 maps from $\mathbb{F}^n$ to $\mathbb{F}^n$ are then characterized as follows. For $1 \le s \le n$ and $2 \le m \le n$, let $\tilde{\mathcal{B}}_{\mathbf{k}(s,m)}$ be the zero-set of one of the following maps:

$$(Q_{\mathbf{k}(s,m)}, \bar{Q}_{\mathbf{k}(s,m)}) : \mathcal{V} \times \mathbb{F}^{n+s} \to \mathbb{F}^{n+s+1}$$

(for families of projections) or

$$(G_{\mathbf{k}(s,m)}, \bar{G}_{\mathbf{k}(s,m)}) : \mathbb{F}^d \times \mathbb{F}^{n+s-1} \to \mathbb{F}^{n+s}$$

(for general d-parameter families). And set

$$\tilde{\mathcal{B}}(s) := \bigcup_{m=2}^{n} \bigcup_{\mathbf{k}(s,m)\in\mathcal{P}(s,m)} \tilde{\mathcal{B}}_{\mathbf{k}(s,m)}.$$

And for $s = n+1$, we set $\tilde{\mathcal{B}}(n+1) := Q^{-1}_{\mathbf{k}(n+1,n+1)}(0)$ or $G^{-1}_{\mathbf{k}(n+1,n+1)}(0)$. In other words, $\tilde{\mathcal{B}}(n+1)$ is the closure of the $A_1|\ldots|A_1$-stratum ($n+1$ A_1s). Let, in both cases, π denote the projection onto the parameter space: then $\mathcal{B}(s) := \pi(\tilde{\mathcal{B}}(s))$ is the closure of the s-local bifurcation set and $\mathcal{B} := \bigcup_{s=1}^{n+1} \mathcal{B}(s)$ the full bifurcation set (notice that, by Corollary 1, $s(n,n) = n+1$).

Remark 3. When $n = 2$ the above conditions for an unstable s-germ are equivalent to the presence of an isolated stable singularity of higher multiplicity. In dimension $n = 2$ there are two isolated stable s-germs, namely Whitney cusps and transverse double-folds. They represent the open $\mathcal{A}$-orbits in A_2 and in $A_1|A_1$, respectively. The cusp and double-fold multiplicities of a map-germ f of the plane, denoted by $c(f)$ and $d(f)$ in [Ri87], characterize the unstable germs: f is unstable if and only if $c(f) \ge 2$ or $d(f) \ge 2$. For $n \ge 3$ this is no longer true: the mono-germ $(x, y, z^3 + (x^2+y^2)z)$ has $\mathcal{A}_e$-codimension one, but the multiplicities of the isolated stable singularities A_3, $A_2|A_1$ and $A_1|A_1|A_1$ are all zero.

A natural question concerning the sets $\tilde{\mathcal{B}}(s)$ is the following: given an $\mathcal{A}_e$-versal family of corank-1 maps $F : \mathbb{F}^d \times \mathbb{F}^n \to \mathbb{F}^d \times \mathbb{F}^n$, are the sets $\tilde{\mathcal{B}}(s) \subset \mathbb{F}^d \times \mathbb{F}^{n+s-1}$ smooth submanifolds? For the set $\tilde{\mathcal{B}}(n+1)$ the smoothness follows from Proposition 2. But for the other sets $\tilde{\mathcal{B}}(s)$, $1 \le s \le n$, this turns out to be false: the components $\tilde{\mathcal{B}}_{\mathbf{k}(s,m)}$ have non-empty intersection. In dimension two, however, the components $\tilde{\mathcal{B}}_{\mathbf{k}(s,m)}$ themselves are smooth (as we will show next); in dimension $n \ge 3$ we suspect that the $\tilde{\mathcal{B}}_{\mathbf{k}(s,m)}$, where $m < n+1$, fail to be smooth (at least the corresponding strata in jet-space are singular, see the proof of Proposition 5).

Now consider the geometry of bifurcation sets in the particular case $n = 2$. There are five $\mathcal{A}_e$-codimension-1 singularities (over $\mathbb{C}$): (i) $(x, y^3 + x^2y)$, (ii) $(x, xy + y^4)$, (iii) $\{(x, y^2), (y^2, x), (x, x + y^2)\}$, (iv) $\{(x, xy + y^3), (y^2, x)\}$ and (v) $\{(x, y^2), (x, x^2 + y^2)\}$. The open $\mathcal{A}$-orbits in A_3,

$A_1|A_1|A_1$ and $A_2|A_1$ are (ii), (iii) and (iv), respectively, and the closed codimension-1 orbits within A_2 and $A_1|A_1$ are (i) and (v), respectively. The partitions $\mathbf{k}(s,m)$ appearing in the indices of the sets $\tilde{\mathcal{B}}_{\mathbf{k}(s,m)}$ corresponding to the closures of the $\mathcal{A}_e$-classes (i) to (v) above are given by (2), (3), (1, 1, 1), (2, 1) and (1, 1), respectively. Then

$$\tilde{\mathcal{B}}(1) = \tilde{\mathcal{B}}_{(2)} \cup \tilde{\mathcal{B}}_{(3)}, \quad \tilde{\mathcal{B}}(2) = \tilde{\mathcal{B}}_{(2,1)} \cup \tilde{\mathcal{B}}_{(1,1)}$$

and $\tilde{\mathcal{B}}(3) = \tilde{\mathcal{B}}_{(1,1,1)}$.

Proposition 3. *Let $F : \mathbb{F}^d \times \mathbb{F}^2 \to \mathbb{F}^d \times \mathbb{F}^2$ be an $\mathcal{A}_e$-versal family of corank-1 maps of the plane. (i) Then the five sets $\tilde{\mathcal{B}}_{\mathbf{k}(s,m)} \subset \mathbb{F}^d \times \mathbb{F}^{n+s-1}$ defined above are smooth submanifolds of dimension $d-1$ (or are empty). (ii) The pairs of components $\tilde{\mathcal{B}}_{(2)}, \tilde{\mathcal{B}}_{(3)} \subset \tilde{\mathcal{B}}(1)$ and $\tilde{\mathcal{B}}_{(2,1)}, \tilde{\mathcal{B}}_{(1,1)} \subset \tilde{\mathcal{B}}(2)$ have non-empty intersections for an open set of families F.*

Proof. (i) From the preceding discussion we know that the sets $\tilde{\mathcal{B}}_{(3)}, \tilde{\mathcal{B}}_{(1,1,1)}, \tilde{\mathcal{B}}_{(2,1)}$ correspond to open $\mathcal{A}$-orbits in their respective $\mathcal{K}$-orbit, hence they are smooth by Proposition 2. For $\tilde{\mathcal{B}}_{(2)}$, we have to add the non-transversality condition $\partial^2 g/\partial x \partial y_1 = 0$ to the conditions for an A_2. For $\tilde{\mathcal{B}}_{(1,1)}$, we supplement the conditions $(**)$ in Section 2.2 for an $A_{(1,1)}$ bi-germ by

$$\sum_{i \geq 1} \frac{\partial^{i+1} g_1}{\partial x \partial y_1^i} \epsilon_2^{i-1}/i! = 0,$$

which is the condition for the failure of transversality to the $A_{(1,1)}$ stratum.

(Geometrically this condition is equivalent to the linear dependence of the (limiting) tangent lines of the discriminants of the two A_1 points. Notice that the "naive" condition for the linear dependence of the (limiting) tangent lines to the discriminant at the points $(x, g_1(x, y_1))$ and $(x, g_1(x, y_1 + \epsilon_2)$, given by $\partial g_1(x, y_1 + \epsilon_2)/\partial x - \partial g_1(x, y_1)/\partial x = 0$, vanishes identically for $\epsilon_2 = 0$. Also notice that

$$\tilde{\mathcal{B}}_{(1,1)} \cap \{\epsilon_2 = 0\} = \{\partial^2 g_1/\partial x \partial y_1 = \partial^i g_1/\partial y_1^i = 0, 1 \leq i \leq 3\},$$

the intersection of $\tilde{\mathcal{B}}_{(1,1)}$ with the diagonal therefore corresponds to the closure of the $\mathcal{A}$-class $(x, xy^2 + y^4 + y^5)$, i.e. type 11_5 in the notation of [Ri87].)

In both cases $\tilde{\mathcal{B}}_{(2)}$ and $\tilde{\mathcal{B}}_{(1,1)}$, the conditions $(**)$ and the additional condition clearly define smooth submanifolds of the appropriate jet-space of codimension $n+s$. The pull-back of these submanifolds by a versal family F yields submanifolds of dimension $d-1$ (or empty sets).

(ii) The defining conditions for the non-transverse $A_{(2)}$ stratum and the $A_{(3)}$ stratum (and similarly for the non-transverse $A_{(1,1)}$ stratum and the $A_{(2,1)}$ stratum) imply that these pairs of strata have non-empty intersection I in jet-space. To complete the proof of the assertion it is sufficient to construct examples of versal families F whose jet-extensions meet the intersection locus I (because this will then be the case for a Zariski-open set of jet-extensions): for $\tilde{\mathcal{B}}_{(2)}, \tilde{\mathcal{B}}_{(3)}$ take any versal unfolding F of $(x, xy^2 + y^4 + y^5)$ and for $\tilde{\mathcal{B}}_{(1,1)}, \tilde{\mathcal{B}}_{(2,1)}$ take a versal unfolding of $(x, xy^2 + y^5 + y^6)$. The results in [Ri90] then show that the jet-extension of F meets I (in [Ri90] C^0-$\mathcal{A}_e$-versal unfoldings are considered, but the adjacencies of strata are preserved if one passes to C^∞-versal unfoldings). □

From the smoothness of the components $\tilde{\mathcal{B}}_{\mathbf{k}(s,m)}$ for versal families one can easily deduce the following topological properties of the corresponding real bifurcation sets. Let $\tilde{\pi}$ denote the restriction of the projection $\pi : \mathbb{F}^d \times \mathbb{F}^{s+1} \to \mathbb{F}^d$ to $\tilde{\mathcal{B}}_{\mathbf{k}(s,m)}$ and set $\Delta := \bigcup_{j \geq 2} \{\epsilon_j = 0\}$. By a "free boundary" of a component $\mathcal{B}_{\mathbf{k}(s,m)}$ of the bifurcation set we mean the following: for a versal family, $\mathcal{B}_{\mathbf{k}(s,m)}$ is locally diffeomorphic to a semi-algebraic set which can be triangulated, and we say that an i-simplex is free if it is adjacent to only one $(i+1)$-simplex.

Proposition 4. *Let $F : \mathbb{F}^d \times \mathbb{F}^2 \to \mathbb{F}^d \times \mathbb{F}^2$ be an $\mathcal{A}_e$-versal family of corank-1 maps of the plane. (i) The map $\tilde{\pi} : \tilde{\mathcal{B}}_{\mathbf{k}(s,m)} \to \mathcal{B}_{\mathbf{k}(s,m)}$ is an r-fold covering, where $r = 1$ for $\mathbf{k}(s,m) = (2), (3)$ and $(2,1)$, $r = 6$ for $\mathbf{k}(s,m) = (1,1,1)$ and $r = 2$ for $\mathbf{k}(s,m) = (1,1)$. When $r \geq 2$, the branch-locus is given by $\tilde{\mathcal{B}}_{\mathbf{k}(s,m)} \cap \Delta =: S_{\mathbf{k}(s,m)}$. (ii) For $\mathbb{F} = \mathbb{R}$, the components $\mathcal{B}_{(1,1,1)}$ and $\mathcal{B}_{(1,1)}$ have "free boundaries" in codimension 2 along $\pi(S_{\mathbf{k}(s,m)})$. The full bifurcation set $\mathcal{B} := \bigcup \mathcal{B}_{\mathbf{k}(s,m)}$ does not have free boundaries in codimension 2.*

Proof. (i) Consider $F = (u, f_u)$ as a multi-germ of a family with target $(v, q) \in \mathbb{F}^d \times \mathbb{F}^2$. The versality of F implies that for all $u \in \mathcal{B}_i \setminus C$, where C is a closed subset, f_u has exactly one $\mathcal{A}_e$-codimension-1 singularity at $f_u^{-1}(q')$, for some q' near q. Let $k \leq s$ be the number of source points with identical recognition conditions ($k = 3 = s$ for $\tilde{\mathcal{B}}_{(1,1,1)}$, $k = 2 = s$ for $\tilde{\mathcal{B}}_{(1,1)}$, but $k = 1 \neq s$ for $\tilde{\mathcal{B}}_{(2,1)}$). There is an S_k action on the source points with identical recognition conditions, hence there are $r := k!$ points of $\tilde{\mathcal{B}}_{\mathbf{k}(s,m)}$ in each fibre $\tilde{\pi}^{-1}(u)$, for $u \in \mathcal{B}_{\mathbf{k}(s,m)} \setminus C$. And the branch-locus $S_{\mathbf{k}(s,m)}$ of the r sheets of $\tilde{\mathcal{B}}_{\mathbf{k}(s,m)}$ is $\tilde{\mathcal{B}}_{\mathbf{k}(s,m)} \cap \Delta$ (in the cases $\mathbf{k}(s,m) = (1,1,1)$ and $(1,1)$ where $r \geq 2$).

(ii) Adding the condition $\epsilon_j = 0$ to the defining conditions of $\tilde{\mathcal{B}}_{\mathbf{k}(s,m)}$ in some appropriate multi-jet space (see above) and pulling back by the multi-jet extension of the versal family F, we see that $\tilde{\mathcal{B}}_{\mathbf{k}(s,m)} \cap \Delta$ is a smooth submanifold of dimension $d - 2$ or is empty. The versality of F implies that $\tilde{\pi}$ is finite-to-one, hence $\pi(\tilde{\mathcal{B}}_{\mathbf{k}(s,m)} \cap \Delta)$ has codimension 2 in $\mathbb{R}^d$. In the cases $\mathbf{k}(s,m) = (1,1,1)$ and $(1,1)$, where $S_{\mathbf{k}(s,m)} = \tilde{\mathcal{B}}_{\mathbf{k}(s,m)} \cap \Delta$ is non-empty, let U be any open neighborhood of $\pi(S_{\mathbf{k}(s,m)})$: then, by the versality of F, all the

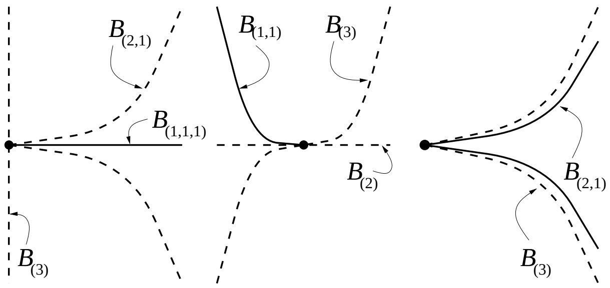

Fig. 1. Multi-local bifurcation sets: the $\mathcal{B}_{(1,1,1)}$ and $\mathcal{B}_{(1,1)}$ components have free boundaries at $\pi(\tilde{\mathcal{B}}_{\mathbf{k}(s,m)} \cap \Delta)$, $\mathbf{k}(s,m) = (1,1,1), (1,1)$ (left and middle diagrams), but $\mathcal{B}_{(2,1)}$ merely has a cusp at $\pi(\tilde{\mathcal{B}}_{(2,1)} \cap \Delta)$ (diagram on the right). The points $\pi(\tilde{\mathcal{B}}_{\mathbf{k}(s,m)} \cap \Delta)$ are marked by a dot and the corresponding components $\mathcal{B}_{\mathbf{k}(s,m)}$, $\mathbf{k}(s,m) = (1,1,1), (2,1), (1,1)$, (to the left, middle and right, respectively) are drawn as solid lines, the other components are drawn as dashed lines.

fibres $\tilde{\pi}^{-1}(u)$, $u \in U$, "correspond" to exactly one $\mathcal{A}_e$-codimension-2 $(s-1)$-germ (i.e. if $(u, x, y_1, \epsilon_2, \dots, \epsilon_s) \in \tilde{\pi}^{-1}(u)$, where some $\epsilon_j = 0$, then f_u is a codimension-2 $(s-1)$-germ at $(x, y_1), \dots, (x, y_1 + \sum_{2 \le k \ne j \le s} \epsilon_k)$). The smoothness of $\tilde{\mathcal{B}}_{\mathbf{k}(s,m)}$ implies that the map $\tilde{\pi}$ is of "folding type" (has even multiplicity) along open subsets of $S_{\mathbf{k}(s,m)}$. Hence $\pi(S_{\mathbf{k}(s,m)})$ is a free boundary of $\mathcal{B}_{\mathbf{k}(s,m)}$. Finally, the defining conditions of $\tilde{\mathcal{B}}_{(1,1,1)}$ and $\tilde{\mathcal{B}}_{(1,1)}$ imply that $\pi(S_{(1,1,1)}) \subset \mathcal{B}_{(3)} \cap \mathcal{B}_{(2,1)}$ and $\pi(S_{(1,1)}) \subset \mathcal{B}_{(2)} \cap \mathcal{B}_{(3)}$. But the sets $\mathcal{B}_{(2)}$, $\mathcal{B}_{(3)}$ and $\mathcal{B}_{(2,1)}$ do not have free boundaries, because $\tilde{\pi} : \tilde{\mathcal{B}}_{\mathbf{k}(s,m)} \to \mathcal{B}_{\mathbf{k}(s,m)}$ is $1:1$ in the complement of some closed subset. It follows that the full bifurcation set does not have free boundaries. □

Remark 4. For non-versal families all the sets $\pi(\tilde{\mathcal{B}}_{\mathbf{k}(s,m)} \cap \Delta)$ are potentially free boundaries, and the sets $\tilde{\mathcal{B}}_{\mathbf{k}(s,m)}$ can also have an "off-diagonal" branch-locus. Non-versal families of projections of a certain class of singular surfaces have been studied in [Ri96]: in this case the full bifurcation set still cannot have free boundaries in codimension 2 and the incidences between components of the bifurcation sets (like, for example, $\pi(\tilde{\mathcal{B}}_{(1,1,1)} \cap \Delta) \subset \mathcal{B}_{(3)} \cap \mathcal{B}_{(2,1)}$) are also valid in this more general situation.

Example 1. Figure 1 shows the bifurcation sets in the base of the miniversal unfoldings of $\{(x, xy+y^4), (y^2, x)\}$ (to the left), $(x, xy^2+y^4+y^5)$ (middle) and $(x, xy + y^5 + y^7)$ (to the right). These examples illustrate the fact that, for versal families, the components $\mathcal{B}_{(1,1,1)}$ and $\mathcal{B}_{(1,1)}$ have free boundaries of codimension 2 at $\pi(\tilde{\mathcal{B}}_{\mathbf{k}(s,m)} \cap \Delta)$, whereas $\mathcal{B}_{(2,1)}$ merely has cuspidal edges at the corresponding locus.

3. Counting $\mathcal{A}_e$-classes of codimension 1 over $\mathbb{C}$

The stable corank-1 s-germs from $\mathbb{C}^n$ to $\mathbb{C}^n$ are all simple, at present it is not known whether all $\mathcal{A}_e$-codimension-1 s-germs are simple (except for the case of mono-germs, see Remark 5 (ii) at the end of the present section). In the present section, $\mathcal{A}_e$-codimension-1 class therefore either refers to a simple $\mathcal{A}_e$-orbit or to a modular stratum of codimension one.

Proposition 5. *For s-germs from $\mathbb{C}^n$ to $\mathbb{C}^n$ there is exactly one connected codimension-1 $\mathcal{A}_e$-orbit (or, in the presence of moduli in codimension 1, one connected modular stratum) for each $\mathcal{K}$-orbit of type $A_{\mathbf{k}(s,m)}$, for $2 \leq m \leq n+1$. The $\mathcal{A}_e$-orbits of $\mathcal{K}$-type $A_{\mathbf{k}(s,m)}$, where $m \geq n+2$, have $\mathcal{A}_e$-codimension greater than one (and, in the presence of moduli, the modular stratum also has codimension greater than one).*

Proof. For $2 \leq m \leq n$, the unstable s-germs in $A_{\mathbf{k}(s,m)}$ are recognized by the map $(G_{\mathbf{k}(s,m)}, \bar{G}_{\mathbf{k}(s,m)})$ defined in Section 2.3. Recall that $G^{-1}_{\mathbf{k}(s,m)}(0)$ is the closure of the $A_{\mathbf{k}(s,m)}$ stratum in the source of the corank-1 map f, and that $(G_{\mathbf{k}(s,m)}, \bar{G}_{\mathbf{k}(s,m)})^{-1}(0)$ consists of non-transverse $A_{\mathbf{k}(s,m)}$-points that do not belong to the closure of $A_{\mathbf{k}(s,m+1)}$. Also recall that $\bar{G}^{-1}_{\mathbf{k}(s,m)}(0)$ is the projection of the set $\{\partial\rho/\partial x_j = 0\}_{1\leq j<n} \subset \mathbb{CP}^{m-2} \times \mathbb{C}^{n+s-1}$. The maps $(G_{\mathbf{k}(s,m)}, \bar{G}_{\mathbf{k}(s,m)})$ and $(G_{\mathbf{k}(s,m)}, \partial\rho/\partial x_1, \ldots, \partial\rho/\partial x_{n-1})$ factor:

$$\mathbb{C}^{n+s-1} \xrightarrow{j^k f} J^k(n+s-1, n) \xrightarrow{\phi_1} \mathbb{C}^{n+s}$$

and

$$\mathbb{CP}^{m-2} \times \mathbb{C}^{n+s-1} \xrightarrow{(\mathrm{id}, j^k f)} \mathbb{CP}^{m-2} \times J^k(n+s-1, n) \xrightarrow{\phi_2} \mathbb{C}^{n+s+m-2}$$

(here $k = \sum_{j=1}^{s}(k_j+1)$). Set $\Lambda := \phi_1^{-1}(0)$ and $\tilde{\Lambda} := \phi_2^{-1}(0)$. The definition of $G_{\mathbf{k}(s,m)}$ and ρ in Section 2.3 implies that $\tilde{\Lambda} \subset \mathbb{CP}^{m-2} \times J^k(n+s-1, n)$ is a smooth connected submanifold of codimension $n+s+m-2$ (in fact, it is the graph of a map). Furthermore, the projection Λ of $\tilde{\Lambda}$ onto $J^k(n+s-1, n)$ is a connected variety of codimension $n+s$, but for $n \geq 3$ Λ fails to be smooth. Deleting certain closed strata S, corresponding to s-germs of $\mathcal{A}_e$-codimension greater than one, yields a connected submanifold $\Lambda \setminus S \subset J^k(n+s-1, n)$ of codimension $n+s$ that corresponds to a single $\mathcal{A}_e$-orbit of codimension one (or, in the presence of moduli in codimension 1, to the modular stratum).

The remaining cases, where $m > n$ are straightforward. The closure of the $A_{\mathbf{k}(s,m)}$ stratum is a connected smooth submanifold of $J^k(n+s-1, n)$ of codimension $m+s-1$, but the $\mathcal{K}$-codimension of the s-germ $A_{\mathbf{k}(s,m)}$ is m (the $s-1$ constant conditions do not contribute to the $\mathcal{K}$-codimension). The $\mathcal{A}_e$-codimension of the open $\mathcal{A}$-orbit (or the modular stratum) in $A_{\mathbf{k}(s,m)}$ is

$m - n$ (by Proposition 1), hence 1 for $m = n + 1$ and ≥ 2 for $m \geq n + 2$. □

Using the above proposition, we can count the $\mathcal{A}_e$-classes of equidimensional codimension-1 s-germs. But first we need some definitions. Let $p(i)$ denote the number of partitions of i. Let (u, f_u) be a mini-versal unfolding of a codimension-1 s-germ $f_0 : \mathbb{C}^m \to \mathbb{C}^m$, then the s-germ $g := (u, f_{u^2}) : \mathbb{C}^{m+1} \to \mathbb{C}^{m+1}$ is called a (quadratic) augmentation of f_0. We need the following fact about such augmentations (see [ACM]): augmentations of $\mathcal{A}_e$-equivalent s-germs of codimension 1 are $\mathcal{A}_e$-equivalent and also have codimension 1. An s-germ that is not (equivalent to) an augmentation is said to be primitive. Notice that all codimension-1 s-germs from $\mathbb{C}^n$ to $\mathbb{C}^n$ are simple if all the primitive codimension-1 s-germs from $\mathbb{C}^m \to \mathbb{C}^m$, $1 \leq m \leq n$, are simple.

Proposition 6. *The number of corank-1 $\mathcal{A}_e$-classes of s-germs from $\mathbb{C}^n$ to $\mathbb{C}^n$ is equal to $\sum_{i=1}^{n} p(i + 1)$. (In the presence of moduli, we count the modular strata of codimension 1 as a single $\mathcal{A}_e$-class.)*

Proof. By induction on n. Each $\mathcal{A}_e$-codimension-1 s-germ $f : \mathbb{C}^n \to \mathbb{C}^n$ is either the $(n - i)$th augmentation of exactly one $\mathcal{A}_e$-codimension-1 s-germ $\tilde{f} : \mathbb{C}^{n-i} \to \mathbb{C}^{n-i}$, $1 \leq i < n$, or is primitive. The number of $\mathcal{A}_e$-classes of s-germs from $\mathbb{C}^n$ to $\mathbb{C}^n$ of codimension 1 is therefore equal to the number of primitive codimension-1 s-germs from $\mathbb{C}^m \to \mathbb{C}^m$, $1 \leq m \leq n$.

We claim that the open $\mathcal{A}$-orbits (or, in the presence of moduli in $\mathcal{A}_e$-codimension 1, the modular strata) within the $p(n+1)$ $\mathcal{K}$-classes $A_{\mathbf{k}(s,n+1)}$ correspond to primitive s-germs $f : \mathbb{C}^n \to \mathbb{C}^n$ of $\mathcal{A}_e$-codimension 1 (or, if the modality is r, of $\mathcal{A}_e$-codimension $r + 1$). Notice that any $\tilde{f} : \mathbb{C}^{n-i} \to \mathbb{C}^{n-i}$ in $A_{\mathbf{k}(s,n+1)}$ has $\mathcal{A}_e$-codimension greater than one (by Proposition 5), hence f cannot be the augmentation of such a $\tilde{f}$.

Finally, there are no primitive s-germs from $\mathbb{C}^n \to \mathbb{C}^n$ of codimension 1 of $\mathcal{K}$-type $A_{\mathbf{k}(s,m)}$, for $m \leq n$. The $(n - m + 1)$st augmentation of a representative $f : \mathbb{C}^{m-1} \to \mathbb{C}^{m-1}$ of the open $\mathcal{A}$-orbit in the $\mathcal{K}$-orbit $A_{\mathbf{k}(s,m)}$ has $\mathcal{A}_e$-codimension 1 and is, by Proposition 5, the only s-germ from $\mathbb{C}^n$ to $\mathbb{C}^n$ in $A_{\mathbf{k}(s,m)}$ of $\mathcal{A}_e$-codimension 1. □

Remark 5. (i) The arguments above show that if the open stratum in $A_{\mathbf{k}(s,n+1)}$ consists of simple $\mathcal{A}_e$-codimension-1 s-germs then all equidimensional s-germs of corank 1 and $\mathcal{A}_e$-codimension 1 are simple. We conjecture that all these codimension-1 s-germs are indeed simple.

(ii) The normal forms in [Go] show that this the case for mono-germs (where $s = 1$). Hence there are n codimension-1 $\mathcal{A}_e$-classes of mono-germs from $\mathbb{C}^n$ to $\mathbb{C}^n$ of corank-1, which are all simple and do not consist of modular strata.

4. The complexity of the complement of $\mathcal{B}$

Throughout this section, the dimension n will be an arbitrary but fixed constant. The upper bound for the number of connected regions in the complement of a bifurcation set $\mathcal{B}$ will be based on the following estimate.

Lemma 1. *Let $\mathcal{B}$ be a semi-algebraic bifurcation set in $P = \mathbb{R}^n$ or $\mathbb{RP}^n$, and let $\hat{\mathcal{B}}$ be a closed real algebraic subset of P containing $\mathcal{B}$. Then $P \setminus \mathcal{B}$ has at most $O((\deg \hat{\mathcal{B}})^n)$ connected components.*

Proof. The bifurcation set $\mathcal{B}$ is a semi-algebraic subset of the closed real algebraic set $\hat{\mathcal{B}} \subset P$, and the number of connected regions cut out by $\mathcal{B}$ is less than or equal to the number of regions cut out by $\hat{\mathcal{B}}$. The number of connected regions of $P \setminus \hat{\mathcal{B}}$ is a linear function of the $(n-1)$st Betti number of $\hat{\mathcal{B}}$: taking a 1-point compactification of $\mathbb{R}^n$ or, in case of $P = \mathbb{RP}^n$, identifying anti-podal points we can consider $\hat{\mathcal{B}}$ as a subset of the n-sphere and obtain the isomorphism of reduced (co-)homology groups $\tilde{H}_0(S^n \setminus \hat{\mathcal{B}}) \cong \tilde{H}^{n-1}(\hat{\mathcal{B}})$ (Alexander duality). The desired upper bound then follows at once from a result of Milnor [Mi], which says that the sum of the Betti number of $\hat{\mathcal{B}}$ is of order $(\deg \hat{\mathcal{B}})^n$. □

Next, we derive a bound for the degree of the bifurcation set of the family of all projections of an algebraic hypersurface (for real hypersurfaces, the bound applies to the complexification of $\mathcal{B}$). Recall the following result of Mather [Ma71] (which is an algebraic-geometric analogue of a well-known result of Mather in the smooth case [Ma73]).

Theorem 1. *Let $M \subset \mathbb{C}^N$ (N sufficiently large) be a regular algebraic surface of dimension n, and let $\pi_\omega(M)$ denote the projection of M onto some p-dimensional linear subspace of $\mathbb{C}^N$ from centre ω. If (n, p) is a nice pair of dimensions, then the set $\hat{\mathcal{B}} := \{\omega \in \mathbb{C}^N : \pi_\omega(M) \text{ is unstable}\}$ has positive codimension for any M.*

Remark 6. The restriction to the nice dimensions (n, p) in the theorem above is necessary, because outside the nice dimensions the stable maps fail to be dense. But projections of hypersurfaces into hyperplanes are equidimensional corank-1 maps, and the stable corank-1 maps are dense for all (n, n). Hence no restrictions on n are required in the results below.

We have the following degree bound for bifurcation sets $\mathcal{B}$ of families of projections of hypersurfaces in $(n+1)$-space into hyperplanes.

Proposition 7. *Let $M \subset \mathbb{F}^{n+1}$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, be a regular algebraic hypersurface of degree d, and consider the family of all central or parallel projections of M into n-planes from centres or directions $\omega \in \mathcal{V}$, where*

$\mathcal{V} = \mathbb{F}^{n+1}$ or $\mathbb{FP}^n$. Let $\hat{\mathcal{B}}$ be either the bifurcation set $\mathcal{B}$ (for $\mathbb{F} = \mathbb{C}$) or the smallest real algebraic set containing the semi-algebraic set $\mathcal{B}$ (for $\mathbb{F} = \mathbb{R}$). Then $\hat{\mathcal{B}}$ is a closed subset of $\mathcal{V}$ of degree at most $O(d^{2(n+1)})$.

Proof. Note that, by Theorem 1 (and Remark 6 following it), $\hat{\mathcal{B}}$ is closed in $\mathcal{V}$. Consider the following diagram (recall the discussion in Section 2.3):

$$\begin{array}{l}\tilde{\mathcal{B}}(s) \subset \mathcal{V} \times \mathbb{F}^{n+s} \\ \big\downarrow \pi_1 \\ \mathcal{B}(s) \subset \hat{\mathcal{B}}(s) \subset \mathcal{V}\end{array}$$

where π_1 is the projection onto the first factor and where $\mathcal{B}(s) = \hat{\mathcal{B}}(s)$ in the case $\mathbb{F} = \mathbb{C}$. There are two distinct cases, (i) $s = n+1$ and (ii) $s = 1, \dots, n$. In the first case (i) $\tilde{\mathcal{B}}(n+1)$ is the zero-set of the map $Q_{\mathbf{k}(n+1,n+1)} : \mathcal{V} \times \mathbb{F}^{2n+1} \to \mathbb{F}^{2n+2}$. In the second case (ii) $\tilde{\mathcal{B}}(s) = \bigcup_{m=2}^{n} \bigcup \tilde{\mathcal{B}}_{\mathbf{k}(s,m)}$, where the second union ranges over $O(1)$ partitions of $m \le n$ having s summands (notice that n is assumed to be a constant). Hence there are $O(1)$ sets $\tilde{\mathcal{B}}(s)$, $1 \le s \le n$, and each such set has $O(1)$ components $\tilde{\mathcal{B}}_{\mathbf{k}(s,m)}$. And each component $\tilde{\mathcal{B}}_{\mathbf{k}(s,m)}$ is the zero-set of some map $(Q_{\mathbf{k}(s,m)}, \bar{Q}_{\mathbf{k}(s,m)}) : \mathcal{V} \times \mathbb{F}^{n+s} \to \mathbb{F}^{n+s+1}$.

Now if d is the degree of M then each component function of $Q_{\mathbf{k}(n+1,n+1)}$ and of $Q_{\mathbf{k}(s,m)}$ has degree $O(d)$, and the degree of the component functions of $\bar{Q}_{\mathbf{k}(s,m)}$ is also $O(d)$ (see Section 2.3 for the definition of $\bar{Q}_{\mathbf{k}(s,m)}$ and recall that n is some given constant). Hence, the degree of each $\tilde{\mathcal{B}}(s)$ is bounded above by $O(d^{n+s+1})$ in both cases (i) and (ii).

Let π_2 denote the projection onto the second factor (i.e. onto $\mathbb{F}^{n+s}$). A "generic" line $L \subset \mathcal{V}$ will cut $\hat{\mathcal{B}}(s)$ in $\delta = \deg \hat{\mathcal{B}}(s)$ points. Let $H \subset \mathbb{F}^{n+s}$ be a "generic" linear subspace whose codimension is equal to the dimension of $\tilde{\mathcal{B}}(s) \cap \pi_1^{-1}(L)$. By Bezout's theorem, the set $\tilde{\mathcal{B}}(s) \cap \pi_1^{-1}(L) \cap \pi_2^{-1}(H)$ consists of at most $O(d^{n+s+1})$ isolated points whose projections onto $\mathcal{V}$ are the δ points of $\hat{\mathcal{B}}(s) \cap L$. Hence $O(d^{n+s+1})$ is an upper bound for the degree of $\hat{\mathcal{B}}(s)$. Finally, note that $s \le n+1$ (by Corollary 1). The degree of $\hat{\mathcal{B}} = \bigcup_{s \le n} \hat{\mathcal{B}}(s)$ is therefore at most $O(d^{2(n+1)})$. □

Remark 7. For regular algebraic surfaces M in 3-space (where $n = 2$) the above bound for the degree of $\hat{\mathcal{B}}$ is asymptotically sharp. This follows from a formula by Petitjean for the degree of the subvariety $\hat{\mathcal{B}}(3) = \hat{\mathcal{B}}_{(1,1,1)}$ of $\hat{\mathcal{B}}$ corresponding to triple fold crossings, which is given by $\frac{1}{3}d(d-3)(d-4)(d-5)(d^2+3d-2)$, see p. 122 of [Pe]. In fact, Petitjean gives formulas for the degrees of all the sets $\hat{\mathcal{B}}_{\mathbf{k}(s,m)}$. The proof of these formulas is based on iterative techniques by Colley for enumerating stationary multiple points [Col] and the recognition conditions for the $\mathcal{A}_e$-codimension-1 singularities for $n = 2$ (i.e. the defining conditions of the sets $\tilde{\mathcal{B}}_{\mathbf{k}(s,m)}$) in [Ri96] in terms of contact between lines and the surface M at a set of points. It would be very interesting to derive a general formula for the degree of the variety $\hat{\mathcal{B}}(n+1) = \hat{\mathcal{B}}_{\mathbf{k}(n+1,n+1)}$ of lines in $\mathbb{F}^{n+1}$ that are tangent to M at $n+1$ points. Notice that $\hat{\mathcal{B}}(n+1)$ is the component of $\hat{\mathcal{B}}$ of maximal degree (for $d = \deg M$ sufficiently large).

The above degree bound for $\hat{\mathcal{B}} \supset \mathcal{B}$, together with the bound in Lemma 1, yields the following

Theorem 2. *Let $M \subset \mathbb{R}^{n+1}$ be a regular algebraic hypersurface of degree d. Then the number of connected regions of $\mathcal{V} \setminus \mathcal{B}$ – and hence the number of distinct stable projections of M – are bounded above by $O(d^{2n(n+1)})$ (for parallel projection) or $O(d^{2(n+1)^2})$ (for central projection).*

Remark 8. The same bounds are valid for certain singular surfaces in 3-space: namely for surfaces with transverse double curves and isolated triple-points [Ri96] and for surfaces with additional cross-caps [Ri98].

5. A final remark

After the present paper had been submitted for publication, a classification by Damon of discriminants of maps of $\mathcal{K}_{V,e}$-codimension 1 has appeared in print (see Sec. 4 of [Da]). This classification and the relation between the $\mathcal{A}_e$-classification of multi-germs and the $\mathcal{K}_{V,e}$-classification of their discriminants (see Sec. 6.2 of [Da]) imply that all corank-1 equidimensional multi-germs of $\mathcal{A}_e$-codimension 1 are simple, which confirms the conjecture in Remark 5 (i). In particular, we now know that all the codimension-1 $\mathcal{A}_e$-orbits in Propositions 5 and 6 are simple.

Acknowledgements. I am grateful for discussions with Roberta Atique, Sylvain Petitjean and Maria Ruas. I also thank the research foundation FAPESP for financial support.

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