Martin–Luther–Universität
Halle–Wittenberg
Institut für Mathematik

Jahn-Graef-Younes type algorithms for discrete vector optimization based on cone-monotone sorting functions

Christian Günther and Nicolae Popovici

Report No. 06 (2017)

Editors:
Professors of the Institute for Mathematics, Martin Luther University Halle-Wittenberg.

Electronic version: see http://www.mathematik.uni-halle.de/institut/reports/

Jahn-Graef-Younes type algorithms for discrete vector optimization based on cone-monotone sorting functions

Christian Günther and Nicolae Popovici

Report No. 06 (2017)

Christian Günther
Martin-Luther-Universität Halle-Wittenberg
Naturwissenschaftliche Fakultät II
Institut für Mathematik
Theodor-Lieser-Str. 5
D-06120 Halle/Saale, Germany

Nicolae Popovici
Babeş-Bolyai University
Faculty of Mathematics and Computer Science
Department of Mathematics
1, M. Kogalniceanu St.
400084 Cluj-Napoca, Romania

Email: christian.guenther@mathematik.uni-halle.de
popovici@math.ubbcluj.ro

Jahn-Graef-Younes type algorithms for discrete vector optimization based on cone-monotone sorting functions

Christian Günther[a] and Nicolae Popovici[b]

[a]Martin Luther University Halle-Wittenberg, Faculty of Natural Sciences II, Institute for Mathematics, 06099 Halle (Saale), Germany;
[b]Babeş-Bolyai University, Faculty of Mathematics and Computer Science, 400084 Cluj-Napoca, Romania

ARTICLE HISTORY
Compiled June 16, 2017

ABSTRACT

In this paper we present new Jahn-Graef-Younes type algorithms for solving discrete vector optimization problems. In order to determine all minimal elements of a finite set with respect to an ordering cone, the original approach proposed by Jahn in 2006 (known as the Jahn-Graef-Younes method) consists of a forward iteration (Graef-Younes method), followed by a backward iteration. Our methods involve additional sorting procedures based on scalar cone-monotone functions. In particular, we analyze the case where the ordering cone is polyhedral. Computational results, obtained in MATLAB, allow us to compare our new algorithms with the original Jahn-Graef-Younes method.

KEYWORDS
Multiobjective optimization; Discrete optimization; Minimal elements; Domination property; Jahn-Graef-Younes method; Cone-monotone functions

Mathematics Subject Classification: 90C29; 90C26; 90C56; 68T20

1. Introduction

In this paper we develop new methods for computing all minimal elements of a finite set of points in $\mathbb{R}^n$ with respect to a pointed convex cone. In practice, such a set consists of many points, so for complexity reasons, it does not make sense to use only the definition of minimality (cf. Jahn [1, Sec. 12.4]).

A reduction approach that eliminates some of the non-minimal elements is given by the Graef-Younes method (see Jahn [1, Alg. 12.17]). This method was originally proposed by Younes [2] and is based on an algorithmic conception by Graef (see Jahn [1, p. 349]). As mentioned by Jahn [1, Sec. 12.4], the Graef-Younes method is a self learning method which becomes better and better step by step, leading to a drastic reduction of the initial set in many concrete instances.

In order to determine exactly the set of all minimal elements, the original algorithmic

---

Dedicated to Professor Johannes Jahn in honor of his 65th birthday

Corresponding author: Christian Günther. Email: Christian.Guenther@mathematik.uni-halle.de

approach by Jahn [1, Alg. 12.20] consists of a forward iteration (Graef-Younes method) in a first phase, followed by a backward iteration in a second one. This method is known in the literature as the "Graef-Younes method with backward iteration" or the "Jahn-Graef-Younes method". Actually, as far as we know, it has been proposed by Jahn [3] and also considered by Jahn and Rathje [4].

In recent years, Eichfelder generalized the Jahn-Graef-Younes method to partially ordered real linear spaces in [5]) and adapted the method to variable ordering structures [6,7]). Furthermore, Köbis et al. [8] presented extensions of the Jahn-Graef-Younes method to set optimization.

In this paper we present new Jahn-Graef-Younes type methods for computing all minimal elements of a finite set with respect to a pointed convex cone, based on certain sorting procedures via cone-monotone scalar functions. Although many of our results hold within partially ordered real linear spaces, we restrict our study to the particular framework of the Euclidean space $\mathbb{R}^n$, due to practical applications.

The paper is organized as follows. In Section 2 we recall some notions and results of convex analysis, mainly concerning polyhedral cones, that are used in the sequel.

In Section 3 we highlight the role of cone-monotone scalar functions in the context of vector optimization. In particular we establish new results that are relevant for the Jahn-Graef-Younes type methods (especially Lemma 3.5 and Theorem 3.8).

After a brief presentation of the classical Graef-Younes method (Algorithm 2) and of the Jahn-Graef-Younes method (Algorithm 3), in Section 4 we introduce our new methods, that are based on some special properties of cone-monotone sorting functions (given by Theorem 4.1). More precisely, our Algorithm 4 is obtained from Algorithm 2 by considering a pre-sorting procedure, while Algorithm 5 is obtained from Algorithm 3 by considering an intermediate sorting procedure, after the forward iteration. Both methods produce the whole set of minimal points of a finite set with respect to a nontrivial ordering cone.

In order to derive implementable versions of our new methods, in Section 5 we identify appropriate sorting functions, possessing two important features: their values are computable efficiently and they allow to decide whether two points are comparable with respect to the ordering cone. Therefore, we present two special classes of linear and nonlinear cone-monotone functions, that are currently used in scalarization of vector optimization problems.

In Section 6 we derive the implementable versions (Algorithms 6 and 7) of our new algorithms introduced in Section 4, in the particular framework where the ordering cone is polyhedral and the strongly cone-monotone sorting functions are linear.

Among many possible applications, our new Jahn-Graef-Younes type algorithms can be used for approximating the sets of minimal outcomes of certain continuous vector optimization problems, via a discretization approach. In Section 7 we apply our algorithms to approximate the set of minimal outcomes for a particular continuous bi-objective test problem (known in the literature as being very difficult to solve), via the "Multiobjective search algorithm with subdivision technique" (MOSAST), proposed by Jahn [3]. A detailed comparative analysis of our algorithms and other classical methods is provided, based on computational experiments in MATLAB.

Finally, in Section 8 we point out possible directions for further research.

2. Preliminaries

Throughout we denote by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ the sets of real numbers, nonnegative real numbers and positive integers, respectively. We endow the n-dimensional Euclidean space $\mathbb{R}^n$ ($n \in \mathbb{N}$) with the usual inner product $\langle \cdot, \cdot \rangle$.

Given any set $S \subseteq \mathbb{R}^n$, we denote by $\operatorname{int} S$, $\operatorname{cl} S$, $\operatorname{bd} S$ and $\operatorname{conv} S$ the interior, the closure, the boundary and the convex hull of S, respectively; the abbreviation $|S| := \operatorname{card} S$ refers to the cardinality of S.

In what follows we recall some basic notions and results of convex analysis that will be used in the sequel (for a detailed study of convex sets/cones we refer the reader to the books by Rockafellar [9], Göpfert et al. [10], and Aliprantis and Tourky [11]).

We say that $S \subseteq \mathbb{R}^n$ is a polytope if it is the convex hull of a nonempty finite set, i.e., there is $P \subseteq \mathbb{R}^n$ with $|P| \in \mathbb{N}$, such that

$$S = \operatorname{conv} P.$$

A set $C \subseteq \mathbb{R}^n$ is said to be a cone, if $0 \in C = \mathbb{R}_+ \cdot C$. A cone C is called: nontrivial, if $\{0\} \neq C \neq \mathbb{R}^n$; pointed, if $C \cap (-C) = \{0\}$; convex, if $C = \operatorname{conv} C$, i.e., $C + C = C$; solid, if $\operatorname{int} C \neq \emptyset$; closed, if $C = \operatorname{cl} C$. Note that if C is a convex cone, then

$$C + \operatorname{int} C = \operatorname{int} C.$$

As usual, for any nonempty set $S \subseteq \mathbb{R}^n$, we define by

$$\operatorname{cone} S := \mathbb{R}_+ \cdot S,$$
$$S^+ := \{x \in \mathbb{R}^n \mid \forall\, y \in S : \ \langle x, y \rangle \geq 0\},$$

the cone generated by S and the polar cone of S, respectively. Note that S^+ is always convex, while $\operatorname{cone} S$ is convex whenever S is convex.

If $C \subseteq \mathbb{R}^n$ is a closed convex cone, then (by the Bipolar Theorem) we have

$$(C^+)^+ = C.$$

In addition, if the closed convex cone C is pointed, then C^+ is solid, i.e., $\operatorname{int} C^+ \neq \emptyset$.

We say that $C \subseteq \mathbb{R}^n$ is a polyhedral cone if it is the polar cone of a nonempty finite set of non-zero vectors, i.e., there is $U \subseteq \mathbb{R}^n \setminus \{0\}$ with $|U| \in \mathbb{N}$, such that

$$C = U^+. \tag{1}$$

A polyhedral cone of type (1) satisfies the following properties:

- C is closed, convex, and $C \neq \mathbb{R}^n$, hence it is nontrivial if and only if

$$0 \notin \operatorname{int} \operatorname{conv} U. \tag{2}$$

- C is pointed if and only if

$$U^\perp := U^+ \cap (-U^+) = \{x \in \mathbb{R}^n \mid \forall\, y \in U : \ \langle x, y \rangle = 0\} = \{0\}. \tag{3}$$

  Of course this requires that $|U| \geq n$.

- the interior of C is given by

$$\operatorname{int} C = \{x \in \mathbb{R}^n \mid \forall\, u \in U :\ \langle u, x\rangle > 0\}. \tag{4}$$

- the polar of C is a polyhedral cone, namely

$$C^+ = (U^+)^+ = \operatorname{cone}\operatorname{conv} U,$$

  whose interior is

$$\operatorname{int} C^+ = \{\lambda \in \mathbb{R}^n \mid \forall\, x \in C \setminus \{0\} :\ \langle \lambda, x\rangle > 0\}. \tag{5}$$

A set $X \subseteq \mathbb{R}^n$ is said to be polyhedral if it can be written as the sum of a polytope and a polyhedral cone, i.e.,

$$X = Y + C, \tag{6}$$

where $Y = \operatorname{conv} P$ for some nonempty finite set $P \subseteq \mathbb{R}^n$ and C is given by (1).

In the sequel it will be convenient to denote, for any $k \in \mathbb{N}$, the index set

$$I_k := \{1, \ldots, k\}.$$

As usual in vector optimization, the concept of minimality can be defined with respect to an arbitrary pointed convex cone. In particular, the standard ordering cone

$$\mathbb{R}^n_+ := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, \forall\, i \in I_n :\ x_i \geq 0\}$$

is of special interest for concrete multiobjective optimization problems. Note that it is a solid polyhedral pointed cone.

3. The role of cone-monotone scalar functions in vector optimization

Throughout this paper we assume that $K \subseteq \mathbb{R}^n$ is a nontrivial pointed convex cone. It induces an ordering (i.e., a reflexive, transitive and antisymmetric binary relation) defined for any $x, y \in \mathbb{R}^n$ by

$$x \leqq_K y :\iff\ y \in x + K.$$

Its irreflexive part is defined for any $x, y \in \mathbb{R}^n$ by

$$x \lneqq_K y :\iff\ y \in x + K \setminus \{0\}.$$

Definition 3.1. Let A be a nonempty subset of $\mathbb{R}^n$. The elements of the set

$$\operatorname{MIN}(A \mid K) := \{x^0 \in A \mid \nexists\, x \in A :\ x \lneqq_K x^0\}$$

are called minimal points of A with respect to K.

Remark 1. By Definition 3.1, we actually have

$$\begin{aligned}\mathrm{MIN}(A \mid K) &= \{x^0 \in A \mid x^0 \notin (A \setminus \{x^0\}) + K\} \\ &= \{x^0 \in A \mid (x^0 - K) \cap A = \{x^0\}\}.\end{aligned}$$

Remark 2. It is easy to check that for any set $A \subseteq \mathbb{R}^n$ the following relation holds:

$$\mathrm{MIN}(A \mid K) = \mathrm{MIN}(A + K \mid K).$$

In general, the numerical methods of vector optimization are aimed to compute the entire set $\mathrm{MIN}(A \mid K)$ or to produce an inner/outer approximation of it. In what follows we highlight the role of cone-monotone functions in developing such methods. The next definition recalls some concepts introduced by Jahn [1, Def. 5.1]) by using a slightly different terminology.

Definition 3.2. A function $\varphi : D \to \mathbb{R}$, defined on a nonempty set $D \subseteq \mathbb{R}^n$, is called:

- K-increasing, if for any $x, y \in D$,

$$x \leqq_K y \implies \varphi(x) \leq \varphi(y);$$

- strongly K-increasing, if for any $x, y \in D$,

$$x \lneqq_K y \implies \varphi(x) < \varphi(y).$$

- (strongly) K-decreasing if $-\varphi$ is (strongly) K-increasing.

Obviously, strongly K-increasing/decreasing functions are K-increasing/decreasing.

The next result (see, e.g., Jahn [1, Lem. 5.14] for a more general framework) shows that cone-monotone functions can be used for developing scalarization methods that provide an inner approximation of $\mathrm{MIN}(A \mid K)$, that is, a set

$$S \subseteq \mathrm{MIN}(A \mid K).$$

Proposition 3.3. *Let $A \subseteq D \subseteq \mathbb{R}^n$ be nonempty sets and let $\varphi : D \to \mathbb{R}$ be a function. Denote $S_\varphi := \mathrm{argmin}_{x \in A}\ \varphi(x)$. If either φ is strongly K-increasing or φ is K-increasing and $|S_\varphi| = 1$, then*

$$S_\varphi \subseteq \mathrm{MIN}(A \mid K).$$

Other numerical methods of vector optimization are rather conceived to provide an outer approximation of $\mathrm{MIN}(A \mid K)$, that is, a set $B \subseteq \mathbb{R}^n$ such that

$$\mathrm{MIN}(A \mid K) \subseteq B \subseteq A.$$

In other words, an outer approximation B of $\mathrm{MIN}(A \mid K)$ is obtained from A by removing a part of its non-minimal points.

The following results are relevant for this approach. In preparation we recall a basic concept of vector optimization (see, e.g. Göpfert et al. [10] and references therein).

Definition 3.4. We say that a set $A \subseteq \mathbb{R}^n$ satisfies the domination property with respect to K if

$$A \subseteq \mathrm{MIN}(A \mid K) + K.$$

Remark 3. A set $A \subseteq \mathbb{R}^n$ satisfies the domination property w.r.t. K if and only if

$$A + K = \mathrm{MIN}(A \mid K) + K.$$

Remark 4. Every finite set $A \subseteq \mathbb{R}^n$ satisfies the domination property w.r.t. K.

Lemma 3.5. *Let $A \subseteq \mathbb{R}^n$ be a set that satisfies the domination property w.r.t. K. Then, for any set $B \subseteq A$ the following assertions are equivalent:*

1°. $\mathrm{MIN}(A \mid K) \subseteq B$.
2°. $\mathrm{MIN}(A \mid K) = \mathrm{MIN}(B \mid K)$.

Proof. Obviously (even in the absence of the domination property) 2° implies 1°, since $\mathrm{MIN}(B \mid K) \subseteq B$.

Conversely, assume that 1° holds. Then, for any $x \in \mathrm{MIN}(A \mid K)$ we have $x \in B$ and $(x - K) \cap A = \{x\}$. Since $B \subseteq A$, we infer that $(x - K) \cap B = \{x\}$, i.e., $x \in \mathrm{MIN}(B \mid K)$. Thus inclusion $\mathrm{MIN}(A \mid K) \subseteq \mathrm{MIN}(B \mid K)$ holds. In order to prove the converse inclusion, let $x' \in \mathrm{MIN}(B \mid K)$. Then we have $x' \in B \subseteq A$ and also $x' \in \mathrm{MIN}(B + K \mid K) = \mathrm{MIN}(B \mid K)$. By the domination property of A and 2°, it follows that $x' \in (x' - K) \cap A \subseteq (x' - K) \cap (\mathrm{MIN}(A \mid K) + K) \subseteq (x - K) \cap (B + K) = \{x'\}$. Therefore we have $(x' - K) \cap A = \{x'\}$, i.e., $x' \in \mathrm{MIN}(A \mid K)$. Thus the inclusion $\mathrm{MIN}(B \mid K) \subseteq \mathrm{MIN}(A \mid K)$ in 2° also holds. □

Remark 5. The domination property assumption imposed on A in Lemma 3.5 is essential for the implication 1° ⇒ 2°, as shown by the following example.

Example 3.6. Let the Euclidean plane $\mathbb{R}^2$ be endowed with the standard ordering cone $K := \mathbb{R}^2_+$. Consider the sets

$$A := \{(0, 1)\} \cup \big(]0, 1] \times \{0\}\big) \quad \text{and} \quad B := \{(0, 1); (1, 0)\}.$$

It is easily seen that $B \subseteq A$ and

$$\mathrm{MIN}(A \mid K) = \{(0, 1)\} \subseteq B = \mathrm{MIN}(B \mid K) \neq \mathrm{MIN}(A \mid K).$$

Next we show that strongly cone-monotone functions can be used to generate the whole set $\mathrm{MIN}(A \mid K)$.

Lemma 3.7. *Let $\varphi : D \to \mathbb{R}$ be a strongly K-increasing function, defined on a nonempty set $D \subseteq \mathbb{R}^n$. For any set $B \subseteq D$ the following assertions are equivalent:*

1°. $B = \mathrm{MIN}(B \mid K)$.
2°. *For any points $b, b' \in B$ with $\varphi(b) < \varphi(b')$ we have $b' \notin b + K$ (i.e., $b \nleqq_K b'$).*
3°. *For any points $b, b' \in B$ with $b \leqq_K b'$ we have $\varphi(b) \geq \varphi(b')$, i.e., the restriction of φ to B is K-decreasing.*

Proof. The equivalence 2° $\iff$ 3° is obvious.

Assume that 1° holds and let $b, b' \in B$ be such that $\varphi(b) < \varphi(b')$. Supposing by the contrary that $b' \in b + K$ we would have $b \in (b' - K) \cap B$. Since $b' \in B = \mathrm{MIN}(B \mid K)$, we infer $b = b'$ hence $\varphi(b) = \varphi(b')$, a contradiction. Thus 1° implies 2°.

Conversely, assume that 2° holds. In order to prove 1°, we just have to show that $B \subseteq \mathrm{MIN}(B \mid K)$. Suppose by the contrary that there is $b' \in B \setminus \mathrm{MIN}(B \mid K)$. Then it would exist $b \in B$ such that $b' \in b + K \setminus \{0\}$. Since φ is strongly K-increasing we infer $\varphi(b) < \varphi(b')$. This contradicts 2°. □

Theorem 3.8. *Let $A \subseteq \mathbb{R}^n$ be a nonempty set satisfying the domination property w.r.t. K and let $B \subseteq A$ be such that $\mathrm{MIN}(A \mid K) \subseteq B$. If $\varphi : D \to \mathbb{R}$ is a strongly K-increasing function, with $A \subseteq D$, then the following assertions are equivalent:*

1°. $B = \mathrm{MIN}(A \mid K)$.
2°. *For any points $b, b' \in B$ with $\varphi(b) < \varphi(b')$ we have $b' \notin b + K$ (i.e., $b \nleqq_K b'$).*
3°. *For any points $b, b' \in B$ with $b \leqq_K b'$ we have $\varphi(b) \geq \varphi(b')$, i.e., the restriction of φ to B is K-decreasing.*

Proof. Follows from Lemmas 3.5 and 3.7. □

Remark 4.b) suggests to use Theorem 3.8 in order to develop new numerical methods for solving discrete vector optimization problems.

4. Jahn-Graef-Younes type algorithms for discrete vector optimization

As stated in the previous section, in what follows $K \subseteq \mathbb{R}^n$ is a nontrivial pointed convex cone. Moreover, since in the sequel we focus on discrete vector optimization problems, throughout this section the notation A will represent a nonempty finite set

$$A := \{a^1, \dots, a^p\} \subseteq \mathbb{R}^n,$$

where$|A| = p \in \mathbb{N}$, i.e., the points $a^1, \dots, a^p$ are pairwise distinct.

In this section we will develop new numerical methods for computing the set $\mathrm{MIN}(A \mid K)$ of all minimal points of A w.r.t. K. In preparation we recall three well-known methods, among which the simplest one (yet naive) is Algorithm 1, which only requires Definition 3.1.

Algorithm 1: NAIVE METHOD, BY PAIRWISE COMPARISON

```
Input: The set A := {a^1, ..., a^p}.
T ← ∅;
for j ← 1 to p do
    if a^j ∉ (A \ {a^j}) + K then
        T ← T ∪ {a^j};
    end
end
Output: The set T (representing the set of all minimal elements of A w.r.t. K).
```

Remark 6. The computational complexity of Algorithm 1 is $O(p^2)$. However, in practice the cardinality of the set A is very high. Hence, one needs methods for reducing the number of necessary pairwise comparisons of the given points with respect to the ordering induced by K.

The next Algorithm 2 is known in the literature as the Graef-Younes method (see Younes [2] and Jahn [1, Sec. 12.4]).

Algorithm 2: GRAEF-YOUNES METHOD

Input: The set $A := \{a^1, \ldots, a^p\}$.
$B \leftarrow \{a^1\}$;
for $j \leftarrow 2$ **to** p **do**
  if $a^j \notin B + K$ **then**
    $B \leftarrow B \cup \{a^j\}$;
  end
end
Output: The set B (that satisfies $\text{MIN}(B \mid K) = \text{MIN}(A \mid K) \subseteq B \subseteq A$).

Remark 7. Since A is finite, the domination property holds in view of Remark 4.b). Therefore, according to Lemma 3.5, the Graef-Younes method (Algorithm 2) generates a reduced set $B \subseteq A$ with the property

$$\text{MIN}(B \mid K) = \text{MIN}(A \mid K) \subseteq B. \tag{7}$$

As pointed out by Jahn [1, Ex. 12.19], in some particular instance the Graef-Younes method can reduce a set A containing 5×10^6 points to a set B containing around 3×10^3 points. However, simple examples in $\mathbb{R}^2$ show that the set B generated by the Graef-Younes method may be very large (sometimes $B = A$), hence the computation of $\text{MIN}(B \mid K)$ is not easier than the computation of $\text{MIN}(A \mid K)$. In the worst-case scenario when $B = A$ we need to perform at most

$$1 + 2 + \ldots + (p - 1) = \frac{(|A| - 1) \cdot |A|}{2}$$

pairwise comparisons of points with respect to the ordering induced by K.

A very interesting approach to overcome this drawback has been proposed by Jahn [3] (see also Jahn and Rathje [4] and Jahn [1]). The so-called Jahn-Graef-Younes method (also known as the Graef-Younes method with backward iteration) is given in Algorithm 3.

Algorithm 3: JAHN-GRAEF-YOUNES METHOD

Input: The set $A := \{a^1, \dots, a^p\}$.

/* Forward iteration */

$i \leftarrow 1$;

$b^i \leftarrow a^i$;

$B \leftarrow \{b^i\}$;

for $j \leftarrow 2$ **to** p **do**

  if $a^j \notin B + K$ **then**

    $i \leftarrow i + 1$;

    $b^i \leftarrow a^j$;

    $B \leftarrow B \cup \{b^i\}$;

  end

end

/* Backward iteration */

$T \leftarrow \{b^i\}$;

for $j \leftarrow 1$ **to** $i - 1$ **do**

  if $b^{i-j} \notin T + K$ **then**

    $T \leftarrow T \cup \{b^{i-j}\}$;

  end

end

Output: The set T (representing the set of all minimal elements of A w.r.t. K).

Simple examples show that the original Jahn-Graef-Younes method (Algorithm 3) is very efficient when the enumeration of the elements of A (that is, $a^1, a^2, \dots$) starts with some minimal elements of A with respect to K. In the worst-case, the computational complexity of Algorithm 3 is $O(p^2)$, depending essentially on the cardinality of the set B, generated after the forward iteration. Therefore a natural idea arises, namely to adapt the Jahn-Graef-Younes method, by considering an appropriate pre-sorting of the elements of A.

Given any function $\varphi : D \to \mathbb{R}$ with $A \subseteq D$ it is easy to find an enumeration $\{a^{j_1}, \dots, a^{j_p}\}$ of the a priori given set $A = \{a^1, \dots, a^p\}$ such that

$$\varphi(a^{j_1}) \leq \varphi(a^{j_2}) \leq \dots \leq \varphi(a^{j_p}). \tag{8}$$

Thus we can always pre-sort the elements of A according to (8) and thereafter apply the algorithms stated above to $\{a^{j_1}, \dots, a^{j_p}\}$ instead of the original enumeration of A. When φ is a strongly K-increasing function we obtain an interesting property, highlighted in the next theorem.

Theorem 4.1. *Let $\varphi : D \to \mathbb{R}$ be a strongly K-increasing function with $A \subseteq D$. If*

$$\varphi(a^1) \leq \varphi(a^2) \leq \dots \leq \varphi(a^p), \tag{9}$$

then Algorithm 2 (the Graef-Younes method) generates the set $B = \mathrm{MIN}(A \mid K)$.

Proof. It is a direct consequence of Theorem 3.8. Indeed, the set B is constructed within Algorithm 2 by eliminating a part of the non-minimal points of A, hence $\mathrm{MIN}(A \mid K) \subseteq B \subseteq A$. Therefore, it suffices to check that the set B satisfies the property 2° of Theorem 3.8. To this end, let any $b, b' \in B$ with $\varphi(b) < \varphi(b')$. Since

$b, b' \in A$, there exist $i, i' \in I_p$ such that $b = a^i$ and $b' = a^{i'}$, hence $\varphi(a^i) < \varphi(a^{i'})$. By (9) it follows that $i < i'$, which shows that during Algorithm 2, the point $b' = a^{i'}$ is added to B after $b = a^i$, hence $b' \notin b + K$. □

Remark 8. If the pre-sorting (8) is given by an arbitrary scalar function φ, then the "reduced" set B generated by applying Algorithm 2 to $\{a^{j_1}, \dots, a^{j_p}\}$ instead of the original enumeration of A may be very large.

In particular, if the function φ is strongly K-decreasing and (9) holds, then the output set coincide with the initial one, i.e., $B = A$ (no reduction occurs). Indeed, by construction of B we have $a^1 \in B$. Moreover, for any $j \in I_p$ with $j \geq 2$ we have $a^j \notin a^i + K$ for all $i \in I_p$ with $i < j$ (otherwise we should have $a^i \lneqq_K a^j$, hence $\varphi(a^i) > \varphi(a^j)$, a contradiction). This means that $a^j \in B$.

Remark 9. If φ is K-increasing and all inequalities in (9) are strict, then Algorithm 2 generates the set $B = \mathrm{MIN}(A \mid K)$, which actually means that $B \subseteq \mathrm{MIN}(B \mid K)$ in view of (7). Indeed, assuming by the contrary that $a^j \in B \setminus \mathrm{MIN}(B \mid K)$ for some $j \in I_p$, we can deduce (by the domination property) the existence of a point $a^i \in \mathrm{MIN}(B \mid K)$ with $i \in I_p$ such that $a^i \lneqq_K a^j$. Since φ is K-increasing and all inequalities in (9) are strict, we infer that $i < j$, contradicting the construction of B (because $a^j \in B$ and $a^j \in a^i + K$ with $a^i \in B$).

Theorem 4.1 suggests to design a new algorithm for solving discrete vector optimization problems, by adding a pre-sorting phase to the original Graef-Younes method, that provides an enumeration $\{a^{j_1}, \dots, a^{j_p}\}$ of A satisfying (8). Therefore we propose the following Algorithm 4, which generates the set $\mathrm{MIN}(A \mid K)$, according to Theorem 4.1 (applied for $\{a^{j_1}, \dots, a^{j_p}\}$ in the role of A).

Algorithm 4: GRAEF-YOUNES TYPE METHOD INVOLVING A PRE-SORTING PHASE

Input: The set $A := \{a^1, \dots, a^p\}$ and the strongly K-increasing function $\varphi : \mathbb{R}^n \to \mathbb{R}$.
/* Phase 1 */
Compute an enumeration $A = \{a^{j_1}, \dots, a^{j_p}\}$ which satisfies (8).
/* Phase 2 */
$B \leftarrow \{a^{j_1}\}$;
for $k \leftarrow 2$ **to** p **do**
  if $a^{j_k} \notin B + K$ **then**
    $B \leftarrow B \cup \{a^{j_k}\}$;
  end
end
$T \leftarrow B$;
Output: The set T (representing the set of all minimal elements of A w.r.t. K).

Remark 10. In Phase 1 of Algorithm 4 we generate an enumeration of the initial set A with respect to the sorting function φ. In the literature there exist effective algorithms for sorting p real numbers with worst-case computational complexity $O(p \cdot \log(p))$.

In Phase 2 of Algorithm 4 the original Graef-Younes method (Algorithm 2) with worst-case computational complexity $O(p^2)$ is applied to the sorted set $\{a^{j_1}, \dots, a^{j_p}\}$ in the role of A in order to generate all minimal elements of A with respect to K.

Notice that Algorithm 4 has a worst-case computational complexity of $O(p^2)$ too.

After Phase 1, we have to perform at most

$$\frac{(|T|-1)\cdot|T|}{2}+(|A|-|T|)\cdot|T|$$

comparisons of points, where $T=\mathrm{MIN}(A\mid K)$.

In what follows we present another new method (Algorithm 5) for solving discrete vector optimization problems obtained by the original Jahn-Graef-Younes method (Algorithm 3) by implementing a sorting procedure of the set B after the forward iteration with respect to a strongly K-increasing function $\varphi:\mathbb{R}^n\to\mathbb{R}$.

Algorithm 5: JAHN-GRAEF-YOUNES TYPE METHOD INVOLVING A SORTING PHASE AFTER THE FORWARD ITERATION

Input: The set $A:=\{a^1,\dots,a^p\}$ and the strongly K-increasing function $\varphi:\mathbb{R}^n\to\mathbb{R}$.

/* Phase 1 */

Apply the forward iteration of Algorithm 3 for the set A as input to get the set $B=\{b^1,\dots,b^i\}$.

/* Phase 2 */

Compute an enumeration $B=\{b^{j_1},\dots,b^{j_i}\}$ such that

$$\varphi(b^{j_1})\geq\varphi(b^{j_2})\geq\cdots\geq\varphi(b^{j_i}).$$

/* Phase 3 */

$T\leftarrow\{b^{j_i}\}$;

for $k\leftarrow 1$ **to** $i-1$ **do**

  if $b^{j_{i-k}}\notin T+K$ **then**

    $T\leftarrow T\cup\{b^{j_{i-k}}\}$;

  end

end

Output: The set T (representing the set of all minimal elements of A w.r.t. K).

Remark 11. In Phase 1 of Algorithm 5 we apply the Graef-Younes method (forward iteration of Algorithm 3) for the a priori given set A.

In Phase 2 the elements of the reduced set B (generated in Phase 1) are sorted by the strongly K-increasing function φ.

Finally, in Phase 3 the backward iteration of Algorithm 3 is applied to the sorted set $\{b^{j_1},\dots,b^{j_i}\}$ in the role of B in order to generate all minimal elements of A with respect to K.

Note that Algorithm 5 has a worst-case computational complexity of $O(p^2)$.

5. Special classes of cone-monotone scalar functions

Throughout this section $K\subseteq\mathbb{R}^n$ will be a nontrivial closed pointed convex cone, hence

$$\operatorname{int}K^+\neq\emptyset\ \text{ and }\ (K^+)^+=K.$$

We are going to study some special classes of K-monotone functions, currently used in scalarization methods for vector optimization. We will use them as sorting functions within Phase 1 of Algorithm 4 and Phase 2 of Algorithm 5. Moreover, we show that these functions can be also used to compare points with respect to the ordering induced by K. Therefore, they allows us to implement the procedures described in Phase 2 of Algorithm 4 and in Phases 1 and 3 of Algorithm 5.

5.1. *Linear cone-monotone sorting functions*

For any $\lambda \in \mathbb{R}^n$ we denote by $\varphi_\lambda : \mathbb{R}^n \to \mathbb{R}$ the linear function defined by

$$\varphi_\lambda(x) := \langle \lambda, x \rangle \quad \text{for all } x \in \mathbb{R}^n. \tag{10}$$

The next well-known result concerns the cone-monotonicity of φ_λ.

Proposition 5.1. *The following statements hold:*

1°. *φ_λ is K-increasing for $\lambda \in K^+ \setminus \{0\}$.*
2°. *φ_λ is strongly K-increasing for $\lambda \in \operatorname{int} K^+$.*

The linear function φ_λ is often used in scalarization methods for vector optimization due to the following direct consequence of Propositions 3.3 and 5.1.

Corollary 5.2. *Let $A \subseteq \mathbb{R}^n$ be a nonempty set and let $\lambda \in K^+$. Then, for any minimizer $x^0 \in \operatorname{argmin}_{x\in A} \varphi_\lambda(x)$, the following assertions hold:*

1°. *If $\lambda \in \operatorname{int} K^+$, then $x^0 \in \operatorname{MIN}(A \mid K)$.*
2°. *If $\lambda \in K^+ \setminus \{0\}$ and $|\operatorname{argmin}_{x\in A} \varphi_\lambda(x)| = 1$, then $x^0 \in \operatorname{MIN}(A \mid K)$.*

In particular, for the standard ordering cone $K = \mathbb{R}^n_+$, Corollary 5.2 corresponds to the classical weighted sum scalarization method (see Jahn [1, Sec. 11.2.1]). Next we illustrate how φ_λ can be also used as a sorting function.

Example 5.3. Let $\mathbb{R}^2$ be endowed with the standard ordering cone $K := \mathbb{R}^2_+$ and let $A := \{a^1, \dots, a^6\} \subseteq \mathbb{R}^2$ be a finite set, where $a^1 = (2,5)$, $a^2 = (1,2)$, $a^3 = (4,4.5)$, $a^4 = (2,3)$, $a^5 = (4,2)$ and $a^6 = (6,1)$. Consider the linear sorting function φ_λ with $\lambda := (1,1)$. Note that φ_λ is strongly $\mathbb{R}^2_+$-increasing by Proposition 5.1. Then for $a^{j_1} := a^2$, $a^{j_2} := a^4$, $a^{j_3} := a^5$, $a^{j_4} := a^6$, $a^{j_5} := a^1$, $a^{j_6} := a^3$, we have

$$\varphi_\lambda(a^{j_1}) \le \varphi_\lambda(a^{j_2}) \le \dots \le \varphi_\lambda(a^{j_6})$$

and hence $a^{j_1} \in \operatorname{MIN}(A \mid \mathbb{R}^2_+)$ by Corollary 5.2. In Figure 1 we illustrate the level lines of the sorting function φ_λ through the points $a^1, \dots, a^6$.

The linear functions φ_λ with $\lambda \in K^+$ can be used to evaluate the ordering relation induced by the ordering cone $K \subseteq \mathbb{R}^n$, as shown by the following straightforward consequence of the fact that $(K^+)^+ = K$.

Proposition 5.4. *For any points $x, y \in \mathbb{R}^n$ we have*

$$x \leqq_K y \iff \forall \lambda \in K^+ : \ \varphi_\lambda(x) \le \varphi_\lambda(y). \tag{11}$$

Corollary 5.5. *Assume that the cone K is polyhedral, i.e., $K = U^+$ for some nonempty finite set $U \subseteq \mathbb{R}^n \setminus \{0\}$ satisfying the conditions* (2) *and* (3)*. Then, for*

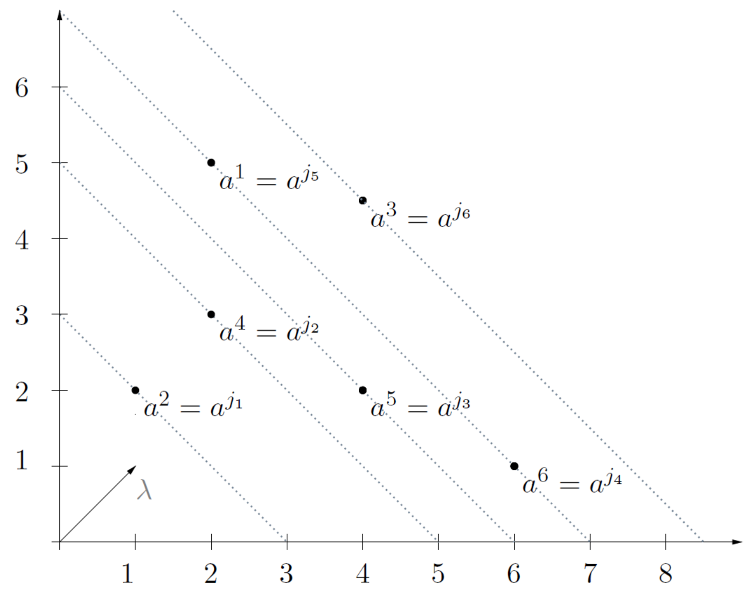

Figure 1. Level lines of the linear sorting function φ_λ for $\lambda = (1, 1)$.

any points $x, y \in \mathbb{R}^n$ *we have*

$$x \leqq_K y \iff \forall u \in U: \ \varphi_u(x) \leq \varphi_u(y). \tag{12}$$

5.2. *Nonlinear cone-monotone sorting functions*

Throughout this section let X be a closed proper subset of $\mathbb{R}^n$, i.e., $\emptyset \neq X = \operatorname{cl} X \neq \mathbb{R}^n$, let $C \subseteq \mathbb{R}^n$ be a nontrivial closed solid convex cone, such that $X + \operatorname{int} C \subseteq X$, and let $c^0 \in \operatorname{int} C$.

Following Tammer and Weidner [12], we define a function $\phi_{X,c^0} : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ for all $x \in \mathbb{R}^n$ by

$$\phi_{X,c^0}(x) := \inf\{s \in \mathbb{R} \,|\, x \in sc^0 - X\}. \tag{13}$$

The next result collects some important properties of this function ϕ_{X,c^0} (see Göpfert et al. [10, Th. 2.3.1 and Prop. 2.3.4]):

Proposition 5.6. *The following statements hold:*

1°. ϕ_{X,c^0} *is well-defined, finite-valued, and continuous.*
2°. ϕ_{X,c^0} *is convex if and only if* X *is convex.*
3°. ϕ_{X,c^0} *is* K*-increasing if and only if* $X + K \subseteq X$.
4°. ϕ_{X,c^0} *is strongly* K*-increasing if and only if* $X + (K \setminus \{0\}) \subseteq \operatorname{int} X$.

As a direct consequence of Propositions 3.3 and 5.6 we get the following result that motivates the use of ϕ_{X,c^0} in nonlinear scalarization methods for solving vector optimization problems.

Corollary 5.7. *Let* $A \subseteq \mathbb{R}^n$ *be a nonempty set. For any* $x^0 \in \operatorname{argmin}_{x\in A} \ \phi_{X,c^0}(x)$ *the following assertions hold:*

1°. *If* $X + (K \setminus \{0\}) \subseteq \operatorname{int} X$, *then* $x^0 \in \mathrm{MIN}(A \mid K)$.
2°. *If* $X + K \subseteq X$ *and* $|\operatorname{argmin}_{x\in A}\ \phi_{X,c^0}(y)| = 1$, *then* $x^0 \in \mathrm{MIN}(A \mid K)$.

By imposing additional assumptions on X and/or C we obtain new insights on the function ϕ_{X,c^0}. In this regard we have the following two results:

Proposition 5.8. *Assume that* $X = Y + C$, *where* $Y \subseteq \mathbb{R}^n$ *is a nonempty compact convex set. Then the following statements hold:*

1°. ϕ_{X,c^0} *is well-defined, finite-valued, continuous and convex.*
2°. *If* $K \subseteq C$, *then* ϕ_{X,c^0} *is* K*-increasing.*
3°. *If* $K \setminus \{0\} \subseteq \operatorname{int} C$, *then* ϕ_{X,c^0} *is strongly* K*-increasing.*

Proof. Clearly X satisfies all initial assumptions of Section 5.2. Indeed, X is closed (as the sum of a compact set and a closed one) and proper (since $\emptyset \neq C \neq \mathbb{R}^n$), while

$$X + \operatorname{int} C \subseteq Y + C + C = Y + C = X. \tag{14}$$

Moreover, X is convex (as being the sum of two convex sets). Therefore assertion 1° follows from Proposition 5.6 (1° and 2°).

If $K \subseteq C$, then we obviously have $X + K \subseteq Y + C + C = Y + C = X$, hence assertion 2° holds in view of Proposition 5.6 (3°).

Finally, when $K \setminus \{0\} \subseteq \operatorname{int} C$, then (14) yields $X + (K \setminus \{0\}) \subseteq X + \operatorname{int} C \subseteq X$, hence assertion 3° follows by Proposition 5.6 (4°). □

Proposition 5.9. *Assume that* $X = Y + C$ *is a polyhedral set given by* (6), *where* $Y = \operatorname{conv} P$ *and* $C = U^+$ *for some nonempty finite sets* $P \subseteq \mathbb{R}^n$ *and* $U \subseteq \mathbb{R}^n \setminus \{0\}$. *Then for all* $x \in \mathbb{R}^n$ *we have*

$$\phi_{X,c^0}(x) = \min_{p\in P} \max_{u\in U} \frac{\langle u, x\rangle + \langle u, p\rangle}{\langle u, c^0\rangle}.$$

Proof. Under the general assumptions stated at the beginning of Section 5.2, we have $c^0 \in \operatorname{int} C$, hence $\langle u, c^0\rangle > 0$ for all $u \in U$, in view of (4). By Proposition 5.8 (1°), the function ϕ_{X,c^0} is well-defined. Moreover, for all $x \in \mathbb{R}^n$ we have

$$\begin{aligned}
\phi_{X,c^0}(x) &= \phi_{Y+C,c^0}(x) \\
&= \inf\{s \in \mathbb{R} \mid sc^0 - x \in Y + C\} \\
&= \inf\{s \in \mathbb{R} \mid \exists\, y \in Y :\ sc^0 - x - y \in C\} \\
&= \inf\{s \in \mathbb{R} \mid \exists\, y \in Y\ \forall\, u \in U :\ \langle u, sc^0 - x - y\rangle \geq 0\} \\
&= \inf\{s \in \mathbb{R} \mid \exists\, y \in Y\ \forall\, u \in U :\ \langle u, x\rangle + \langle u, y\rangle \leq s\langle u, c^0\rangle\} \\
&= \inf\{s \in \mathbb{R} \mid \exists\, y \in Y :\ \max_{u\in U} \frac{\langle u, x\rangle + \langle u, y\rangle}{\langle u, c^0\rangle} \leq s\} \\
&= \min_{y\in Y} \max_{u\in U} \frac{\langle u, x\rangle + \langle u, y\rangle}{\langle u, c^0\rangle} \\
&= \min_{p\in P} \max_{u\in U} \frac{\langle u, x\rangle + \langle u, p\rangle}{\langle u, c^0\rangle}.
\end{aligned}$$

Note that the last equality holds, since we minimize a concave function (namely the maximum of a finite number of affine functions) over the polytope S, and therefore the minimum is attained at some extreme point of S hence an element of P, by a classical argument in convex analysis (see, e.g., Rockafellar [9, Cor. 32.3.4]). □

Remark 12. In the particular case when $X = C = K$ (i.e., X is given by (6) where $P = \{0\}$), we have

$$\begin{aligned} x \leqq_K y &\Longleftrightarrow \forall u \in U: \ \varphi_u(x) \leq \varphi_u(y) \\ &\Longleftrightarrow \forall\, u \in U: \ \langle u, x-y\rangle \leq 0 \\ &\Longleftrightarrow \forall\, u \in U: \ \frac{\langle u, x-y\rangle}{\langle u, c^0\rangle} \leq 0 \\ &\Longleftrightarrow \max_{u\in U} \frac{\langle u, x-y\rangle}{\langle u, c^0\rangle} \leq 0 \\ &\Longleftrightarrow \phi_{K,c^0}(x-y) \leq 0 \end{aligned}$$

for all $x, y \in \mathbb{R}^n$, in view of Propositions 5.5 and 5.9.

5.3. *Nonlinear sorting functions defined by means of oblique norms*

In order to derive optimality conditions in vector optimization based on scalarization, Tammer and Winkler [13] considered the particular framework where $K = \mathbb{R}^n_+$ is the standard ordering cone and $X \subseteq \mathbb{R}^n$ is a special polyhedral set, defined by means of a certain block norm. More precisely, let $\gamma : \mathbb{R}^n \to \mathbb{R}$ be a norm and let

$$B_\gamma := \{x \in \mathbb{R}^n \,|\, \gamma(x) \leq 1\}$$

the corresponding unit ball. Notice that B_γ is convex, compact and symmetric with respect to the origin. In addition, we suppose that γ is a block norm, meaning that B_γ is polyhedral, i.e.,

$$B_\gamma = \{x \in \mathbb{R}^n \,|\, \forall i \in I_q: \ \langle v^i, x\rangle \leq 1\},$$

where $v^1, \ldots, v^q \in \mathbb{R}^n \setminus \{0\}$, $q \in \mathbb{N}$, with $\mathbb{R}_+ \cdot v^i \neq \mathbb{R}_+ \cdot v^j$ for all $i, j \in I_q$, $i \neq j$. Let

$$I^{\mathrm{act}} := \left\{ i \in I_q \;\middle|\; \{x \in \mathbb{R}^n \,|\, \langle v^i, x\rangle = 1\} \cap B_\gamma \cap \operatorname{int} \mathbb{R}^n_+ \neq \emptyset \right\}. \tag{15}$$

Notice that I^{act} is nonempty. Consider the polyhedral set

$$X_\gamma := \{x \in \mathbb{R}^n \,|\, \forall i \in I^{act} : \langle v^i, x\rangle \leq 1\}$$

and let $w^0 \in \mathbb{R}^n$. Tammer and Winkler [13] considered the (polyhedral) set

$$X := -X_\gamma - w^0 = \{x \in \mathbb{R}^n \,|\, \forall i \in I^{act} : \langle -v^i, x\rangle \leq 1 + \langle -v^i, w^0\rangle\}. \tag{16}$$

In order to give an easier representation for the index set I^{act}, we will impose additional assumptions on the block norm γ. Recall (see, e.g., Schandl et al. [14]) that γ is called absolute if for every $\overline{x} := (\overline{x}_1, \ldots, \overline{x}_n) \in \mathbb{R}^n$ we have

$$\gamma(x) = \gamma(\overline{x}) \quad \text{for all } x \in R(\overline{x}),$$

where

$$R(\overline{x}) := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid \forall i \in I_n : \ |x_i| = |\overline{x}_i|\}.$$

Furthermore, the block norm γ is called oblique if it is absolute and moreover for every $x \in \mathbb{R}^n_+ \cap \operatorname{bd} B_\gamma$ we have

$$(x - \mathbb{R}^n_+) \cap \mathbb{R}^n_+ \cap \operatorname{bd} B_\gamma = \{x\}.$$

In the sequel, we assume that the block norm γ is absolute.

The next lemma presents equivalent characterizations for the index set I^{act} defined in (15) (see Tammer and Winkler [13, Lem. 3.2]).

Lemma 5.10. *We have*

$$I^{\text{act}} = \{i \in I_q \mid v^i \in \mathbb{R}^n_+ \setminus \{0\}\},$$

and if γ is oblique, then

$$I^{\text{act}} = \{i \in I_q \mid v^i \in \operatorname{int} \mathbb{R}^n_+\}.$$

According to Tammer and Winkler [13], we introduce the set

$$V := \{v^i \mid i \in I^{\text{act}}\}.$$

Then, for a given $c^0 \in \operatorname{int} \mathbb{R}^n_+$, we consider a function $\varphi_{V,c^0,w^0} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\varphi_{V,c^0,w^0}(x) := \inf\{s \in \mathbb{R} \mid x \in sc^0 + X_\gamma + w^0\} \tag{17}$$

for all $x \in \mathbb{R}^n$. Notice that $\varphi_{V,c^0,w^0} = \phi_{X,c^0}$, where X is given by (16), while the new notation φ_{V,c^0,w^0} is motivated by the next result, based on Tammer and Winkler [13, Cor. 3.1, 3.2 and 3.3].

Proposition 5.11. *The following statements hold:*

1°. *φ_{V,c^0,w^0} is well-defined, finite-valued, continuous, convex and $\mathbb{R}^n_+$-increasing.*
2°. *For all $x \in \mathbb{R}^n$ we have*

$$\varphi_{V,c^0,w^0}(x) = \max_{i \in I^{\text{act}}} \frac{\langle v^i, x\rangle - \langle v^i, w^0\rangle - 1}{\langle v^i, c^0\rangle}.$$

3°. *If γ is oblique, then φ_{V,c^0,w^0} is strongly $\mathbb{R}^n_+$-increasing.*

Remark 13. Notice, for any $c^0 \in \operatorname{int} \mathbb{R}^n_+$, we have $\langle v^i, c^0\rangle > 0$ for all $i \in I^{act}$, since $v^i \in \mathbb{R}^n_+ \setminus \{0\}$ by Lemma 5.10.

Example 5.12. Let us consider the cone $K = \mathbb{R}^2_+$ and the set $A = \{a^1, \ldots, a^6\} \subseteq \mathbb{R}^2$ as given in Example 5.3. Now, we consider the nonlinear sorting function φ_{V,c^0,w^0} with

$$V := \{(1, 2), (2, 1)\}, \ c^0 := (1, 1) \ \text{ and } \ w^0 := (0, 0).$$

Notice that φ_{V,c^0,w^0} is strongly $\mathbb{R}^2_+$-increasing by Proposition 5.11. It is easily seen that

$$\varphi_{V,c^0,w^0}(a^{j_1}) \leq \varphi_{V,c^0,w^0}(a^{j_2}) \leq \ldots \leq \varphi_{V,c^0,w^0}(a^{j_6}),$$

where $a^{j_1} := a^2$, $a^{j_2} := a^4$, $a^{j_3} := a^5$, $a^{j_4} := a^1$, $a^{j_5} := a^3$ and $a^{j_6} := a^6$. The level lines of the sorting function φ_{V,c^0,w^0} through the points $a^1, \ldots, a^6$ are shown in Figure 2.

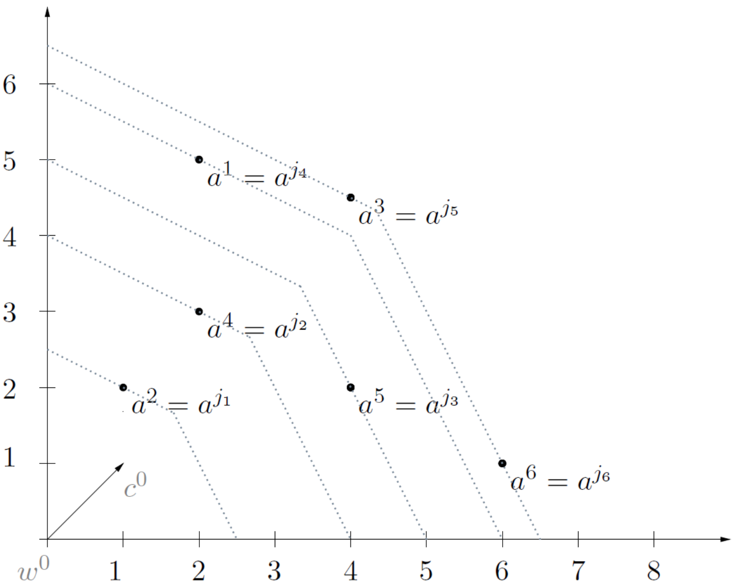

Figure 2. Level lines of the nonlinear sorting function φ_{V,c^0,w^0}.

6. Implementation of the new Jahn-Graef-Younes type algorithms for polyhedral ordering cones and linear sorting functions

In this section we will develop implementable versions of our new Jahn-Graef-Younes type methods (namely Algorithm 4 and Algoritm 5) for computing the minimal points of a finite set with respect to a polyhedral cone, by using linear sorting functions.

As in Section 4, we consider a finite set

$$A := \{a^1, \ldots, a^p\} \subseteq \mathbb{R}^n,$$

where the points $a^1, \ldots, a^p$ are pairwise distinct. We are interested to compute the set $\mathrm{MIN}(A \mid K)$ of all minimal points of A with respect to a polyhedral cone

$$K := U^+ \text{ with } U := \{u^1, \ldots, u^m\} \subseteq \mathbb{R}^n \setminus \{0\}, \tag{18}$$

where $u^1, \ldots, u^m$ are pairwise distinct and U satisfies the conditions (2) and (3).

In what follows, for any $\alpha := (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$, we denote

$$\lambda_\alpha := \sum_{t \in I_m} \alpha_t u^t$$

and we introduce the linear scalar function $\eta_\alpha := \varphi_{\lambda_\alpha}$, defined according to (10) as

$$\eta_\alpha(x) := \varphi_{\lambda_\alpha}(x) = \langle \lambda_\alpha, x\rangle = \sum_{t\in I_m} \alpha_t \langle u^t, x\rangle = \sum_{t\in I_m} \alpha_t \varphi_{u^t}(x) \tag{19}$$

for all $x \in \mathbb{R}^n$. The following result shows that, by an appropriate choice of α, the corresponding function η_α can be used for the sorting procedures our algorithms.

Proposition 6.1. *If $\alpha = (\alpha_1, \dots, \alpha_m) \in \operatorname{int} \mathbb{R}^m_+$, then the function η_α (i.e., φ_{λ_α}) is strongly K-increasing.*

Proof. By Proposition 5.1 it suffices to prove that $\lambda_\alpha \in \operatorname{int} K^+$, which in view of (5) (applied to K in the role of C) reduces to show that

$$\langle \lambda_\alpha, x\rangle > 0 \;\text{ for any }\; x \in K \setminus \{0\}. \tag{20}$$

To this aim, consider an arbitrary point $x \in K \setminus \{0\}$. Since

$$\alpha_t > 0 \;\text{ and }\; \langle u^t, x\rangle \geq 0 \;\text{ for all }\; t \in I_m, \tag{21}$$

we can easily deduce by (19) that

$$\langle \lambda_\alpha, x\rangle \geq 0.$$

Actually, this inequality is strict. Indeed, assume to the contrary that $\langle \lambda_\alpha, x\rangle = 0$. This means that $\sum_{t\in I_m} \alpha_t \langle u^t, x\rangle = 0$ by (19). In view of (21), it follows that $\langle u^t, x\rangle = 0$ for all $t \in I_m$. By assumption (3), we get $x = 0$, a contradiction. Thus (20) holds. □

Now, we are ready to present the implementation of Algorithm 4 for computing the minimal elements of A with respect to K.

Algorithm 6: GRAEF-YOUNES TYPE METHOD INVOLVING A PRE-SORTING PHASE (FOR A POLYHEDRAL ORDERING CONE AND A LINEAR SORTING FUNCTION)

Input: The cone $K = U^+$ where $U = \{u^1, \dots, u^m\}$, the set $A := \{a^1, \dots, a^p\}$, and some $\alpha := (\alpha_1, \dots, \alpha_m) \in \operatorname{int} \mathbb{R}^m_+$.

```
/* Phase 1                                                        */
for j ← 1 to p do
    η^j ← 0;
    for t ← 1 to m do
        φ_t^j ← ⟨u^t, a^j⟩;
        η^j ← α_t · φ_t^j + η^j;
    end
end
/* Phase 2                                                        */
Compute an enumeration of the given points of the set A
such that A = {a^{j_1}, ..., a^{j_p}} and η^{j_1} ≤ η^{j_2} ≤ ... ≤ η^{j_p}
for k ← 1 to p do
    a^k ← a^{j_k};
    for t ← 1 to m do
        φ_t^k ← φ_t^{j_k};
    end
end
/* Phase 3                                                        */
i ← 1;
ind_i ← 1;
B ← {a^{ind_i}};
for j ← 2 to p do
    s ← 0;
    for l ← 1 to i do
        for t ← 1 to m do
            if φ_t^{ind_l} > φ_t^j then
                s ← s + 1;
                break ;
            end
        end
        if s < l then
            break ;
        end
    end
    if s = i then
        i ← i + 1;
        ind_i ← j;
        B ← B ∪ {a^{ind_i}};
    end
end
T ← B;
```

Output: The set T (representing the set of all minimal elements of A w.r.t. K).

Remark 14. In the preparatory Phase 1 of Algorithm 6 we compute the values $\varphi_{u^t}(a^j)$ and $\eta_\alpha(a^j)$ for all $t \in I_m$ and $j \in I_p$.

In Phase 2, we generate an enumeration $A = \{a^{j_1}, \ldots, a^{j_p}\}$ such that

$$\eta_\alpha(a^{j_1}) \leq \eta_\alpha(a^{j_2}) \leq \ldots \leq \eta_\alpha(a^{j_p}).$$

Moreover, we put $a^k := a^{j_k}$ and $\varphi_{u^t}(a^k) := \varphi_{u^t}(a^{j_k})$ for all $k \in I_p$ and all $t \in I_m$.

Finally, in Phase 3 the Graef-Younes method (Algorithm 2) is applied to the set $A = \{a^1, \ldots, a^p\}$ (generated in Phase 2) in order to compute all minimal elements of the initial set A with respect to K. For the evaluation of the ordering relation of $\leqq_K$ we use the characterization (12) taking into account that all values $\varphi_{u^t}(a^k)$ with $t \in I_m$ and $k \in I_p$ are already computed in Phase 2.

Notice that the worst-case computational complexity of Algorithm 6 is $O(p^2 \cdot m)$.

The corresponding algorithmic implementation to Algorithm 5 for computing $\mathrm{MIN}(A \mid K)$ is formulated in Algorithm 7.

Algorithm 7: Jahn-Graef-Younes type method involving a sorting phase after the forward iteration (for a polyhedral ordering cone and a linear sorting function)

Input: The cone $K = U^+$ where $U = \{u^1, \ldots, u^m\}$, the set $A := \{a^1, \ldots, a^p\}$, and some $\alpha := (\alpha_1, \ldots, \alpha_m) \in \operatorname{int} \mathbb{R}^m_+$.

/* Phase 1 */
for $j \leftarrow 1$ **to** p **do**
  for $t \leftarrow 1$ **to** m **do**
    $\varphi^j_t \leftarrow \langle u^t, a^j \rangle$;
  end
end
/* Phase 2 */
$i \leftarrow 1$;
$ind_i \leftarrow 1$;
$b^i \leftarrow a^{ind_i}$;
$B \leftarrow \{b^i\}$;
for $j \leftarrow 2$ **to** p **do**
  $s \leftarrow 0$;
  for $l \leftarrow 1$ **to** i **do**
    for $t \leftarrow 1$ **to** m **do**
      if $\varphi^{ind_l}_t > \varphi^j_t$ **then**
        $s \leftarrow s + 1$;
        break ;
      end
    end
    if $s < l$ **then**
      break ;
    end
  end
  if $s = i$ **then**
    $i \leftarrow i + 1$;
    $ind_i \leftarrow j$;
    $b^i \leftarrow a^{ind_i}$;
    $B \leftarrow B \cup \{b^i\}$;
  end
end
/* Phase 3 */
for $j \leftarrow 1$ **to** i **do**
  $\eta^j \leftarrow 0$;
  for $t \leftarrow 1$ **to** m **do**
    $\eta^j \leftarrow \alpha_t \cdot \varphi^{ind_j}_t + \eta^j$;
  end
end
Compute an enumeration of the given points of the set B such that $B = \{b^{j_1}, \ldots, b^{j_i}\}$ and $\eta^{j_1} \geq \eta^{j_2} \geq \ldots \geq \eta^{j_i}$

```
for k ← 1 to i do
    b^k ← b^{j_k};
    for t ← 1 to m do
        φ_t^k ← φ_t^{ind_{j_k}};
    end
end
/* Phase 4                                                    */
q ← 1;
ind_q ← i;
T ← {b^{ind_q}};
for j ← 1 to i − 1 do
    s ← 0;
    for l ← 1 to q do
        for t ← 1 to m do
            if φ_t^{ind_l} > φ_t^{i−j} then
                s ← s + 1;
                break ;
            end
        end
        if s < l then
            break ;
        end
    end
    if s = q then
        q ← q + 1;
        ind_q ← i − j;
        T ← T ∪ {b^{ind_q}};
    end
end
Output: The set T (representing the set of all minimal elements of A w.r.t. K).
```

Remark 15. In Phase 1 of Algorithm 7 we compute $\varphi_{u^t}(a^j)$ for all $t \in I_m$ and $j \in I_p$.

In Phase 2 we apply the Graef-Younes method (forward iteration of Algorithm 3) for the a priori given set A to get the reduced set $B = \{b^1, \ldots, b^i\}$. For the evaluation of the ordering relation $\leqq_K$ we use the characterization (12), taking into account that the values $\varphi_{u^t}(a^j)$ for all $t \in I_m$ and all $j \in I_p$ are already computed in Phase 1.

In Phase 3 we compute the values $\eta_\alpha(b^k)$ for all $k \in I_i$. Moreover, we generate an enumeration $B = \{b^{j_1}, \ldots, b^{j_i}\}$ such that

$$\eta_\alpha(b^{j_1}) \geq \eta_\alpha(b^{j_2}) \geq \ldots \geq \eta_\alpha(b^{j_i}).$$

Notice that η_α is strongly K-increasing, since $\alpha \in \operatorname{int} \mathbb{R}^m_+$ by Proposition 6.1. Moreover, we put $b^k := b^{j_k}$ for all $k \in I_i$ and $\varphi_{u^t}(b^k) := \varphi_{u^t}(b^{j_k})$ for all $k \in I_i$ and all $t \in I_m$.

Finally, in Phase 4 the backward iteration of Algorithm 3 is applied to the set $B = \{b^1, \ldots, b^i\}$ (generated in Phase 3) in order to compute all minimal points of A with respect to K. Again, we use (12) to evaluate $\leqq_K$, taking into account that the values $\varphi_{u^t}(b^k)$ with $t \in I_m$ and $k \in I_i$ are already computed in Phase 3.

The worst-case computational complexity of Algorithm 7 is $O(p^2 \cdot m)$.

7. Applications

Our new Jahn-Graef-Younes type algorithms introduced in Sections 4 and 6 can be used for approximating the sets of minimal outcomes of certain continuous vector optimization problems, via a discretization approach proposed by Jahn [3], namely the "Multiobjective search algorithm with subdivision technique" (MOSAST).

In this section we will apply our algorithms to a particular continuous bi-objective test problem (known in the literature as being very difficult to solve). A detailed comparative analysis of our algorithms and other classical methods is provided, based on computational experiments in MATLAB.

7.1. *Continuous bi-objective optimization problems and the approximation of their solution sets*

Consider a vector-valued function $f = (f_1, f_2) : S \to \mathbb{R}^2$, defined on a nonempty bounded set $S \subseteq \mathbb{R}^2$, and let $f(S) := \{f(x) \mid x \in S\}$. Assume that $K \subseteq \mathbb{R}^2$ is a nontrivial pointed polyhedral cone. An element $x \in S$ is said to be an efficient solution of the vector optimization problem

$$\begin{cases} f(x) = (f_1(x), f_2(x)) \to \min \quad \text{w.r.t. } K \\ x \in S \end{cases} \tag{22}$$

if $f(x) \in \text{MIN}(f(S) \mid K)$. In what follows we denote by

$$\text{EFF}(S \mid f, K) := f^{-1}(\text{MIN}(f(S) \mid K))$$

the set of all efficient solutions to problem (22).

Since the computation of the set $\text{MIN}(f(S) \mid K)$ is in general a very difficult task, we are interested in finding a good enough approximation of $\text{MIN}(f(S) \mid K)$. One possibility is to generate a finite set $\widetilde{S} \subseteq S$ and to compute the set $\text{MIN}(f(\widetilde{S}) \mid K)$. Of course, in general there is no containment relation between the sets $\text{MIN}(f(S) \mid K)$ and $\text{MIN}(f(\widetilde{S}) \mid K)$, but we always have

$$\text{MIN}(f(\widetilde{S}) \mid K) \subseteq \text{MIN}(f(S) \mid K) + K,$$

by the domination property.

In what follows $\widetilde{S}$ will be generated by an iterative procedure, as a union of sets

$$\widetilde{S} := \widetilde{S}_0 \cup \widetilde{S}_1 \cup \ldots \cup \widetilde{S}_l.$$

Then, for each $i \in \{0, 1, \ldots, l\}$ we will compute the set

$$T_i := \text{MIN}(f(\widetilde{S}_i) \mid K)$$

by our methods from the previous sections for the finite set $A := f(\widetilde{S}_i)$. Then, denoting

$$T := T_0 \cup T_1 \cup \ldots \cup T_l,$$

we have

$$\mathrm{MIN}(f(\widetilde{S}) \mid K) = \mathrm{MIN}\,(T \mid K)\,, \tag{23}$$

in view of Remarks 2, 3 and 4. The right-hand side term of (23) will be computed by applying our methods for the set $A := T$. Next, we present the main steps of this iterative procedure:

Step 1. Compute a first approximation of $\mathrm{MIN}(f(S) \mid K)$:

- Consider a box (i.e., an axis-parallel rectangle) $B_0 \subseteq \mathbb{R}^2$ such that $S \subseteq B_0$. Generate a finite set of random points $\widetilde{B_0} \subseteq B_0$ and define $\widetilde{S}_0 := \widetilde{B}_0 \cap S$.
- Compute the sets $T_0 := \mathrm{MIN}(f(\widetilde{S}_0) \mid K)$ and $\mathrm{EFF}(\widetilde{S}_0 \mid f, K) := f^{-1}(T_0)$.

Step 2. Apply a subdivision technique similar to that introduced by Dellnitz et al. [15] (see also Jahn [3]) in order to improve the approximation of the set $\mathrm{MIN}(f(S) \mid K)$:

- Consider a system $\mathcal{B}$ of boxes in $\mathbb{R}^2$ that covers the set S, i.e., $S \subseteq \bigcup_{B\in\mathcal{B}} B$.
- Determine the subsystem of boxes

$$\mathcal{B}_{act} := \{B \in \mathcal{B} \mid B \cap \mathrm{EFF}(\widetilde{S}_0 \mid f, K) \neq \emptyset\}$$

  and let $\{B_1, \ldots, B_l\} = \mathcal{B}_{act}$ be an enumeration of $\mathcal{B}_{act}$ with $l = |\mathcal{B}_{act}|$.
- For each $i \in I_l$ generate a finite set of random points $\widetilde{B}_i \subseteq B_i$ and let $\widetilde{S}_i := \widetilde{B}_i \cap S$.
- For every $i \in I_l$ compute the set $T_i := \mathrm{MIN}(f(\widetilde{S}_i) \mid K)$.

Step 3. Compute the set $\mathrm{MIN}\,(T \mid K)\,.$

Remark 16. The iterative procedure described above represents a counterpart of the MOSAST method. In contrast to the classical approach by Jahn [3] (see also Limmer et al. [16]), we use a fixed number of randomly generated points.

In order to get a good approximation of the set $\mathrm{MIN}(f(S) \mid K)$, the cardinality of $\widetilde{S}_0$ should be high enough. In Step 2 we have to solve a family of l discrete vector optimization problems. Since l can be very high, it is convenient to consider sets $\widetilde{S}_i$, $i \in I_l$, with reasonable small cardinality (significantly smaller than $|\widetilde{S}_0|$).

In what follows we present the pseudo-code of our iterative procedure.

Algorithm 8: Special instance of MOSAST

Input: The nonempty bounded set $S \subseteq \mathbb{R}^2$, the objective function $f : S \to \mathbb{R}^2$, a box $B_0 := [P_1^{min}, P_1^{max}] \times [P_2^{min}, P_2^{max}] \subseteq \mathbb{R}^2$, such that $S \subseteq B_0$, and three positive integers (custom parameters) $\#_{step1}$, $\#_{step2}$, $\#_{int}$.

/* Step 1 */

Generate a set of random points, $\widetilde{B_0} \subseteq B_0$ with $|\widetilde{B_0}| = \#_{step1}$;

$\widetilde{S}_0 \leftarrow \widetilde{B_0} \cap S$;

$T_0 \leftarrow \text{MIN}(f(\widetilde{S}_0) \mid K)$;

$\text{EFF}(\widetilde{S}_0 \mid f, K) \leftarrow f^{-1}(T_0)$;

/* Step 2 */

$T \leftarrow T_0$;

for $k \leftarrow 1$ **to** $\#_{int} + 1$ **do**

$P_1^k \leftarrow P_1^{min} + \frac{k-1}{\#_{int}} \cdot (P_1^{max} - P_1^{min})$;

$P_2^k \leftarrow P_2^{min} + \frac{k-1}{\#_{int}} \cdot (P_2^{max} - P_2^{min})$;

end

$l \leftarrow 0$;

for $k \leftarrow 1$ **to** $\#_{int}$ **do**

for $t \leftarrow 1$ **to** $\#_{int}$ **do**

$B \leftarrow [P_1^k, P_1^{k+1}] \times [P_2^t, P_2^{t+1}]$;

if $|B \cap \text{EFF}(\widetilde{S}_0 \mid f, K)| > 0$ **then**

$l \leftarrow l + 1$;

$B_l \leftarrow B$;

Generate a set of random points, $\widetilde{B_l} \subseteq B_l$ with $|\widetilde{B_l}| = \#_{step2}$;

$\widetilde{S}_l \leftarrow \widetilde{B_l} \cap S$;

$T_l \leftarrow \text{MIN}(f(\widetilde{S}_l) \mid K)$;

$T \leftarrow T \cup T_l$;

end

end

end

/* Step 3 */

$\widetilde{T} \leftarrow \text{MIN}(T \mid K)$;

$\text{EFF}(\widetilde{S} \mid f, K) \leftarrow f^{-1}(\widetilde{T})$;

Output: The set $\widetilde{T}$ (representing the minimal elements of the set $f(\widetilde{S})$ with respect to K) and the set $\text{EFF}(\widetilde{S} \mid f, K)$ (representing the corresponding set of efficient solutions).

7.2. *Comparative analysis of our algorithms for Jahn's test problem*

As test problem for our numerical experiments we will consider a particular vector optimization problem of type (22), known in the literature for being difficult to solve (see Jahn [3, Ex 2]).

Example 7.1 (Jahn's test problem)**.** Consider the feasible set $S \subseteq \mathbb{R}^2$ of all points $x = (x_1, x_2) \in \mathbb{R}^2$ that satisfy the following constraints

$$\begin{cases} -1.5 \le x_1 \le 1 \\ 0 \le x_2 \le 2.25 \\ x_1^2 - x_2 \le 0 \\ x_1 + 2x_2 - 3 \le 0. \end{cases}$$

Define the objective function $f : S \to \mathbb{R}^2$ for all $x = (x_1, x_2) \in S$ by

$$f(x) = (f_1(x), f_2(x)) := (-x_1, x_1 + x_2^2 - \cos(50x_1)).$$

The set S is illustrated in Figure 3 (color light grey) while the outcome set $f(S)$ is illustrated in Figure 4 (color light grey).

In the following two sections, 7.2.1 and 7.2.2, we will study this problem when $K = \mathbb{R}^2_+$ is the standard ordering cone, and when $K = U^+$ is a polyhedral ordering cone, respectively.

We start our analysis by applying Algorithm 8 to Jahn's test problem, where $B_0 := [-1.5, 1] \times [0, 2.25]$, while the sets $\mathrm{MIN}(f(\widetilde{S}_0) \mid K)$, $\mathrm{MIN}(f(\widetilde{S}_i) \mid K)$, $i \in I_l$, and $\mathrm{MIN}\,(T \mid K)$, are computed by means of Algorithm 4 with:

- $\varphi := \varphi_\lambda$, $\lambda = (1, 1)$, when $K = \mathbb{R}^2_+$ (in Section 7.2.1)
- $\varphi := \varphi_{\lambda_\alpha} = \eta_\alpha$, $\alpha = (1, 1)$, when $K = U^+$ (in Section 7.2.2).

Then, we use the sets $\widetilde{S}_0, \widetilde{S}_1, \ldots, \widetilde{S}_l$ as input data for the following algorithm.

Algorithm 9: PROCEDURE $*$

Input: The sets $\widetilde{S}_0, \widetilde{S}_1, \ldots, \widetilde{S}_l$.
$T \leftarrow \mathrm{MIN}(f(\widetilde{S}_0) \mid K)$;
for $t \leftarrow 1$ **to** l **do**
  $T \leftarrow T \cup \mathrm{MIN}(f(\widetilde{S}_t) \mid K)$;
end
$\widetilde{T} \leftarrow \mathrm{MIN}(T \mid K)$;
Output: The set $\widetilde{T}$ (representing an approximation of the set $\mathrm{MIN}(f(S) \mid K)$).

Within Algorithm 9 all the sets $\mathrm{MIN}(f(\widetilde{S}_0) \mid K)$, $\mathrm{MIN}(f(\widetilde{S}_i) \mid K)$, $i \in I_l$, and $\mathrm{MIN}\,(T \mid K)$ are computed by the same algorithm, corresponding to the Procedure $*$ indicated in Table 1 (within Section 7.2.1) and Table 3 (within Section 7.2.2) .

We mention that our comparative analysis is based on numerical results obtained by implementing the algorithms in MATLAB 2016a on a Core i5-7200U 2x 2.50GHz CPU, 16GB ram computer.

7.2.1. *Jahn's test problem when K is the standard ordering cone*

In this subsection we consider the standard ordering cone $K = \mathbb{R}^2_+$. We analyse twelve different procedures, as listed in Table 1. Among them, Procedures III-VIII use linear sorting functions of type $\varphi = \varphi_\lambda : \mathbb{R}^2 \to \mathbb{R}$, $\lambda \in \mathrm{int}\,\mathbb{R}^2_+$, given by (10), while the Procedures IX - XII use nonlinear sorting functions of type $\varphi = \varphi_{V,c^0,w^0} : \mathbb{R}^2 \to \mathbb{R}$, given by (17), where $V \subseteq \mathbb{R}^2$, $c^0 := (1,1)$ and $w^0 := (0,0)$. All considered sorting functions are strongly $\mathbb{R}^2_+$-increasing, in view of Propositions 5.1 and 5.11.

Proc.∗	Alg.	Presorting	Sorting after for. iter.	Parameters
I	1	-	-	-
II	3	-	-	-
III	4	$\varphi := \varphi_\lambda$	-	$\lambda = (1,2)$
IV	5	-	$\varphi := \varphi_\lambda$	$\lambda = (1,2)$
V	4	$\varphi := \varphi_\lambda$	-	$\lambda = (1,1)$
VI	5	-	$\varphi := \varphi_\lambda$	$\lambda = (1,1)$
VII	4	$\varphi := \varphi_\lambda$	-	$\lambda = (2,1)$
VIII	5	-	$\varphi := \varphi_\lambda$	$\lambda = (2,1)$
IX	4	$\varphi := \varphi_{V,c^0,w^0}$	-	$V = \{(1,2),(2,1)\}$
X	5	-	$\varphi := \varphi_{V,c^0,w^0}$	$V = \{(1,2),(2,1)\}$
XI	4	$\varphi := \varphi_{V,c^0,w^0}$	-	$V = \{(1,3),(2.5,2.5),(3,1)\}$
XII	5	-	$\varphi := \varphi_{V,c^0,w^0}$	$V = \{(1,3),(2.5,2.5),(3,1)\}$

Table 1. Procedures applied for Jahn's test problem when $K = \mathbb{R}^2_+$ is the standard ordering cone.

For convenience let us denote by $\widetilde{B} := \widetilde{B}_1 \cup \widetilde{B}_2 \cup \ldots \cup \widetilde{B}_l$ the set of all randomly generated points, hence $\widetilde{S} = \widetilde{B} \cap S$. Also, denote by $\#_{\leqq_K}$ the number of pairwise comparisons with respect to the ordering relation $\leqq_K$.

In what follows we will analyse the computational results obtained by applying Algorithm 9 for Jahn's test problem.

The running times (in seconds) and the number of pairwise comparisons needed to compute the set $\widetilde{T} = \mathrm{MIN}(f(\widetilde{S}) \mid \mathbb{R}^2_+)$ by Algorithm 9 are listed in Table 2.

Figure 3 illustrates the initial feasible set S (color lightgrey), the set $\widetilde{S}$ (color darkgrey), the system of boxes $\mathcal{B}_{act} = \{B_1, \ldots, B_l\}$ (color red) and the set $\mathrm{EFF}(\widetilde{S} \mid f, \mathbb{R}^2_+)$ (color black), generated by Algorithm 8 for Jahn's test problem ($K = \mathbb{R}^2_+$, $\#_{step1} = 10^5$, $\#_{step2} = 10^4$, $\#_{int} = 30$).

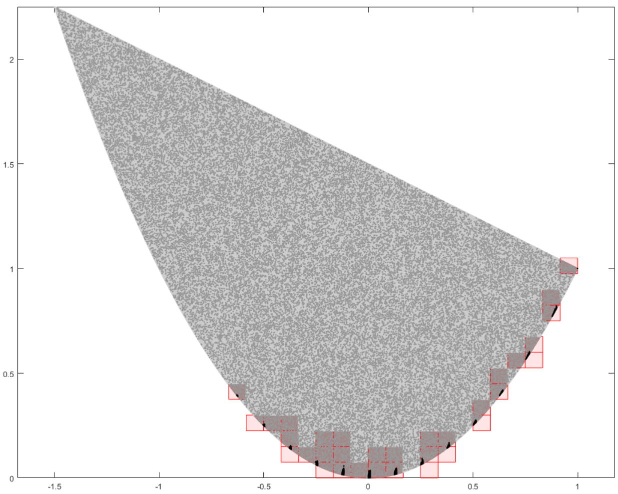

Figure 3. The set S (color lightgrey), the set $\widetilde{S}$ (color darkgrey), the system of boxes $\mathcal{B}_{act} = \{B_1, \ldots, B_{33}\}$ (color red) and the set $\mathrm{EFF}(\widetilde{S} \mid f, \mathbb{R}^2_+)$ (color black) for Jahn's test problem with $K = \mathbb{R}^2_+$.

	$\#_{step1}$	10^6	10^7	10^7
	$\#_{step2}$	10^4	10^4	10^5
	$\#_{int}$	30	30	30
	$\lvert\widetilde{B}\rvert$	1.260.000	10.220.000	12.200.000
	$\lvert\widetilde{S}\rvert$	627.047	4.758.085	5.896.073
	l	26	22	22
	$\lvert T\rvert$	15.786	14.832	49.596
	$\lvert\widetilde{T}\rvert$	1.964	2.808	6.986
I	Runtime	27,6	672,8	940,0
	$\#_{\leqq_K}$	684510944	16.952.327.907	23.032.520.468
II	Runtime	2,4	5,3	37,9
	$\#_{\leqq_K}$	50.301.957	107.473.149	843.505.826
III	Runtime	4,9	86,4	127,3
	$\#_{\leqq_K}$	129.438.579	2.440.717.924	4.123.833.895
IV	Runtime	2,8	7,8	37,7
	$\#_{\leqq_K}$	54.701.638	117.968.982	907.403.128
V	Runtime	2,8	45,6	87,6
	$\#_{\leqq_K}$	87.408.106	1.192.388.508	2.823.993.735
VI	Runtime	2,3	6,3	35,3
	$\#_{\leqq_K}$	52.968.556	113.309.910	889.035.465
VII	Runtime	2,4	28,7	72,4
	$\#_{\leqq_K}$	67.748.612	501.887.794	2.125.652.797
VIII	Runtime	2,3	6,7	35,9
	$\#_{\leqq_K}$	52.950.686	113.172.781	888.846.269
IX	Runtime	2,9	50,8	86,1
	$\#_{\leqq_K}$	77.731.403	856.876.870	2.492.171.225
X	Runtime	2,3	9,9	39,6
	$\#_{\leqq_K}$	53.035.032	113.716.565	889.756.299
XI	Runtime	2,9	40,7	86,3
	$\#_{\leqq_K}$	77.565.368	858.921.388	2.457.803.337
XII	Runtime	2,3	6,8	35,9
	$\#_{\leqq_K}$	52.995.315	113.590.919	889.759.216

Table 2. Computational results for Jahn's test problem with standard ordering cone $K = \mathbb{R}^2_+$.

The outcome sets $f(S)$ (color lightgrey) and $f(\widetilde{S})$ (color darkgrey) as well as the set of minimal points $\widetilde{T} = \mathrm{MIN}(f(\widetilde{S}) \mid \mathbb{R}^2_+)$ (color black) are represented in Figure 4.

By analyzing the Table 2, one can see that the running times depend essentially on the corresponding sorting functions (listed in Table 1). More precisely, the results produced by Algorithm 5 with a strongly $\mathbb{R}^2_+$-increasing sorting function (Procedures IV, VI, VIII, X, XII) are comparable with those produced by the original Jahn-Graef-Younes method (Procedure II). In contrast, Procedures III, V, VII, IX and XI, based on Algorithm 4, seem to be slower than Procedure II for our test instances, since the number of pairwise comparisons with respect to the ordering relation $\leqq_K$ are higher. Naturally, all Jahn-Graef-Younes type algorithms are significantly faster than

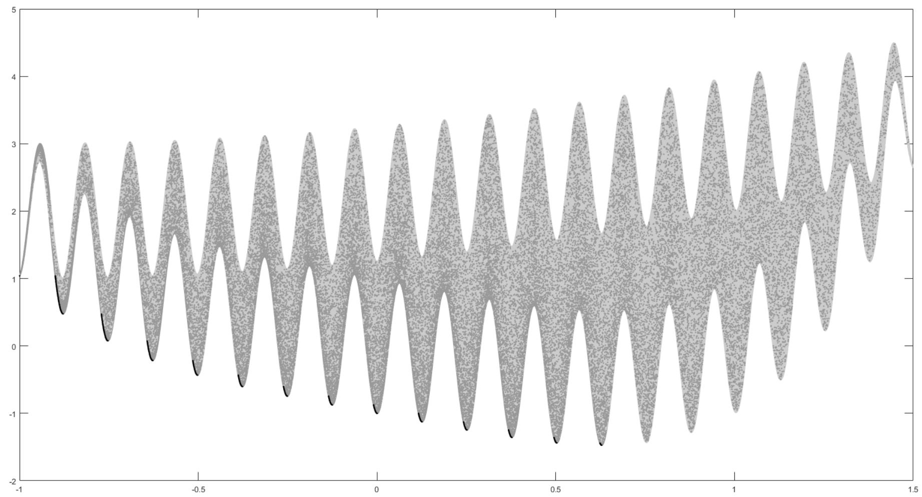

Figure 4. The set $f(S)$ (color lightgrey), the set $f(\widetilde{S})$ (color darkgrey) and the set of minimal elements $\widetilde{T} = \text{MIN}(f(\widetilde{S}) \mid \mathbb{R}^2_+)$ (color black) for Jahn's test problem with $K = \mathbb{R}^2_+$.

Algorithm 1 (the naive method). Notice that sometimes a procedure may be faster than another one (in running time), even if it requires a larger number of pairwise comparisons with respect to the ordering $\leqq_K$. This is due to the different costs for the evaluation of $\leqq_K$.

7.2.2. Jahn's test problem when K is a polyhedral ordering cone

In this section we consider a polyhedral ordering cone $K = U^+$ where

$$U = \{u^1, u^2\} := \{(100, 1), (-100, 1)\}.$$

We analyze eight different procedures, as listed in Table 3. Among them, the last six procedures use linear sorting functions as defined in Section 6. More precisely, for any $\alpha := (\alpha_1, \alpha_2) \in \text{int}\,\mathbb{R}^2_+$, we consider the function $\eta_\alpha : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\eta_\alpha(x) := \langle \alpha_1 \cdot u^1 + \alpha_2 \cdot u^2, x \rangle \quad \text{for all } x \in \mathbb{R}^2,$$

which is strongly K-increasing, in view of Proposition 6.1.

Proc.∗	Alg.	Presorting	Sorting after for. iter.	Parameters
I	1	-	-	-
II	3	-	-	-
III	4, 6	$\varphi := \varphi_{\lambda_\alpha} = \eta_\alpha$	-	$\alpha = (1, 2)$
IV	5 , 7	-	$\varphi := \varphi_{\lambda_\alpha} = \eta_\alpha$	$\alpha = (1, 2)$
V	4, 6	$\varphi := \varphi_{\lambda_\alpha} = \eta_\alpha$	-	$\alpha = (1, 1)$
VI	5 , 7	-	$\varphi := \varphi_{\lambda_\alpha} = \eta_\alpha$	$\alpha = (1, 1)$
VII	4, 6	$\varphi := \varphi_{\lambda_\alpha} = \eta_\alpha$	-	$\alpha = (2, 1)$
VIII	5 , 7	-	$\varphi := \varphi_{\lambda_\alpha} = \eta_\alpha$	$\alpha = (2, 1)$

Table 3. Procedures applied for Jahn's test problem when $K = U^+$ is a polyhedral ordering cone.

Remark 17. All procedures described in Table 3 use the characterization given in Corollary 5.5 for the pairwise comparison of points with respect to the ordering $\leqq_K$.

In what follows we will analyse the computational results obtained by applying Algorithm 9 for Jahn's test problem.

The running times (in seconds) and the number of pairwise comparisons needed to compute the set $\widetilde{T} = \text{MIN}(f(\widetilde{S}) \mid K)$ by Algorithm 9 are listed in Table 4.

	$\#_{step1}$	10^5	10^6	10^6
	$\#_{step2}$	10^4	10^4	10^5
	$\#_{int}$	30	30	30
	$\vert\widetilde{B}\vert$	1.200.000	1.870.000	9.700.000
	$\vert\widetilde{S}\vert$	836.057	992.983	5.664.607
	l	110	87	87
	$\vert T\vert$	113.601	94.200	291.481
	$\vert\widetilde{T}\vert$	42.784	45.485	154.145
I	Runtime	346,1	698,5	5356,9
	$\#_{\leqq_K}$	8.830.661.499	16.371.742.342	101.963.388.244
II	Runtime	148,6	194,9	1762,2
	$\#_{\leqq_K}$	3.429.003.410	4.111.754.987	38.244.516.704
III	Runtime	93,1	160,0	1198,2
	$\#_{\leqq_K}$	2.902.570.705	4.794.331.614	35.225.370.700
IV	Runtime	122,9	178,6	1520,1
	$\#_{\leqq_K}$	3.485.894.921	4.243.658.305	38.897.471.354
V	Runtime	116,7	201,8	1451,4
	$\#_{\leqq_K}$	2.901.559.998	4.438.000.337	35.456.839.150
VI	Runtime	130,3	194,9	1638,5
	$\#_{\leqq_K}$	3.491.033.748	4.272.269.882	38.996.647.665
VII	Runtime	137,4	296,2	1697,6
	$\#_{\leqq_K}$	3.023.916.712	5.813.641.555	36.856.821.409
VIII	Runtime	138,4	196,9	1722,5
	$\#_{\leqq_K}$	3.497.988.473	4.278.940.881	38.992.693.160

Table 4. Computational results for Jahn's test problem with polyhedral ordering cone $K = U^+$.

Figure 5 illustrates the set S (color lightgrey), the set $\widetilde{S}$ (color darkgrey), the system of boxes $\mathcal{B}_{act} = \{B_1, \ldots, B_l\}$ (color red) and the set $\text{EFF}(\widetilde{S} \mid f, K)$ (color black), generated by Algorithm 8 for Jahn's test problem ($K = U^+$, $\#_{step1} = 10^5$, $\#_{step2} = 10^4$, $\#_{int} = 30$). The corresponding outcome sets $f(S)$ (color lightgrey) and $f(\widetilde{S})$ (color darkgrey) as well as the set of minimal elements $\widetilde{T} = \text{MIN}(f(\widetilde{S}) \mid K)$ (color black) are represented in Figure 6. Notice that the set of minimal elements of $f(S)$ with respect to K can be analytically described by

$$\text{MIN}(f(S) \mid K) = \{(-s, s + s^4 - \cos(50s)) \in \mathbb{R}^2 \mid s \in [-1.5, 1]\}.$$

Finally, observe that the computational results listed in Table 4 neatly show that a pre-sorting phase can improve the performance of the original Jahn-Graef-Younes

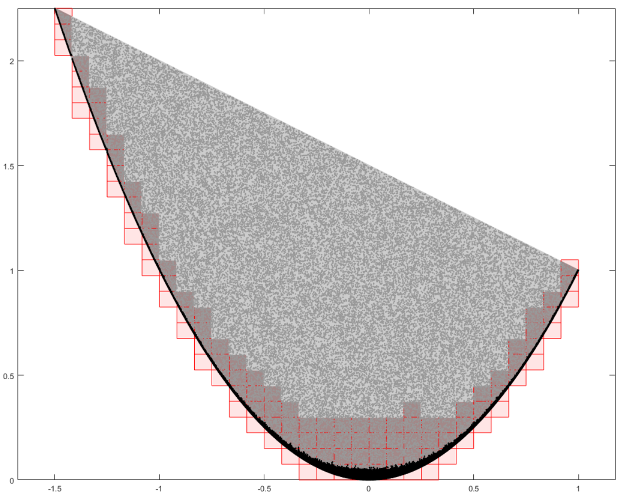

Figure 5. The set S (color lightgrey), the set $\widetilde{S}$ (color darkgrey), the system of boxes $\mathcal{B}_{act} = \{B_1, \ldots, B_{110}\}$ (color red) and the set $\mathrm{EFF}(\widetilde{S} \mid f, K)$ (color black) for Jahn's test problem with $K = U^+$.

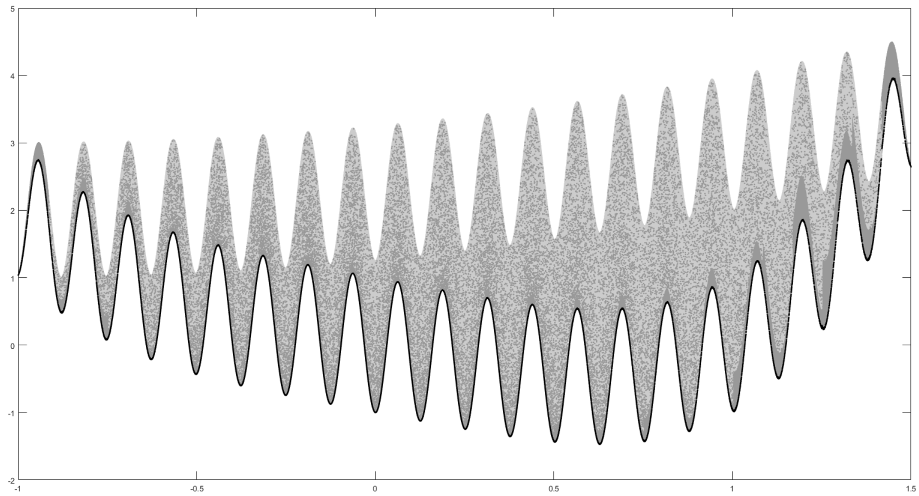

Figure 6. The set $f(S)$ (color lightgrey), the set $f(\widetilde{S})$ (color darkgrey) and the set of minimal elements $\widetilde{T} = \mathrm{MIN}(f(\widetilde{S}) \mid K)$ (color black) for Jahn's test problem with $K = U^+$.

method. In this regard Procedures III and V (based on Algorithm 6) seem to be the best choices. Also our Algorithm 7 performs quite well within Procedures IV and VI.

8. Conclusions

We developed new algorithms for determining all minimal elements of a finite set A with respect to a nontrivial pointed convex cone K in $\mathbb{R}^n$. They are obtained from the original Graef-Younes and Jahn-Graef-Younes methods by considering additional sorting procedures via strongly K-increasing scalar functions.

An interesting open question is how to choose the strongly K-increasing function in concrete applications. In what concerns the functions φ_λ, φ_{V,c^0,w^0} and η_α considered in Section 7, we should identify appropriate choices of the input data $\lambda \in \operatorname{int} K^+$, $V \subseteq \mathbb{R}^n$, $c^0 \in \operatorname{int} K$, $w^0 \in \mathbb{R}^n$, and $\alpha \in \operatorname{int} \mathbb{R}^m_+$. In this regard we will investigate location problems involving block norms, following Alzorba et al. [17].

Moreover, in a forthcoming work we aim to derive new Jahn-Graef-Younes type methods for set optimization problems, based on this paper as well as on the works by Eichfelder [5–7], Jahn [18], and Köbis et al. [8].

References

[1] Jahn J. Vector optimization - theory, applications, and extensions. 2nd ed. Berlin, Heidelberg: Springer; 2011.

[2] Younes YM. Studies on discrete vector optimization [dissertation]. Eypt: University of Demiatta; 1993.

[3] Jahn J. Multiobjective Search Algorithm with Subdivision Technique. Comput Optim Appl. 2006;35(2):161–175.

[4] Jahn J, Rathje U. Graef-Younes Method with Backward Iteration. In: Küfer KH, Rommelfanger H, Tammer C, Winkler K. Multicriteria Decision Making and Fuzzy Systems - Theory, Methods and Applications. Aachen: Shaker Verlag; 2006. p. 75–81.

[5] Eichfelder G. Ordering Structures in Vector Optimization and Applications in Medical Engineering. Ilmenau: Ilmenau University of Technology; 2013. (Preprint-Series of the Institute of Mathematics).

[6] Eichfelder G. Numerical Procedures in Multiobjective Optimization with Variable Ordering Structures. J Optim Theory Appl. 2014;162(2):489–514.

[7] Eichfelder G. Variable Ordering Structures in Vector Optimization. Berlin, Heidelberg: Springer; 2014.

[8] Köbis E, Kuroiwa D, Tammer C. Generalized set order relations and their numerical treatment. Applied Anal Optim. 2017;1(1): 45–66.

[9] Rockafellar RT. Convex Analysis. Princeton: Princeton University Press; 1970.

[10] Göpfert A, Riahi H, Tammer C, Zălinescu C. Variational Methods in Partially Ordered Spaces. New York: Springer; 2003.

[11] Aliprantis CD, Tourky R. Cones and Duality. American Mathematical Society; 2007.

[12] Gerth (Tammer) C, Weidner P. Nonconvex separation theorems and some applications in vector optimization. J Optim Theory Appl. 1990;67(2):297–320.

[13] Tammer C, Winkler K. A new scalarization approach and applications in multicriteria d.c. optimization. J Nonlinear Convex Anal. 2003;4(3):365–380.

[14] Schandl B, Klamroth, Wiecek MM. Introducing oblique norms into multiple criteria programming. J Global Optimization. 2002;23(1):81–97.

[15] Dellnitz M, Schütze O, Hestermeyer T. Covering Pareto Sets by Multilevel Subdivision Techniques. J Optim Theory Appl. 2005;124(1):113–136.

[16] Limmer S, Fey D, Jahn J. GPU implementation of a multiobjective search algorithm. Positivity. 2012;16(3):397–404.

[17] Alzorba S, Günther C, Popovici N, Tammer C. A new algorithm for solving planar multiobjective location problems involving the Manhattan norm. European J Oper Res. 2017;258(1):35–46.

[18] Jahn J. Vectorization in set optimization. J Optim Theory Appl. 2015;167(3):783–795.

Reports of the Institutes 2017

01-17. M. Klinge and R. Weiner, *Strong stability preserving explicit peer methods for discontinuous Galerkin discretizations*

02-17. Chr. Günther and Chr. Tammer, *On generalized-convex constrained multi-objective optimization*

03-17. E. Köbis, L. Th. Tam, Chr. Tammer and J.-Ch. Yao, *Necessary conditions for solutions of set optimization problems with respect to variable domination structures*

04-17. E. Köbis, Chr. Tammer and J.-Ch. Yao, *Optimality conditions for set-valued optimization problems based on set approach and applications in uncertain optimization*

05-17. L. Th. Tam, *Multiobjective approaches based on variable ordering structures for intensity problems in radiotherapy treatment*

Reports are available via WWW: http://www.mathematik.uni-halle.de/institut/reports/