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Scalarization of approximate solutions of vector optimization problem with variable order structure based on nonlinear scalarization

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Abstract

Approximate minimizer, approximate nondominated and approximate minimal solutions of vector optimization problems with respect to the variable order structure are defined in [36]. Here we characterize these solution concepts by a nonlinear scalarization method by means of nonlinear functionals. In the case of exact solutions, specially in variable order case, authors use cone or pointed convex cone valued map for domination but here we will use just set-valued map without any cone or pointed convex cone assumptions. This set-valued map associates a set with certain properties to each element of the image space and present a characterization of approximate minimal, minimizer and nondominated solutions by using this scalarization method. In the last section, we will give an extension of Ekeland's theorem for vector optimization problem with variable order structure.

Key Words: Vector optimization, variable order structure, approximate solution, Ekeland's variational principle.

Mathematics subject classifications (MSC 2000): 90C29, 90C30,90C48,90C59.

1 Introduction

Vector optimization problems are useful and have many applications in economics theory, engineering design, management science and many other fields. For having an order in vector optimization problem, normally authors use a partial ordering by a nontrivial cone C in the image space. In 1974, Yu [40] defined a definition of optimal element of vector optimization with respect to the variable order structure named nondominated elements. A nondominated element is an element which is not dominated by other point with respect to the associated set to this other point by

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a set-valued map. Also, Chen, Huang and Yang [5, 9, 6] introduced another kind of optimal element which is called minimal. An minimal point is a candidate element which is not dominated by other point with respect to the associated set to this candidate point by a set-valued map. From the definitions, it is obvious to see that in nondominated element, we use a set associate to the other points but in minimal element, an ordering set is set associate to the minimal point. Some properties of these points can be found in [5, 9, 6, 14, 15, 25, 40]. Here, we also discuss about minimizers [9]. A minimizer is an element which is not dominated by another point with respect to the any set. Recently, we can see applications of vector optimization problem with respect to the variable order structure in decision making problems and some other areas [13]. Here, we will not discuss about exact solutions but approximate solutions and we know that exact solutions are special case of approximate solutions and our result can be used for exact solutions. Approximate solutions are a kind of solutions which we use for solving a optimization problems. We know that we need compactness assumption to have a exact solution. Also, we know that under the weak assumptions and without compactness condition, we need to make approximate solutions and we can have these solutions without any compactness assumptions. Also, if we apply numerical algorithms for solving optimization problem, then these algorithms usually generate approximate solutions which are close to the theoretical solutions. Several authors wrote about different approximate solutions with respect to the fixed order structure. See [24, 29, 31, 33, 37, 38, 39] for different definitions, concepts and properties of these elements. Also, Gutiérrez , Jiménez and Novo in [22] introduce a new concept of approximate solution of vector optimization problem and they unified some different concepts of approximate solutions with respect to the fixed order structure. But in this paper and [35], we deal with approximate nondominated, approximate minimal and approximate minimizers with respect to the variable order structure. Here, we use a set valued map which associate a set to the each element of the image space. By this ordering, we can define different kind of approximate elements like ϵk^0-nondominated, ϵk^0-minimizer and ϵk^0-minimal elements. In the case of fixed order structure all these definitions coincide and there is no difference between these elements. After showing some properties of approximate elements, we will define a nonlinear scalarization method for characterizing the approximate elements with respect variable order structure. We characterize approximate elements via this nonlinear scalarization method. We will not use convex hypothesis or cone here and our results are general and applicable for more general sets. In fact, we will use sets with some properties instead of cone for defining variable order and will not restrict ourself to the cone for order structure. The scalarization method is based on the suitable nonlinear functionals on the image space. Later, we will give an extension of Ekeland's theorem for vector optimization problem with variable order structure. Application of Ekeland's variational principle can be seen in economics, control theory, game theory, nonsmooth analysis and many others.

2 Preliminaries

Let Y be a linear topological space. A set $C \subseteq Y$ is called a cone if $\lambda c \in C$ for all $\lambda \geq 0$ and $c \in C$. The set C is said to be pointed if $C \cap (-C) \subseteq \{0\}$. Let Ω be a nonempty subset of linear topological space Y, we denote the topological interior of the set Ω by int (Ω), cl (Ω) denotes the topological closure, $\partial\Omega$ the topological boundary of Ω, Cone (Ω) the cone generated by Ω and conv (Ω) denotes the convex hull of a set Ω. A nonempty set $C \subset \mathbb{R}^m$ is said to be convex if $\lambda c^1 + (1-\lambda)c^2 \in C$ for all $c^1, c^2 \in C$ and $0 \leq \lambda \leq 1$. A set $C \subset Y$ is said to be solid if $\text{int}(C) \neq \emptyset$ and a set $C \subset Y$ is a proper set if $\emptyset \neq C \neq Y$. See [19, 20, 28] for basic definitions and concepts for vector optimization. Also, see [17, 18, 34] for some scalarization methods for solving vector optimization with respect to the fix ordering and some properties of these scalarization methods.

Also, suppose that $C : Y \rightrightarrows Y$ is a set valued map where $C(y)$ is a closed set with $0 \in \partial C(y)$ for every $y \in Y$. We define three different following domination relations: for $y^1, y^2, y^3 \in Y$

$$y^1 \leq_1 y^2 \quad \text{if} \quad y^2 \in y^1 + (C(y^1) \backslash \{0\}), \tag{1}$$

$$y^1 \leq_2 y^2 \quad \text{if} \quad y^2 \in y^1 + (C(y^2) \backslash \{0\}), \tag{2}$$

$$y^1 \leq_3 y^2 \quad \text{if} \quad y^2 \in y^1 + (C(y^3) \backslash \{0\}). \tag{3}$$

If $C(y^1) = C(y^2) = C(y^3)$ for all $y^1, y^2, y^3 \in Y$, then these three domination relations are same and problem reduces to the optimization with standard domination structure.

3 Different concepts of approximate solutions with variable order

Suppose that $\epsilon \geq 0$ and $k^0 \in Y \backslash \{0\}$. We define ϵk^0-nondominated, ϵk^0-minimal and ϵk^0-minimizers with respect to variable order structure. Furthermore, we define weakly (strongly) ϵk^0-minimal (nondominated) elements and weakly (strongly) ϵk^0-minimizers. For sure there is no difference between ϵk^0-nondominated, ϵk^0-minimal elements and ϵk^0-minimizers in the case of fixed order structure. This statement is also true for weakly (strongly) ϵk^0-optimal elements. In this section, we show that this statement can not be true in variable order structure and all these three definitions define different elements. This will be shown by several examples. In the following, we suppose that Y is a linear topological space.

Assumption (A1). Let Y be a linear topological space and $\Omega \subset Y$. Suppose that $C : Y \rightrightarrows Y$ is a set valued map where $C(y)$ is a closed set with $0 \in \partial C(y)$. We assume that $k^0 \in Y \backslash \{0\}$ such that $C(y) + [0, \infty)k^0 \subseteq C(y)$ for all $y \in \Omega$ and $\epsilon \geq 0$.

We define the first concept of approximate solution of variable order structure based on the domination relation (1). More details and properties of these points are given in [36].

Definition 3.1. *Let* Assumption (A1) *holds.*

1. $y_\epsilon \in \Omega$ *is said to be an* ϵk^0*-nondominated element of* Ω *with respect to the ordering map* $C : Y \rightrightarrows Y$ *if*
$$\forall y \in \Omega : \qquad (y_\epsilon - \epsilon k^0 - (C(y)\backslash\{0\})) \cap \{y\} = \emptyset.$$

2. *Suppose that* int $C(y) \neq \emptyset$ for all $y \in \Omega$. $y_\epsilon \in \Omega$ *is said to be a weakly* ϵk^0*-nondominated element of* Ω *with respect to the ordering map* $C : Y \rightrightarrows Y$ *if*
$$\forall y \in \Omega : \qquad (y_\epsilon - \epsilon k^0 - \text{int } C(y)) \cap \{y\} = \emptyset.$$

3. $y_\epsilon \in \Omega$ *is said to be a strongly* ϵk^0*-nondominated element of* Ω *with respect to the ordering map* $C : Y \rightrightarrows Y$ *if*
$$\forall y \in \Omega : \qquad y_\epsilon - \epsilon k^0 \in y - C(y).$$

If $\epsilon = 0$, then all these definitions coincide with the usual definitions of nondominated points (see [15, 40]). We denote the set of ϵk^0-nondominated, weakly ϵk^0-nondominated and strongly ϵk^0-nondominated elements by ϵk^0-$N(\Omega, C)$, ϵk^0-$WN(\Omega, C)$ and ϵk^0-$SN(\Omega, C)$ respectively.

Now, we define the second concept of approximate solution of variable order structure based on the domination relation (2). More details and properties of these points are given in [36].

Definition 3.2. *Let* Assumption (A1) *holds.*

1. $y_\epsilon \in \Omega$ *is said to be an* ϵk^0*-minimal element of* Ω *with respect to the ordering map* $C : Y \rightrightarrows Y$ *if*
$$(y_\epsilon - \epsilon k^0 - (C(y_\epsilon)\backslash\{0\})) \cap \Omega = \emptyset.$$

2. *Suppose that* int $C(y_\epsilon) \neq \emptyset$. $y_\epsilon \in \Omega$ *is said to be a weakly* ϵk^0*-minimal element of* Ω *with respect to the ordering map* $C : Y \rightrightarrows Y$ *if*
$$(y_\epsilon - \epsilon k^0 - \text{int } C(y_\epsilon)) \cap \Omega = \emptyset.$$

3. $y_\epsilon \in \Omega$ *is said to be a strongly* ϵk^0*-minimal element of* Ω *with respect to the ordering map* $C : Y \rightrightarrows Y$ *if*
$$\forall y \in \Omega : \qquad y_\epsilon - \epsilon k^0 \in y - C(y_\epsilon).$$

If $\epsilon = 0$, then all these definitions coincide with the usual definitions of minimal points (see [15, 25]). We denote the set of ϵk^0-minimal, weakly ϵk^0-minimal and strongly ϵk^0-minimal elements by ϵk^0-$M(\Omega, C)$, ϵk^0-$WM(\Omega, C)$ and ϵk^0-$SM(\Omega, C)$ respectively.

Now we define ϵk^0-minimizers based on the domination relation (3). More details and properties of these points are given in [36].

Definition 3.3. *Let* Assumption (A1) *holds.*

1. $y_\epsilon \in \Omega$ *is said to be an* ϵk^0*-minimizer of* Ω *with respect to the ordering map* $C : Y \rightrightarrows Y$ *if*

$$\forall y, y^1 \in \Omega : \qquad (y_\epsilon - \epsilon k^0 - (C(y)\backslash\{0\})) \cap \{y^1\} = \emptyset.$$

2. *Suppose that* int $C(y) \neq \emptyset$ *for all* $y \in \Omega$. $y_\epsilon \in \Omega$ *is said to be a weakly* ϵk^0*-minimizer of* Ω *with respect to the ordering map* $C : Y \rightrightarrows Y$ *if*

$$\forall y, y^1 \in \Omega : \qquad (y_\epsilon - \epsilon k^0 - \text{int } C(y)) \cap \{y^1\} = \emptyset.$$

3. $y_\epsilon \in \Omega$ *is said to be a strongly* ϵk^0*-minimizer of* Ω *with respect to the ordering map* $C : Y \rightrightarrows Y$ *if*

$$\forall y^1, y^2 \in \Omega : \qquad y_\epsilon - \epsilon k^0 \in y^1 - C(y^2).$$

If $\epsilon = 0$, then these definitions are the definitions of the exact minimizer, weakly minimizer and strongly minimizer elements and coincide with Definition 1.11 [9]. We denote the set of ϵk^0-minimizers, weakly ϵk^0-minimizers and strongly ϵk^0-minimizers by ϵk^0-$MZ(\Omega, C)$, ϵk^0-$WMZ(\Omega, C)$ and ϵk^0-$SMZ(\Omega, C)$ respectively.

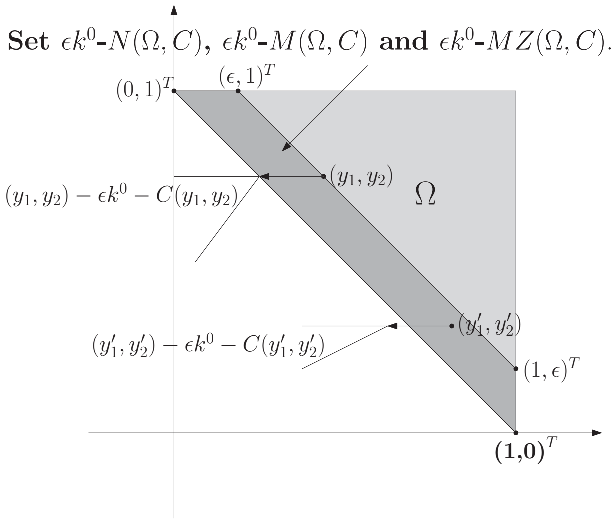

Figure 1: Example 3.4 where the set of ϵk^0-$N(\Omega, C)$, ϵk^0-$MZ(\Omega, C)$ and ϵk^0-$M(\Omega, C)$ of Ω coincide.

Example 3.4. *Let* $\epsilon = \frac{1}{100}$ *and* $k^0 = (1, 0)^T$. *Also, suppose that*

$$\Omega = \{(y_1, y_2)|\ \{(y_1 + y_2 \geq 1)\} \cap \{0 \leq y_1, y_2 \leq 1\}\}$$

and

$$C(y_1, y_2) = \begin{cases} \mathbb{R}^2_+ & \textit{if } y_1 = 0 \\ \textit{Cone conv}\{(1, 0)^T, (y_1, y_2)\} & \textit{otherwise.} \end{cases}$$

It is easy to see that cl $C(y) + [0,\infty)k^0 \subseteq$ cl $C(y)$ *for all* $y \in \Omega$. *Then* $\{(y_1, y_2) \in \Omega |\ y_1 + y_2 \leq 1 + \frac{1}{100}\}$ *are* ϵk^0*-nondominated,* ϵk^0*-minimizer and also* ϵk^0*-minimal elements and the sets of all these points coincide (see Figure 1).*

In the case of fixed order structure, ϵk^0-nondominated (weakly ϵk^0-nondominated) elements, ϵk^0-minimizer (weakly ϵk^0-minimizer) and ϵk^0-minimal (weakly ϵk^0-minimal) elements coincide but the following examples shows that this is not true when we are talking about variable order structure.

Example 3.5. *Let* $\epsilon = \frac{1}{100}$ *and* $k^0 = (1, 0)^T$. *Also suppose that*

$$\Omega = \{(y_1, y_2)|\ 0 \leq y_1, y_2 \leq 1\}.$$

and

$$C(y_1, y_2) = \begin{cases} \{(d_1, d_2) \in \mathbb{R}^2 |\ d_1 \geq 0,\ d_2 \leq 0\} & \text{if } y_1 = 0 \\ Cone\ conv\{(1,0)^T, (y_1, y_2)\} & \text{otherwise.} \end{cases}$$

It is easy to see that cl $C(y) + [0,\infty)k^0 \subseteq$ cl $C(y)$ *for all* $y \in \Omega$. *Then* $\{(y_1, y_2) \in \Omega |\ y_1 \leq \epsilon\}$ *is the set of* ϵk^0*-minimal points but just the elements of the set* $\{(y_1, y_2) \in \Omega |\ y_1 < \epsilon\} \bigcup \{(\epsilon, 1)^T\}$ *are* ϵk^0*-nondominated and* ϵk^0*-minimizers (see Figure 2).*

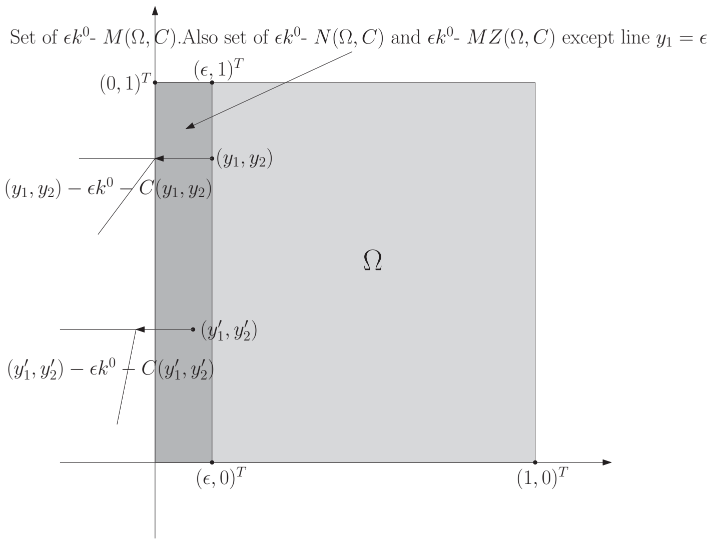

Figure 2: Example 3.5 where there exists an ϵk^0-minimal element of the set Ω which is not ϵk^0-minimizer and ϵk^0-nondominated element.

Example 3.6. *Let* $\epsilon = \frac{1}{100}$ *and* $k^0 = (1, 1)^T$. *Also suppose that*

$$\Omega = \Big\{(y_1, y_2) \in \mathbb{R}^2 |\ \{y_1 + y_2 \geq -1\} \cap \{y_1 \leq 0,\ y_2 \leq 0\}\Big\}$$

and

$$C(y_1, y_2) = \begin{cases} \{(d_1, d_2) \in \mathbb{R}^2 |\ d_2 \geq 0,\ d_1 + d_2 \geq -1\} & \text{for } (y_1, y_2) = (-1, 0)^T \\ \mathbb{R}^2_+ & \text{otherwise.} \end{cases}$$

It is easy to see that $\text{cl } C(y) + [0, \infty)k^0 \subseteq \text{cl } C(y)$ for all $y \in \Omega$. Then $\{(y_1, y_2) \in \Omega \mid y_1 + y_2 \leq -\frac{98}{100},\ y_1 \neq -1\}$ is the set of ϵk^0-minimal elements, but $(-1, 0)^T$ is not minimal. However $(-1, 0)^T$ is an ϵk^0-nondominated point and $\{(y_1, y_2) \in \Omega |\ y_1 + y_2 \leq -\frac{98}{100}\}$ is the set of all ϵk^0-nondominated elements. Obviously, $(-1, 0)^T$ is not a ϵk^0-minimizer and $\{(y_1, y_2) \in \Omega |\ -1 < y_2 \leq -1 + \epsilon\}$ is the set of ϵk^0-minimizers (see Figure 3).

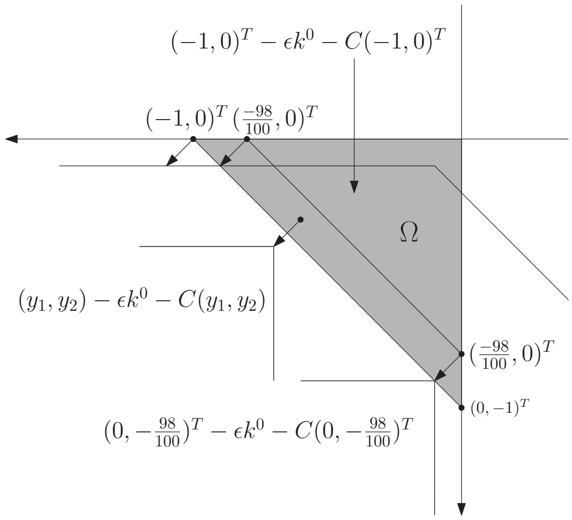

Figure 3: Example 3.6 where $(-1, 0)^T$ is an ϵk^0-nondominated of the set Ω, but it is neither ϵk^0-minimizer nor ϵk^0-minimal element.

In the following example, we show that there are some points which are ϵk^0-nondominated and also ϵk^0-minimal but they are not ϵk^0-minimizers.

Example 3.7. *Let* $\epsilon = \frac{1}{100}$ *and* $k^0 = (0, 1)^T$. *Also suppose that*

$$\Omega = \left\{(y_1, y_2) \in \mathbb{R}^2_+ |\ \{y_1 + y_2 \geq 2\} \cap \{y_1 \geq 0,\ 0 \leq y_2 \leq 2\}\right\}.$$

and

$$C(y_1, y_2) = \begin{cases} \{(d_1, d_2) \in \mathbb{R}^2 |\ d_1 \leq 0,\ d_2 \geq 0\} & \text{for } (y_1, y_2) = (4, 2)^T \\ \mathbb{R}^2_+ & \text{otherwise.} \end{cases}$$

It is easy to see that $\text{cl } C(y) + [0, \infty)k^0 \subseteq \text{cl } C(y)$ *for all* $y \in \Omega$. *Then* $\{(y_1, y_2) \in \Omega \mid y_1 + y_2 \leq 2 + \epsilon\}$ *is the set of* ϵk^0*-minimal and* ϵk^0*-nondominated points. But only points of the set* $\left\{(y_1, y_2) \in \Omega |\ y_2 < \epsilon = \frac{1}{100}\right\}$ *are* ϵk^0*-minimizer. This shows that there are some points which are both* ϵk^0*-nondominated and* ϵk^0*-minimal but not* ϵk^0*-minimizer (see Figure 4).*

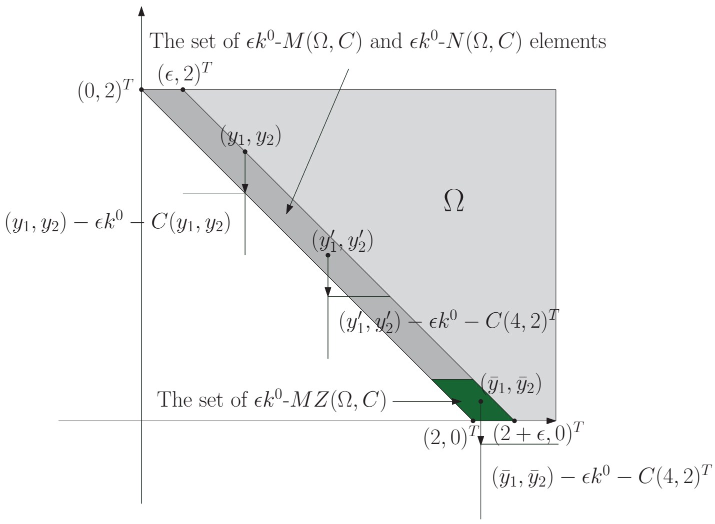

Figure 4: Example 3.7 where there exists an element which is both ϵk^0-nondominated and ϵk^0-minimal but not ϵk^0-minimizer.

In [36], approximate solutions with respect to variable order structure are studied with more details. Relation between different kind of approximate solutions, relation between exact and approximate solution with variable order structure and properties of approximate solution with variable order are discussed in [36].

4 Scalarization via Nonlinear Functionals

In this section, we present a scalarization method with the help of nonlinear functionals. This scalarization was introduced by Tammer and Weinder [17] in 1983 and one year later by Pascoletti-Serafini [34] in 1984. Some generalization of this scalarization method for variable order structure where the ordering map is pointed, closed convex cone-valued can be found in [9, 10, 11, 16]. Here, we give a generalization of Tammer-Weinder functional without any cone or convexity assumption and we use it for characterization of all our three different approximate solutions. In fact, our ordering map is just a set-valued map with some properties. For sure, our scalarization also works in the case that map is cone and convex cone valued.

Assumption (A2). Let $k^0 \in Y\backslash\{0\}$ and $C : Y \rightrightarrows Y$ be a set-valued map where $C(y)$ is a proper, closed set with $0 \in \partial C(y)$, $C(y) + (0, \infty)k^0 \subseteq \text{int } C(y)$ for all $y \in Y$.

For $y \in Y$, we define a functional $\theta_y : \Omega \to \mathbb{R}$ in the following way

$$\theta_y(y^1) := \inf\{t \in \mathbb{R} \mid y^1 \in tk^0 - C(y)\}. \tag{4}$$

Eichfelder [16] used the following generalization of Tammer-Weidner functional:

$$\psi(y) := \inf\{t \in \mathbb{R} \mid y \in tk^0 - C(y)\} \tag{5}$$

for characterization of exact nondominate and minimal solutions where $C(y)$ is pointed closed convex cone for all $y \in Y$. Note that in (5), pointed closed convex cone $C(y)$ is the cone which associated to the same y but for the functional in (4), $C(y)$ is independent from y^1. Chen, Huang and Yang [9], Chen and Yang [10], Chen, Yang and Yu [11] used the following generalization of Tammer-Weidner functional:

$$\zeta(y, z) := \inf\{t \in \mathbb{R} \mid z \in tk(y) - C(y)\} \tag{6}$$

for characterization of exact nondominate and minimal solutions where $C(y)$ is pointed closed convex cone for all $y \in Y$ and $k(y) \in \text{int } C(y)$. Chen and Yang [10] proved similar result to Theorem 4.5. Chen, Yang and Yu [11] proved that if $C : Y \rightrightarrows Y$ is linear set-valued map and if $C(y)$ is pointed closed convex cone for all $y \in Y$, then the functional (6) is subadditive in second variable. Chen, Huang and Yang [9] proved that if $C : Y \rightrightarrows Y$ is continuous set-valued map and if $C(y)$ is pointed closed convex cone for all $y \in Y$, then the functional (6) is lower semicontinuous and homogenous in second variable. We begin this section with the following lemma.

Lemma 4.1. *Let* Assumption (A2) *holds,* $y_1, y_2 \in Y$ *and suppose that* $\theta_{y^2}(y^1) = t_1$, *then for any* $t_2 > t_1$, *the following holds:*

$$y^1 \in t_2k^0 - C(y^2). \tag{7}$$

Proof. By $C(y) + (0, \infty)k^0 \subseteq \text{int } C(y)$, $y^1 \in t_1k^0 - C(y^2)$ and $t_2 - t_1 > 0$, we can write:

$$y^1 \in t_1k^0 - C(y^2) = t_2k^0 - [(t_2 - t_1)k^0 + C(y^2)] \subseteq t_2k^0 - C(y^2). \quad \square$$

Two important properties which we need to show for our scalarizing functional are well-define and properness. In the following two theorems, we will show these properties. First, we show that under some assumptions, our scalarizing functional is proper.

Theorem 4.2. *The functional* θ_y *defined in (4) is proper for any* $y \in Y$ *if and only if for any* $y \in Y$, $C(y)$ *does not contain any line parallel to* k^0, *i.e.*

$$\forall y^1, y^2 \in Y, \quad \exists t \in \mathbb{R} \ : \quad y^1 \notin tk^0 - C(y^2).$$

Proof. Suppose that $\theta_{y^2}(y^1) = -\infty$, then by Lemma 4.1, for any $t > -\infty$, we have $y^1 \in tk^0 - C(y^2)$ and $\{tk^0 - y^1 \mid t \in \mathbb{R}\} \subset C(y^2)$ and this means that there exists $y^2 \in Y$ such that $C(y^2)$ contains a line parallel to k^0 and this leads to contradiction. Obviously, if there exists a parallel line to k^0, then $\{tk^0 - y^1 \mid t \in \mathbb{R}\} \subset C(y^2)$ for any t and $\theta_{y^2}(y^1) = -\infty$ $\square$.

Now we show that our scalarizing functional is well-defined. For showing this property, we need to assume some assumptions. The following theorem shows us if just one of the assumptions (1), (2) or (3) holds, then our functional is well-defined.

Theorem 4.3. *Let* Assumption (A2) *holds. If one of the following properties hold for each* $y \in \Omega$, *then* $\theta_y : \Omega \to \mathbb{R}$ *in (4) is well-defined.*

1. $C(y)$ *does not contain any parallel line to* k^0.
2. $C(y)$ *is pointed.*
3. *There exists a cone* $D \subseteq Y$ *such that* $k^0 \in \text{int } D$ *and* $C(y) + \text{int } D \subseteq \text{int } C(y)$.

Proof.

1. For any $y^1, y^2 \in Y$, let $S = \{t \in \mathbb{R} \mid y^1 \in tk^0 - C(y^2)\}$. We just need to prove that this set is closed and it is bounded from below. Since $C(y)$ does not contain any parallel line to k^0, then by Theorem 4.2, we know that there is $t_1 \in \mathbb{R}$ such that $y^1 \notin t_1k^0 - C(y^2)$. Also by Lemma 4.1, we know that for any $t_2 < t_1$, $y^2 \notin t_2k^0 - C(y^2)$, otherwise since $t_2 < t_1$ then $y^1 \in t_1k^0 - C(y^2)$. Therefore S is bounded from below. Now, we prove that S is closed. Suppose that $t_n \in S$ is a sequence such that $t_n \to t$. Since $t_n \in S$, then
$$y^1 \in t_nk^0 - C(y^2) \Rightarrow t_nk^0 - y^1 \in C(y^2).$$
Since $C(y^2)$ is closed for all $y \in Y$, then the limit point $tk^0 - y^1 \in C(y^2)$ and this implies that S is closed and proof is done.
2. Pointedness of $C(y)$ implies that $C(y)$ does not contain any parallel line to k^0 and the rest of proof is similar.
3. By Proposition 2.3.4 of [19], we know that $C(y)$ does not contain any parallel line to k^0 and the rest of proof is exactly same as part 1. □

In the special case that $C(x)$ is pointed convex cone we have the following corollary [10].

Corollary 4.4. *Suppose that* $C(y)$ *is a convex cone for all* $y \in Y$ *in Theorem 4.3 and* $k^0 \in \bigcap_{y \in Y} \text{int } C(y)$, *then* θ_y *is well-defined.*

Proof. Obviously, when $C(y)$ is a convex cone, then by $C(y) = D$ and part 3 of Theorem 4.3, we have $C(y) + \text{int } C(y) \subseteq \text{int } C(y)$ and similar to proof of part 1 of Theorem 4.3, θ_y is well-defined. □

The following theorem shows some important properties of the functional θ_y in (4) and we will use it to prove other properties of our scalarizing functional like subadditivity, monotonicity and other theorems in the next sections. In the following theorem, we give some properties of our scalarizing functional. Also, see Chen and Yang [10] for the case that C is pointed, convex cone-valued map.

Theorem 4.5. *Let* Assumption (A2) *holds. For any* $y^1, y^2 \in Y$, *we have the following properties concerning* θ_{y^2},

1. $\theta_{y^2}(y^1) < t \Leftrightarrow y^1 \in tk^0 - \text{int } C(y^2)$.
2. $\theta_{y^2}(y^1) \leq t \Leftrightarrow y^1 \in tk^0 - \text{cl } C(y^2)$.
3. $\theta_{y^2}(y^1) = t \Leftrightarrow y^1 \in tk^0 - \partial C(y^2)$.
4. $\theta_{y^2}(y^1) \geq t \Leftrightarrow y^1 \notin tk^0 - \text{int } C(y^2)$.
5. $\theta_{y^2}(y^1) > t \Leftrightarrow y^1 \notin tk^0 - C(y^2)$.

Proof.

1. Suppose that $\theta_{y^2}(y^1) < t$, then there exists $\gamma > 0$ such that $\theta_{y^2}(y^1) + \gamma = t$. By definition of $\theta_{y^2}(y^1)$, we can write

   $$y^1 \in \theta_{y^2}(y^1)k^0 - C(y^2) \Rightarrow \theta_{y^2}(y^1)k^0 + \gamma k^0 - y^1 \in C(y^2) + \gamma k^0.$$

   By the assumption $C(y) + (0, \infty)k^0 \subseteq \text{int } C(y)$ for any $y \in Y$, we have $tk^0 - y^1 \in \text{int } C(y^2)$.

   Now suppose that $y^1 \in tk^0 - \text{int } C(y^2)$, therefore there exists $c^1 \in \text{int } C(y^2)$ such that

   $$y^1 = tk^0 - c^1. \quad (8)$$

   Since $c^1 \in \text{int } C(y^2)$, then there exists $\gamma > 0$ such that $c^1 - \gamma k^0 \in C(y^2)$. By this and (8):

   $$y^1 = (t - \gamma)k^0 - (c^1 - \gamma k^0) \Rightarrow y^1 \in (t - \gamma)k^0 - C(y^2).$$

   Hence $\theta_{y^2}(y^1) \leq (t - \gamma) < t$.

2. Suppose that $\theta_{y^2}(y^1) \leq t$, then $\theta_{y^2}(y^1) = t$ or $\theta_{y^2}(y^1) < t$. In the case of $\theta_{y^2}(y^1) < t$, by part 1:

   $$y^1 \in tk^0 - \text{int } C(y^2) \Rightarrow y^1 \in tk^0 - C(y^2).$$

   Now suppose that $\theta_{y^2}(y^1) = t$ and there exists a sequence $t_n \to t$ such that $t < t_n$ and $\theta_{y^2}(y^1) < t_n$. By part 1,

   $$y^1 \in t_n k^0 - \text{int } C(y^2) \Rightarrow t_n k^0 - y^1 \in C(y^2).$$

   By $t_n k^0 - y^1 \to tk^0 - y^1$ and since $C(y^2)$ is a closed set, then $tk^0 - y^1 \in C(y^2)$ and $y^1 \in tk^0 - C(y^2)$.

   Now suppose $y^1 \in tk^0 - C(y^2)$, then obviously from definition $\theta_{y^2}(y^1) \leq t$.

3. Suppose that $\theta_{y^2}(y^1) = t$, then $y^1 \in tk^0 - C(y^2)$ and this means $y^1 \in tk^0 - \partial C(y^2)$ or $y^1 \in tk^0 - \text{int } C(y^2)$. If $y^1 \in tk^0 - \partial C(y^2)$, then we are done. But suppose that $y^1 \in tk^0 - \text{int } C(y^2)$, then by part 1, $\theta_{y^2}(y^1) < t$ and this is contradiction to our assumption.

   Now suppose that $y^1 \in tk^0 - \partial C(y^2)$, then obviously $y^1 \in tk^0 - C(y^2)$ and $\theta_{y^2}(y^1) \leq t$. If $\theta_{y^2}(y^1) \neq t$, then $\theta_{y^2}(y^1) < t$ and by part 1, $y^1 \in tk^0 - \text{int } C(y^2)$ which is contradiction to our assumption.

4. This follows from part 1.

5. This follows from part 2. □

In the following, we will prove some important properties of our scalarization functional. In fact, we prove that our scalarizing functional is lower semicontinuous, subadditive, homogenous, monotone and continuous in the case that some assumptions hold. These properties are important for us and we will use them in the next sections and later they will help us to write a generalization of variational principle for vector optimization with variable order structure. First, we show lower semicontinuity.

Theorem 4.6. *Let* Assumption (A2) *holds, then θ_z in (4) is lower semicontinuous for any $z \in Y$.*

Proof. We need to show that for any $t \in \mathbb{R}$, the set

$$S_t = \{y \in Y \mid \theta_z(y) \leq t\}$$

is a closed set. For this, we suppose that $y^n \to y^0$ is a sequence and $y^n \in S_t$. We show that the limit point of this sequence belongs to the set S_t and this proves that S_t is a closed set. Since $y^n \in S_t$, then $\theta_z(y^n) \leq t$. Now by part 2 of Theorem 4.5, we have,

$$y^n \in tk^0 - C(z) \Rightarrow \ tk^0 - y^n \in C(z).$$

Since $C(z)$ is a closed set, then the limit point of the sequence $tk^0 - y^n \to tk^0 - y^0$ also belongs to $C(z)$ and $y^0 \in tk^0 - C(z)$ and by part 2 of Theorem 4.5, we can write $\theta_z(y^0) \leq t$. This means that S_t is a closed set for any $t \in \mathbb{R}$ and θ_z is lower semicontinuous for any $z \in Y$. □

Theorem 4.7. *Let* Assumption (A2) *holds. For each $y \in \Omega$, θ_y in (4) is homogeneous if and only if $C(y)$ is a cone.*

Proof. Suppose that $\lambda > 0$, then for any $y^1, y^2 \in Y$, we have:

$$\theta_{y^2}(\lambda y^1) = \inf \ \{t \in \mathbb{R} \mid \lambda y^1 \in tk^0 - C(y^2)\}.$$

Since C is a cone, we have $C(y^2) = \lambda C(y^2)$ and

$$\theta_{y^2}(\lambda y^1) = \inf\{t \in \mathbb{R} \mid \lambda y^1 \in tk^0 - \lambda C(y^2)\} = \lambda \inf\{\frac{t}{\lambda} \in \mathbb{R} \mid y^1 \in \frac{t}{\lambda}k^0 - C(y^2)\},$$

so by $t' = \frac{t}{\lambda}$, we can write:

$$\theta_{y^2}(\lambda y^1) = \lambda \inf\{t' \in \mathbb{R} \mid y^1 \in t'k^0 - C(y^2)\} = \lambda\theta_{y^2}(y^1).$$

Now assume that θ_y is homogenous and take $y^1 \in C(y)$. Obviously $0 \in C(y)$ and by part 2 of Theorem 4.5, $\theta_y(-y^1) \leq 0$. Since θ_y is homogeneous, we obtain

$$\theta_y(-\lambda y^1) \leq \lambda\theta_y(-y^1) \leq 0.$$

Again by part 2 of Theorem 4.5, $-\lambda y^1 \in C(y)$ and $\lambda C(y) \subseteq C(y)$. Now suppose that $y^1 \in C(y)$, then by part 2 of Theorem 4.5

$$\theta_y(-y^1) \leq 0 \Rightarrow \lambda\theta_{y^2}(-\frac{y^1}{\lambda}) \leq 0.$$

By $\lambda > 0$, we get $\frac{y^1}{\lambda} \in C(y)$ and $y^1 \in \lambda C(y)$ and this implies $C(y) \subseteq \lambda C(y)$. Hence, $C(y) = \lambda C(y)$ for any $\lambda > 0$ and $y \in Y$ and C is a cone. □

Subadditivity of the scalarizing functional is one of important properties. We need this property in the next section for the characterization of approximate non-dominated, minimal and minimizers. Also, subadditivity is a important property for the writing variational principle for vector optimization with variable order structure.

Theorem 4.8. *Let* Assumption (A2) *holds.* θ_y *in (4) is subadditive if and only if* $C(y) + C(y) \subseteq C(y)$ *for all* $y \in Y$.

Proof. Suppose that $C(y) + C(y) \subseteq C(y)$ for all $y \in Y$ and take $y^1, y^2 \in Y$. Let $t_1, t_2 \in \mathbb{R}$, then by part 2 of Theorem 4.5

$$\theta_y(y^1) = t_1 \Rightarrow y^1 \in t_1k^0 - C(y). \tag{9}$$

$$\theta_y(y^2) = t_2 \Rightarrow y^2 \in t_2k^0 - C(y). \tag{10}$$

By (9), (10) and since $C(y) + C(y) \subseteq C(y)$, we can write:

$$y^1 + y^2 \in (t_1 + t_2)k^0 - (C(y) + C(y)) \subseteq (t_1 + t_2)k^0 - C(y).$$

By part 2 of Theorem 4.5, $\theta_y(y^1 + y^2) \leq t_1 + t_2 = \theta_y(y^1) + \theta_y(y^2)$.

Now assume that θ_y is subadditive. We show that $C(y) + C(y) \subseteq C(y)$. Take $y^1, y^2 \in C(y)$. By part 2 of Theorem 4.5 and $y^1 \in C(y)$, then $\theta_y(-y^1) \leq 0$ and by $y^2 \in C(y)$, then $\theta_y(-y^2) \leq 0$. Since θ_y is subadditive, we obtain

$$\theta_y(-y^1 - y^2) \leq \theta_y(-y^1) + \theta_y(-y^2) \leq 0.$$

Again by part 2 of Theorem 4.5, $-y^1 - y^2 \in C(y)$ and this proofs that $C(y) + C(y) \subseteq C(y)$. □

For sure, there are a lot of new things about variable order structure and one of the important things is how to define the convexity of the functional with respect to the ordering map. In the vector optimization problem with fixed order structure, convexity of the functional is equal to the convexity of the epigraph, i.e. the scalarizing functional is convex if and only if its epigraph is convex. But unfortunately, for the definitions of convexity for variable order structure in literature, this is not true and convexity of epigraph does not imply a convex functional. See [9] for more details about convexity in variable order structure but still there exist no unified definition of convexity in variable order structure and relation between the convexity of epigraph and convexity of the functional is not known yet. We say that our functional $\theta_y : \Omega \to \mathbb{R}$ is convex if for all $y^1, y^2 \in Y$ and $0 < \lambda < 1$ the following inequality holds,

$$\theta_y(\lambda y^1 + (1-\lambda)y^2)) \leq \lambda\theta_y(y) + (1-\lambda)\theta_y(y^2).$$

Theorem 4.9. *Let* Assumption (A2) *holds. For all $y \in \Omega$, θ_y is convex if and only if $C(y)$ be a convex cone for each $y \in \Omega$.*

Proof. Suppose that $\lambda \in [0,1]$, $y^1, y^2 \in Y$ such that $\theta_y(y^1) = t_1$ and $\theta_y(y^2) = t_2$. By part 2 of Theorem 4.5, $y^1 \in t_1k^0 - C(y)$ and $y^2 \in t_2k^0 - C(y)$ and since C is a convex cone, we can write,

$$\begin{aligned}\lambda y^1 + (1-\lambda)y^2 &\in \lambda t_1k^0 + (1-\lambda)t_2k^0 - (C(y) + C(y))\\ &\subseteq (\lambda t_1 + (1-\lambda)t_2)k^0 - C(y).\end{aligned}$$

Therefore

$$\theta_y(\lambda y^1 + (1-\lambda)y^2)) \leq \lambda\theta_y(y) + (1-\lambda)\theta_y(y^2),$$

this means that θ_y is convex.

Now suppose that θ_y is convex, $y^1, y^2 \in C(y)$ and $\lambda \in (0,1)$. By $y^1, y^2 \in C(y)$ and part 2 of Theorem 4.5, $\theta_y(y^1) \leq 0$ and $\theta_y(y^2) \leq 0$ and by convexity of θ_y, we can write

$$\theta_y(\lambda y^1 + (1-\lambda y^2)) \leq \lambda\theta_y(y^1) + (1-\lambda)\theta_y(y^2) \leq 0.$$

Therefore, by part 2 of Theorem 4.5, $\lambda y^1 + (1-\lambda y^2) \in C(y)$ and $C(y)$ is convex. □

Last theorem of this section is about the monotonicity of our scalarizing functional. In this theorem, we will prove some monotonicity properties of our scalarization functional and these properties will be used in the next section for characterization of approximate optimal points of vector optimization problem with variable order structure and also in the last section about variational principle of vector optimization with variable order structure.

Theorem 4.10. *Let Y be a linear topological space, $k^0 \in Y\backslash\{0\}$, $D \subseteq Y$ and* Assumption (A2) *holds, then the following properties hold for θ_y in (4):*

1. *θ_y is D-monotone $\Leftrightarrow C(y) + D \subset C(y)$ for all $y \in Y$.*

2. $\forall\, y^1, y^2 \in Y,\ t_1 \in \mathbb{R}:$ $\quad \theta_{y^2}(y^1+t_1k^0) = \theta_{y^2}(y^1)+t_1$ (translation property).

   Let furthermore $C(y) + (0,\infty)k^0 \subseteq \operatorname{int} C(y)$ for all $y \in Y$. Then

3. *θ_y is continuous for all $y \in Y$.*

4. *If θ_y is proper, then θ_y is D-monotone $\Leftrightarrow C(y) + D \subseteq C(y) \Leftrightarrow \partial C(y) + D \subseteq C(y)$. Moreover, if θ_y is finite-valued, then θ_y is strictly D-monotone $\Leftrightarrow C(y)\backslash\{0\} + D \subseteq \operatorname{int} C(y) \Leftrightarrow \partial(C(y)\backslash\{0\}) + D \subseteq \operatorname{int} C(y)$.*

Proof.

1. Suppose that $C(y) + D \subset C(y)$ for all $y \in Y$ and $y^1 \leq_D y^2$, we prove that $\theta_y(y^1) \leq \theta_y(y^2)$ for any $y \in Y$. W.L.O.G, choose $y \in Y$ arbitrarily and suppose that $\theta_y(y^2) = t$. By part 2 of Theorem 4.5

   $$y^2 \in tk^0 - C(y). \tag{11}$$

   Since $y^1 \leq_D y^2$, then there exists $d \in D$ such that $y^1 + d = y^2$. By (11), we can write

   $$y^2 = y^1 + d \in tk^0 - C(y) \Rightarrow y^1 \in tk^0 - (C(y) + d) \subseteq tk^0 - C(y).$$

   Again by part 2 of Theorem 4.5, $\theta_y(y^1) \leq t = \theta_y(y^2)$.

   Now suppose that θ_y is D-monotone and choose $d \in D$ and $y^1 \in C(y)$ arbitrarily. Since $y^1 \in C(y)$, then by part 2 of Theorem 4.5, $\theta_y(-y^1) \leq 0$. Also, since θ_y is D-monotone, then $\theta_y(-y^1 - d) \leq 0$ and again by part 2 of Theorem 4.5

   $$-y^1 - d \in -C(y) \Rightarrow y^1 + d \in C(y) \qquad \forall y^1 \in C(y),\ \forall d \in D.$$

   Since y, y^1, d were chosen arbitrarily, then $C(y) + D \subseteq C(y)$.

2. Suppose that $\theta_{y^2}(y^1) = t$. By part 2 of Theorem 4.5,

   $$y^1 \in tk^0 - C(y^2) \Rightarrow y^1 + t_1k^0 \in (t + t_1)k^0 - C(y^2) \Rightarrow \theta_{y^2}(y^1 + t_1k^0) = t + t_1,$$

   and this means that $\theta_{y^2}(y^1 + t_1k^0) = \theta_{y^2}(y^1) + t_1$.

3. By Theorem 4.6, we know that θ_y is lower semicontinuous and we just need to prove that it is also upper semicontinuous. Therefore we need to show that for any $t \in \mathbb{R}$, the set

   $$\bar{S}_t = \{y^1 \in Y |\ \theta_y(y^1) \geq t\}$$

   is a closed set. For this, we suppose that $y^n \to y^0$ is a sequence and $y^n \in \bar{S}_t$. We show that the limit point of this sequence belongs to the set $\bar{S}_t$ and this proves that $\bar{S}_t$ is a closed set. Since $y^n \in \bar{S}_t$, then $\theta_y(y^n) \geq t$. Now by part 4 of Theorem 4.5, we have,

$$y^n \notin tk^0 - \text{int } C(y) \Rightarrow tk^0 - y^n \notin \text{int } C(y) \Rightarrow tk^0 - y^n \in (\text{int } C(y))^c.$$

Since int $C(y)$ is open, then the complement $(\text{int } C(y))^c$ and includes all the limit points. Therefore $tk^0 - y^0 \in (\text{int } C(y))^c$ and this means

$$tk^0 - y^0 \notin \text{int } C(y) \Rightarrow y^0 \notin tk^0 - \text{int } C(y).$$

Again by part 4 of Theorem 4.5, we have $\theta_y(y^0) \geq t$ and this implies that $\bar{S}_t$ is a closed set and θ_y is upper semicontinuous. Since θ_y is also lower semicontinuous, then θ_y is continuous.

4. We prove the second part, the first part is similar.

   Assume that θ_y is strictly D-monotone and take $y^1 \in C(y)$ and $d \in D\backslash\{0\}$. Since $y^1 \in C(y)$, then by part 2 of Theorem 4.5, $\theta_y(-y^1) \leq 0$, and so, by hypothesis $\theta_y(-y^1 - d) < 0$. By part 1 of Theorem 4.5

$$-y^1 - d \in -\text{int } C(y) \Rightarrow y^1 + d \in \text{int } C(y) \qquad \forall y^1 \in C(y),\ \forall d \in D.$$

   Now, suppose that $\partial C(y) + (D\backslash\{0\}) \subset \text{int } C(y)$ for all $y \in Y$ and $y^1, y^2 \in Y$ with $y^2 - y^1 \in D\backslash\{0\}$. From part 3 of Theorem 4.5 we have that $y^2 \in \theta_y(y^2)k^0 - \partial C(y)$, and so

$$y^1 + d \in \theta_y(y^2)k^0 - \partial C(y) \Rightarrow$$
$$y^1 \in \theta_y(y^2)k^0 - (\partial C(y) + (D\backslash\{0\})) \subset \theta_y(y^2)k^0 - \text{int } C(y).$$

   By part 1 of Theorem 4.5, we obtain $\theta_y(y^1) < \theta_y(y^2)$. The remaining implication is obvious. □

5 Characterization of ϵk^0-optimal elements by scalarization via nonlinear functionals

In the scalarization of vector optimization problem, we replace the original vector optimization problem with scalar-valued optimization problem to characterize the optimal elements. In this section, we characterize ϵk^0-optimal elements by scalarization via nonlinear functionals. By this scalarization, we show that approximate solution of original vector optimization problem is also a solution for the scalar problem and vice versa. In the following theorem, we show that ϵk^0-minimizer of the set Ω is a solution of scalar optimization problem. For the scalarization, we generalized scalarization method by Tammer and Weidner. For more details and some properties of this scalarization method in the case of fixed order structure, see [17, 18, 34].

Assumption (A3). Let Ω be a subset of Y, $\epsilon \geq 0$ and $k^0 \in Y\backslash\{0\}$. Also, suppose that $C : Y \rightrightarrows Y$ is a set-valued map where $C(y)$ is a proper, pointed, closed and solid set with $0 \in \partial C(y)$, $C(y) + (0, \infty)k^0 \subseteq \text{int } C(y)$ for any $y \in Y$.

Theorem 5.1. *Let* Assumption (A3) *holds.*

1. *If $y_\epsilon \in \Omega$ is an ϵk^0-minimizer of the set $\Omega \subseteq Y$, then $\theta_y(0) \leq \theta_y(z - y_\epsilon) + \epsilon$ for all $y, z \in \Omega$, where $\theta_y(z) = \inf \{t \in \mathbb{R} \mid tk^0 - z \in C(y)\}$.*

2. *If $y_\epsilon \in \Omega$ is a weakly ϵk^0-minimizer of Ω, then $\theta_y(0) \leq \theta_y(z - y_\epsilon) + \epsilon$ for all $y, z \in \Omega$.*

3. *If $y_\epsilon \in \Omega$ is a strongly ϵk^0-minimizer of Ω, then $\theta_y(0) < \theta_y(z - y_\epsilon) + \epsilon$ for all $y, z \in \Omega$.*

Proof.

1. Suppose that y_ϵ is an ϵk^0-minimizer of the set Ω and there exist $y, z \in \Omega$ such that $\theta_y(z - y_\epsilon) + \epsilon < \theta_y(0) = t$. First, we prove that $t = 0$. By part 2 of Theorem 4.5,

$$\theta_y(0) = t \Rightarrow tk^0 - 0 \in C(y) \Rightarrow tk^0 \in C(y), \tag{12}$$

   by $0 \in \partial C(y)$ for all $y \in \Omega$, then $t \leq 0$. Also, by $0 \in \partial C(y)$, $C(y) + [0, \infty)k^0 \subseteq C(y)$ and since $C(y)$ is pointed, then $t \geq 0$ and $t = 0$.

   Since $\theta_y(z - y_\epsilon) + \epsilon < \theta_y(0)$, then there exists $\gamma > 0$ such that $\theta_y(z - y_\epsilon) = -\gamma - \epsilon$ and by part 2 of Theorem 4.5,

$$(-\gamma - \epsilon)k^0 + y_\epsilon - z = c^1 \in C(y) \Rightarrow y_\epsilon - z - \epsilon k^0 \in C(y) + \gamma k^0. \tag{13}$$

   By $\gamma > 0$ and $C(y) + (0, \infty)k^0 \subseteq \text{int } C(y)$, we have $y_\epsilon - \epsilon k^0 - z \in \text{int } C(y)$ and

$$(y_\epsilon - \epsilon k^0 - C(y) \backslash \{0\}) \cap \Omega \neq \emptyset$$

   which is contradiction to our assumption.

2. Proof is similar to the proof of previous part.

3. From the first part, we know that $\theta_y(0) \leq \theta_y(z - y_\epsilon) + \epsilon$ for all $y, z \in \Omega$. We just need to show that $\theta_y(0) \neq \theta_y(z - y_\epsilon) + \epsilon$ for all $y, z \in \Omega$ and this means that we need to show $\theta_y(z - y_\epsilon) + \epsilon \neq 0$ for all $y, z \in \Omega$. If $y_\epsilon = z$ and $\epsilon > 0$, then $\theta_y(z - y_\epsilon) = 0$ and obviously $\theta_y(z - y_\epsilon) + \epsilon \neq 0$. Again, if $y_\epsilon = z$ and $\epsilon = 0$, then our assumption ($\theta_y(0) < \theta_y(z - y_\epsilon) + \epsilon$) can not be fulfilled. So we supposed that $y_\epsilon \neq z$. Suppose that there exist $y, z \in \Omega$ such that $\theta_y(z - y_\epsilon) + \epsilon = 0$ then by part 2 of Theorem 4.5

$$y_\epsilon - \epsilon k^0 - z \in C(y). \tag{14}$$

   Also, by definition of strongly ϵk^0-minimizer, for all $y \in \Omega, z \in \Omega \backslash \{y_\epsilon\}$, we have

$$y_\epsilon - \epsilon k^0 \in z - (C(y) \backslash \{0\}) \Rightarrow y_\epsilon - \epsilon k^0 - z \in -(C(y) \backslash \{0\}). \tag{15}$$

   By (14) and (15), we can imply $(y_\epsilon - \epsilon k^0 - z) \in C(y) \cap -(C(y) \backslash \{0\})$. But this is contradiction to the pointedness of $C(y)$. Therefore $\theta_y(0) \neq \theta_y(z - y_\epsilon) + \epsilon$ and $\theta_y(0) < \theta_y(z - y_\epsilon) + \epsilon$ for all $y, z \in \Omega$. □

In the above theorem, we showed that each ϵk^0-minimizer is a solution for scalar-valued problem. Now, we show that also each solution of scalar problem is at least a weakly ϵk^0-minimizer.

Theorem 5.2. *Let* Assumption (A3) *holds.*

1. *If $y_\epsilon \in \Omega$ and $\theta_y(0) < \theta_y(z-y_\epsilon)+\epsilon$ for all $y, z \in \Omega$, then y_ϵ is an ϵk^0-minimizer of the set Ω.*
2. *If $y_\epsilon \in \Omega$ and $\theta_y(0) \leq \theta_y(z-y_\epsilon)+\epsilon$ for all $y, z \in \Omega$, then y_ϵ is a weakly ϵk^0-minimizer of the set Ω.*

Proof.

1. Suppose that $\theta_y(0) < \theta_y(z - y_\epsilon) + \epsilon$ for all $y, z \in \Omega$ but y_ϵ is not an ϵk^0-minimizer, then there exist $y^1, y^2 \in \Omega$ such that $y_\epsilon - \epsilon k^0 \in y^1 + C(y^2)\backslash\{0\}$. This means that there exists $c^1 \in C(y^2)$ such that $y_\epsilon - \epsilon k^0 = y^1 + c^1$. Similar to the proof of Theorem 5.1, $\theta_{y^2}(0) = 0$ and by part 2 of Theorem 4.5
$$y_\epsilon - \epsilon k^0 - y^1 - c^1 = 0 \Rightarrow -\epsilon k^0 + y_\epsilon - y^1 \in C(y^2).$$
By part 2 of Theorem 4.5, $\theta_{y^2}(y^1-y_\epsilon) = -\epsilon \leq 0 = \theta_{y^2}(0)$ which is contradiction to our assumption.
2. Suppose that $\theta_y(0) \leq \theta_y(z - y_\epsilon) + \epsilon$ for all $y, z \in \Omega$ but y_ϵ is not a weakly ϵk^0-minimizer, then there exist $y^1, y^2 \in \Omega$ such that $y^1 \in y_\epsilon - \epsilon k^0 - \text{int } C(y^2)$. This means that there exists $c^1 \in \text{int } C(y^2)$ such that $y^1 = y_\epsilon - \epsilon k^0 - c^1$. Similar to part 1, we know that $\theta_{y^2}(0) = 0$. Since $c^1 \in \text{int } C(y^2)$, there exists $\gamma > 0$ such that $c^1 - \gamma k^0 \in C(y^2)$ and
$$y_\epsilon - y^1 - (\epsilon + \gamma)k^0 = c^1 - \gamma k^0 \in C(y^2).$$
By part 2 of Theorem 4.5, $\theta_{y^2}(y^1 - y_\epsilon) = -(\gamma + \epsilon) < 0 = \theta_{y^2}(0)$ which is contradiction to our assumption. □

By Theorems 5.3 and 5.4, we characterized approximate minimizer by scalarization via nonlinear functional methods. We can also use this method for characterizing the approximate nondominate elements. In the following theorem, we show that each ϵk^0-nondomiante element of the set Ω is a solution for the scalar-valued optimization problem.

Theorem 5.3. *Let* Assumption (A3) *holds.*

1. *If $y_\epsilon \in \Omega$ is an ϵk^0-nondominated element of the set $\Omega \subseteq Y$, then $\theta_z(0) \leq \theta_z(z - y_\epsilon) + \epsilon$ for all $z, y \in \Omega$ where $\theta_z(y) = \inf \{t \in \mathbb{R} \mid tk^0 - y \in C(z)\}$.*
2. *If $y_\epsilon \in \Omega$ is a weakly ϵk^0-nondominated element of Ω, then $\theta_z(0) \leq \theta_z(z-y_\epsilon)+\epsilon$ for all $z \in \Omega$.*

3. *If $y_\epsilon \in \Omega$ is a strongly ϵk^0-nondominated element of Ω, then $\theta_z(0) < \theta_z(z - y_\epsilon) + \epsilon$ for all $z \in \Omega$.*

Proof.

1. Suppose that y_ϵ is an ϵk^0-nondominated element of the set Ω and there exists $z \in \Omega$ such that $\theta_z(z - y_\epsilon) + \epsilon < \theta_z(0) = t$. Similar to the proof of Theorem 5.1, $t = 0$ and since $\theta_z(z - y_\epsilon) + \epsilon < \theta_z(0)$, then there exists $\gamma \geq 0$ such that $\theta_z(z - y_\epsilon) = -\gamma - \epsilon$ and by part 2 of Theorem 4.5,

   $$(-\epsilon - \gamma)k^0 + y_\epsilon - z = c^1 \in C(z) \Rightarrow y_\epsilon - z - \epsilon k^0 = c^1 + \gamma k^0 \in C(z) + \gamma k^0. \quad (16)$$

   By $\gamma \geq 0$ and $C(z) + (0, \infty)k^0 \subseteq \text{int } C(z)$, we have $y_\epsilon - \epsilon k^0 - z \in \text{int } C(z)$ and

   $$(y_\epsilon - \epsilon k^0 - C(z)\backslash\{0\}) \cap \{z\} \neq \emptyset$$

   which is contradiction to our assumption.

2. Proof is similar to the proof of part 1.

3. By part 1, proof is similar to the proof of part 3 of Theorem 5.1.

We showed that each ϵk^0-nondominated element of the set Ω is a solution for scalar problem. Now, we show that also each solution of scalar-valued problem is at least a weakly ϵk^0-nondomiante element of the set Ω with respect to the ordering map $C : Y \rightrightarrows Y$.

Theorem 5.4. *Let* Assumption (A3) *holds.*

1. *Let $y_\epsilon \in \Omega$ and $\theta_z(0) < \theta_z(z - y_\epsilon) + \epsilon$ for all $z \in \Omega$, then y_ϵ is an ϵk^0-nondominated element of the set Ω.*

2. *Let $y_\epsilon \in \Omega$ and $\theta_z(0) \leq \theta_z(z - y_\epsilon) + \epsilon$ for all $z \in \Omega$, then y_ϵ is a weakly ϵk^0-nondominated element of the set Ω.*

Proof.

1. Suppose that $\theta_z(0) < \theta_z(z - y_\epsilon) + \epsilon$ for all $z \in \Omega$ but y_ϵ is not an ϵk^0-nondominated element, then there exists $z \in \Omega$ such that $y_\epsilon - \epsilon k^0 \in z + C(z)\backslash\{0\}$. This means that there exists $c^1 \in C(z)\backslash\{0\}$ such that $y_\epsilon - \epsilon k^0 = z + c^1$. Same as previous theorems, $\theta_z(0) = 0$. By part 2 of Theorem 4.5 and $y_\epsilon - \epsilon k^0 = z + c^1$, we have

   $$y_\epsilon - \epsilon k^0 - z - c^1 = 0 \Rightarrow -\epsilon k^0 + y_\epsilon - z \in C(z)\backslash\{0\}.$$

   By part 2 of Theorem 4.5, $\theta_z(z - y_\epsilon) + \epsilon \leq 0 = \theta_z(0)$ which is contradiction to our assumption.

2. Proof is similar to the proof of part 2 of Theorem 5.2.

In the special case when $\epsilon = 0$ and C is cone-valued map where each $C(y)$ is pointed convex cone, Eichfelder [16] gives characterization of exact solutions of vector optimization with variable order structure for the nondominated and minimal points. In the following theorem, we characterize approximate minimal elements of the set Ω with respect to the ordering map C by scalarization via nonlinear functionals. We show that each ϵk^0-minimal element of the set Ω is a solution for the scalar-valued optimization problem.

Theorem 5.5. *Let* Assumption (A3) *holds.*

1. *If $y_\epsilon \in \Omega$ is an ϵk^0-minimal element of the set $\Omega \subseteq Y$, then $\theta_{y_\epsilon}(0) \leq \theta_{y_\epsilon}(y - y_\epsilon) + \epsilon$ for all $y \in \Omega$ where $\theta_{y_\epsilon}(y) = \inf \{t \in \mathbb{R} \mid tk^0 - y \in C(y_\epsilon)\}$.*

2. *If $y_\epsilon \in \Omega$ is a weakly ϵk^0-minimal element of Ω, then $\theta_{y_\epsilon}(0) \leq \theta_{y_\epsilon}(y - y_\epsilon) + \epsilon$ for all $y \in \Omega$.*

3. *If $y_\epsilon \in \Omega$ is a strongly ϵk^0-minimal element of Ω, then $\theta_{y_\epsilon}(0) < \theta_{y_\epsilon}(y - y_\epsilon) + \epsilon$ for all $y \in \Omega$.*

Proof.

1. Suppose that y_ϵ is an ϵk^0-minimal element of the set Ω and there exists $y \in \Omega$ such that $\theta_{y_\epsilon}(y - y_\epsilon) + \epsilon < \theta_{y_\epsilon}(0) = t$. Similar to the proof of Theorem 5.1, $t = 0$ and since $\theta_{y_\epsilon}(y - y_\epsilon) + \epsilon < 0$, then there exists $\gamma > 0$ such that $\theta_{y_\epsilon}(y - y_\epsilon) = -\gamma - \epsilon$ and by part 2 of Theorem 4.5,

$$(-\gamma - \epsilon)k^0 + y_\epsilon - y = c^1 \in C(y_\epsilon) \Rightarrow y_\epsilon - \epsilon k^0 - y = c^1 + \gamma k^0 \in C(y_\epsilon) + \gamma k^0. \quad (17)$$

By $\gamma > 0$ and $C(y_\epsilon) + (0, \infty)k^0 \subseteq \text{int } C(y_\epsilon)$, we have $y_\epsilon - \epsilon k^0 - y \in \text{int } C(y_\epsilon)$ and

$$(y_\epsilon - \epsilon k^0 - C(y_\epsilon) \backslash \{0\}) \cap \Omega \neq \emptyset$$

which is contradiction to our assumption.

2. Proof is similar to the part 1.

3. By part 1, proof is similar to the proof of part 3 of Theorem 5.1.

Theorem 5.5 tells us that each ϵk^0-mimimal element of the set Ω is a solution for scalar problem. In the following, we show that also each solution of scalar-valued problem is at least a weakly ϵk^0-minimal element of the set Ω with respect to the ordering map $C : Y \rightrightarrows Y$.

Theorem 5.6. *Let* Assumption (A3) *holds.*

1. *Let $y_\epsilon \in \Omega$ and $\theta_{y_\epsilon}(0) < \theta_{y_\epsilon}(y - y_\epsilon) + \epsilon$ for all $y \in \Omega$, then y_ϵ is an ϵk^0-minimal element of the set Ω.*

2. *Let $y_\epsilon \in \Omega$ such that $\theta_{y_\epsilon}(0) \leq \theta_{y_\epsilon}(y - y_\epsilon) + \epsilon$ for all $y \in \Omega$, then y_ϵ is a weakly ϵk^0-minimal element of the set Ω.*

Proof.

1. Suppose that $\theta_{y_\epsilon}(0) < \theta_{y_\epsilon}(y - y_\epsilon) + \epsilon$ for all $y \in \Omega$ but y_ϵ is not an ϵk^0-minimal element, then there exists $y \in \Omega$ such that $y_\epsilon - \epsilon k^0 \in y + C(y_\epsilon)\backslash\{0\}$. This means that there exists $c^1 \in C(y_\epsilon)$ such that $y_\epsilon - \epsilon k^0 = y + c^1$.

   Similar to the proof of Theorem 5.1, $\theta_{y_\epsilon}(0) = 0$ and by part 2 of Theorem 4.5,
   $$y_\epsilon - \epsilon k^0 - y - c^1 = 0 \Rightarrow -\epsilon k^0 + y_\epsilon - y \in C(y_\epsilon).$$
   Again by part 2 of Theorem 4.5, $\theta_{y_\epsilon}(y - y_\epsilon) + \epsilon \leq 0 = \theta_{y_\epsilon}(0)$ which is contradiction to our assumption.

2. Proof is similar to the proof of part 2 of Theorem 5.2.

6 Vectorial Ekeland's variational principle with variable order structure

Several generalization of Ekeland's variational principle for vector optimization problem with fix order structure are given in [1, 2, 3, 4, 7, 8, 21, 23, 26, 27, 30, 37]. In this section, with the help of some lemmas, we give an extension of Ekeland's variational principle for vector optimization problem and minimal points with variable order structure.

Assumption (A4). Let X be a real Banach space, $S \subseteq X$, $\epsilon \geq 0$ and $k^0 \in Y\backslash\{0\}$. Also, suppose that $f : X \to Y$ and $D : X \rightrightarrows Y$ is a set-valued map where $D(x)$ is a proper, pointed, closed and solid set with $0 \in \partial D(x)$, $D(x) + (0, \infty)k^0 \subseteq \text{int } D(x)$ for any $x \in S$.

Let Assumption (A4) holds. $x_\epsilon \in S$ is said to be an ϵk^0-minimal solution if
$$(f(x_\epsilon) - \epsilon k^0 - D(x_\epsilon)) \cap f(S) = \emptyset.$$
We denote the set of ϵk^0-minimal solutions by ϵk^0-$M(S, f, D)$. For more detail and properties of these points see [35, 36]. When $\epsilon = 0$, this definition coincide with the concept of minimal solution with respect to the variable ordering structure [10, 15]. Definition of weakly ϵk^0-minimal solutions is similar and instead of $D(x_\epsilon)$ we use int $D(x_\epsilon)$. We denote the set of all weakly ϵk^0-minimal solutions by ϵk^0-$WM(S, f, D)$. We will study the following vector optimization problem with respect to the variable order structure and set-valued map $D : X \rightrightarrows Y$:
$$\epsilon k^0 - Min(S, f, D). \tag{VVOP}$$

Definition 6.1. *We say that $f : X \to Y$ is lower semicontinuous with respect to the ordering map $D : X \rightrightarrows Y$, $k^0 \in Y \backslash \{0\}$ and $S \subseteq X$ (for short (k^0, D, S)-lsc), if*

$$M^X_{(\omega,t)} := \{x \in S \mid f(x) \in tk^0 - \text{cl } D(\omega)\}$$

is closed for all $\omega \in S$ and each $t \in \mathbb{R}$.

If $D = D(\omega_1) = D(\omega_2)$ is a fixed set, then Definition 6.1 coincides with definition by Tammer in the page 133 of [37]. Also, if $D = D(\omega_1) = D(\omega_2)$ is a cone, Luc [32] mentioned the following cone continuity. f is D-semicontinuous if at any element $y \in Y$,

$$\{x \in S \mid f(x) \in y - D\}$$

is closed. From definition, it is easy to see that each D-semicontinuous function is (k^0, D, S)-lsc. Also, if $Y = \mathbb{R}$, then our definition coincide with usual definition of lower semicontinuity. In order to prove the main theorem of this section, we need to have the following lemmas.

Lemma 6.2. *Let $D : X \rightrightarrows Y$ be a set valued map and for all $\omega \in S$, $D(\omega)$ be a closed set and $D(\omega) + (0, \infty)k^0 \subseteq \text{int } D(\omega)$. For each $\omega \in S$, consider the functional θ_ω defined by (4). If the objective function $f : X \to Y$ in* (VVOP) *is a (k^0, D, S)-lsc function, then $(\theta_\omega \circ f)(\cdot) = \theta_\omega(f(\cdot))$ is a lower semicontinuous functional for each $\omega \in S$.*

Proof. Since the function $f : X \to Y$ is a (k^0, D, S)-lower semicontinuous, then the set

$$M^X_{(\omega,t)} = \{x \in S \mid f(x) \in tk^0 - D(\omega)\}$$

is closed for all $\omega \in S$ and $t \in \mathbb{R}$.

Now consider that $M^Y_{(\omega,t)} = tk^0 - D(\omega) \subseteq Y$. By $D(\omega) + (0, \infty)k^0 \subseteq \text{int } D(\omega)$ and part 3 of Theorem 4.10, we know that $\theta_\omega : Y \to (-\infty, \infty)$ is a continuous functional for each $\omega \in S$ and by part 2 of Theorem 4.5, we can write

$$M^Y_{(\omega,t)} = tk^0 - D(\omega) = \{y \in Y \mid y \in tk^0 - D(\omega)\} =$$

$$= \{y \in Y \mid \theta_\omega(y) \leqq \theta_\omega(tk^0)\} = \{y \in Y \mid \theta_\omega(y) \leqq t\} := M^Y_{(\omega,\theta_\omega,t)}$$

for each $\omega \in S$ and $t \in \mathbb{R}$. By this we can write,

$$M^X_{(\omega,\theta_\omega,t)} = \{x \in \Omega \mid \theta_\omega(f(x)) \leqq t\} = \{x \in S \mid f(x) \in M^Y_{(\omega,\theta_\omega,t)}\} =$$

$$= \{x \in S \mid f(x) \in M^Y_{(\omega,t)}\} = M^X_{(\omega,t)}$$

is closed for all $\omega \in S$ and $t \in \mathbb{R}$. This means that each $\theta_\omega \circ f$ is a lower semicontinuous for all $\omega \in S$.

Lemma 6.3. *Let* Assumption (A4) *holds and $B : X \rightrightarrows Y$ be a cone-valued map such that for all $\omega \in S$, $k^0 \in \text{int } D(\omega)$ and $D(\omega) + B(\omega)\backslash\{0\} \subseteq D(\omega)$. If $x_\epsilon \in \epsilon k^0$-$M(S, f, D)$, then there exists strictly B-monotone continuous functional $\theta_{x_\epsilon} : Y \to (-\infty, \infty)$ such that*

$$\forall x \in S, \qquad \theta_{x_\epsilon}(f(x_\epsilon)) \leqq \theta_{x_\epsilon}(f(x) + \epsilon k^0).$$

Moreover, if $D(x_\epsilon) + D(x_\epsilon) \subseteq D(x_\epsilon)$ holds, then θ_{x_ϵ} defined by (4) is subadditive on Y and

$$\forall x \in S, \qquad \theta_{x_\epsilon}(f(x_\epsilon)) \leqq \theta_{x_\epsilon}(f(x)) + \theta_{x_\epsilon}(\epsilon k^0).$$

Proof. Suppose that $k^0 \in Y\backslash\{0\}$, $\epsilon > 0$ and $x_\epsilon \in \epsilon k^0$-$M(S, f, D)$. This means that,

$$(f(x_\epsilon) - \epsilon k^0 - (D(x_\epsilon)\backslash\{0\})) \cap f(S) = \emptyset.$$

By this, obviously we can write,

$$(f(x_\epsilon) - (D(x_\epsilon)\backslash\{0\})) \cap (f(S) + \epsilon k^0) = \emptyset.$$

We consider $V(x_\epsilon) := (f(x_\epsilon) - D(x_\epsilon)\backslash\{0\})$ and $f(S) + \epsilon k^0 = U$. By assumptions on the maps D and B, we can apply theorem 2.3.6 of [19] and get desired functional.

Therefore, there exists a continuous functional $\theta_{x_\epsilon} : Y \to (-\infty, \infty)$ such that $\theta_{x_\epsilon}(f(x_\epsilon)) \leqq \theta_{x_\epsilon}(f(S) + \epsilon k^0)$.

Now if $D(x_\epsilon) + D(x_\epsilon) \subseteq D(x_\epsilon)$ holds, then by Theorem 4.8, θ_{x_ϵ} is subadditive functional and

$$\theta_{x_\epsilon}(x_\epsilon) \leqq \theta_{x_\epsilon}(f(S)) + \theta_{x_\epsilon}(\epsilon k^0). \quad \square$$

The following lemma gives some properties of the functional in Lemma 6.3 and we will use these properties later in the proof of other lemmas and our main theorem about vectorial Ekeland variational principle for minimal solutions of (VVOP).

Lemma 6.4. *Let all the assumptions of Lemma 6.3 hold, then we can choose the functional θ_{x_ϵ} defined in Lemma 6.3 in a way such that all the followings holds.*

1. $\theta_{x_\epsilon}(k^0) = 1$.
2. $\theta_{x_\epsilon}(0) = 0$.
3. $\theta_{x_\epsilon}(\epsilon k^0) = \epsilon$ and $\theta_{x_\epsilon}(-\epsilon k^0) = -\theta_{x_\epsilon}(\epsilon k^0) = -\epsilon$.

Proof.

1. By (4), we have

$$\theta_{x_\epsilon}(y) := \inf\{t \in \mathbb{R} \mid y \in tk^0 - D(x_\epsilon)\}.$$

Also by $0 \in \partial D(x_\epsilon)$, we have $k^0 \in k^0 - \partial D(x_\epsilon)$. Therefore by part 3 of Theorem 4.5, we can write $\theta_\omega(k^0) = 1$.

2. We know that $0 \in \partial D(x_\epsilon)$, then by part 3 of Theorem 4.5, $\theta_{x_\epsilon}(0) = 0$.

3. By part 2 of Theorem 4.10, for all $y \in Y, t \in \mathbb{R}, \omega \in S$ the following equation holds.
$$\theta_\omega(y + tk^0) = \theta_\omega(y) + t.$$
Hence $\theta_\omega(0 + \epsilon k^0) = \theta_\omega(0) + \epsilon$ and $\theta_\omega(\epsilon k^0) = \epsilon$. Proof for others is similar. □

Lemma 6.5. *Let $k^0 \in Y$, $\epsilon \in \mathbb{R}_+$, $x_\epsilon \in S$ and $B : X \rightrightarrows Y$ be a cone-valued such that $k^0 \in \text{int } B(\omega)$ for all $\omega \in S$.*

(i) Furthermore, suppose that for strictly B-monotone , continuous, subadditive functional $\theta_{x_\epsilon} : Y \to \mathbb{R}$ the following inequality holds
$$\forall x \in S, \qquad \theta_{x_\epsilon}(f(x_\epsilon)) \leqq \theta_{x_\epsilon}(f(x)) - \theta_{x_\epsilon}(-\epsilon k^0),$$
then $x_\epsilon \in \epsilon k^0$-$WM(S, f, D)$ for some set-valued map $D : X \rightrightarrows Y$ such that $B(x_\epsilon)\backslash 0 \subseteq D(x_\epsilon)$, $0 \in \text{cl } D(x_\epsilon)\backslash D(x_\epsilon)$, $\text{cl } D(x_\epsilon) + (B(x_\epsilon)\backslash\{0\}) \subseteq D(x_\epsilon)$.

Proof. We define $D(x_\epsilon)$ as following,
$$D(x_\epsilon) = \{y \in Y \mid \theta_{x_\epsilon}(-y + f(x_\epsilon) - \epsilon k^0) < \theta_{x_\epsilon}(f(x_\epsilon) - \epsilon k^0)\}, \tag{18}$$
and functional $\hat{\theta}_{x_\epsilon}(y) : Y \to \mathbb{R}$ with
$$\hat{\theta}_{x_\epsilon}(y) := \theta_{x_\epsilon}(y + f(x_\epsilon) - \epsilon k^0). \tag{19}$$
By (19) and (i) and since θ_{x_ϵ} is subadditive, we have
$$\hat{\theta}_{x_\epsilon}(f(S)+\epsilon k^0 - f(x_\epsilon)) = \theta_{x_\epsilon}(f(S)) \geqq \theta_{x_\epsilon}(f(x_\epsilon)) + \theta_{x_\epsilon}(-\epsilon k^0) \geqq \theta_{x_\epsilon}(f(x_\epsilon) - \epsilon k^0) = \hat{\theta}_{x_\epsilon}(0).$$
Now by (18) and (19), we can write
$$\hat{\theta}_{x_\epsilon}(-D(x_\epsilon)) = \theta_{x_\epsilon}(-D(x_\epsilon) + f(x_\epsilon) - \epsilon k^0) < \theta_{x_\epsilon}(f(x_\epsilon) - \epsilon k^0) = \hat{\theta}_{x_\epsilon}(0),$$
therefore
$$(-\text{int } D(x_\epsilon)) \cap (f(S) + \epsilon k^0 - f(x_\epsilon)) = \emptyset \Rightarrow (f(x_\epsilon) - \epsilon k^0 - \text{int } D(x_\epsilon)) \cap f(S) = \emptyset.$$
Since θ_{x_ϵ} is a strictly B-monotone functional, then $B(x_\epsilon)\backslash\{0\} \subseteq D(x_\epsilon)$. Now we show that $\text{cl } D(x_\epsilon) + (B(x_\epsilon)\backslash\{0\}) \subseteq D(x_\epsilon)$. Choose $y \in \text{cl } D(x_\epsilon)$ and $b \in y + B(x_\epsilon)\backslash\{0\}$. Since $\hat{\theta}_{x_\epsilon}$ is strictly B-monotone and $y \in \text{cl } D(x_\epsilon) \subseteq \{y \mid \hat{\theta}_{x_\epsilon}(-y) \leqq \hat{\theta}_{x_\epsilon}(0)\}$, then
$$\hat{\theta}_{x_\epsilon}(-b) < \hat{\theta}_{x_\epsilon}(-y) \leqq \hat{\theta}_{x_\epsilon}(0).$$
Therefore $b \in \text{cl } D(x_\epsilon) + (B(x_\epsilon)\backslash\{0\})$ implies $b \in D(x_\epsilon)$. Assumption $k^0 \in \text{int } B(x_\epsilon)$ and $\text{cl } D(x_\epsilon) + (B(x_\epsilon)\backslash\{0\}) \subseteq D(x_\epsilon)$ implies $D(x_\epsilon) + \epsilon k^0 \subseteq D(x_\epsilon)$. Also since $0 \in \text{cl } (B(x_\epsilon)\backslash\{0\})$, $B(x_\epsilon)\backslash\{0\} \subseteq D(x_\epsilon)$ and $0 \notin D(x_\epsilon)$, therefore $0 \in \text{cl } D(x_\epsilon)\backslash D(x_\epsilon)$. □

The principal result of the next theorem is an extension of Ekeland's theorem to vector optimization problem with variable order structure. In fact, in the following we have an extension for minimal solutions of the problem (VVOP).

Theorem 6.6. *Let* Assumption (A4) *holds,* $D(\omega)+D(\omega) \subseteq D(\omega)$ *for all* $\omega \in S$ *and* $B : X \rightrightarrows Y$ *be a cone-valued map such that* $k^0 \in \text{int } B(\omega)$ *and* $D(\omega) + B(\omega)\backslash\{0\} \subseteq D(\omega)$ *for all* $\omega \in S$. *Consider the problem* (VVOP) *and suppose that the objective function* $f : X \to Y$ *is* (k^0, D, S)*-lsc and bounded from below on the closed subset* S *of* X. *If there exists* $\bar{x} \in S$ *such that* $\bar{x} \in \epsilon k^0\text{-}M(S, f, D)$ *and* $D(\omega) \subseteq D(\bar{x})$ *for all* $\omega \in S$, *then there exists a point* $x_\epsilon \in S$ *such that the following hold,*

1. $x_\epsilon \in \epsilon k^0\text{-}WM(S, f, B)$,
2. $\|\bar{x} - x_\epsilon\| \leqq \sqrt{\epsilon}$,
3. $$x_\epsilon \in WM(S, f_{\epsilon k^0}, B) \quad \text{with } f_{\epsilon k^0}(x) = f(x) + \sqrt{\epsilon}\,\|x - x_\epsilon\|\,k^0. \tag{20}$$

Proof. Suppose that $\bar{x} \in S$ and $\bar{x} \in \epsilon k^0\text{-}M(S, f, D)$, then by the definition of ϵk^0-minimal solutions, we have

$$(f(\bar{x}) - \epsilon k^0 - (D(\bar{x})\backslash\{0\})) \cap f(S) = \emptyset.$$

Now suppose that $\bar{f} := f - f(\bar{x})$, then we have

$$(\bar{f}(\bar{x}) - \epsilon k^0 - (D(\bar{x})\backslash\{0\}) \cap \bar{f}(S) = \emptyset.$$

By $D(\bar{x}) + D(\bar{x}) \subseteq D(\bar{x})$, Lemma 6.3 and 6.4, it is obvious that there exists a strictly B-monotone continuous subadditive functional $\theta_{\bar{x}} : Y \to (-\infty, \infty)$ such that

$$\forall s \in S, \qquad \theta_{\bar{x}}(\bar{f}(\bar{x})) \leqq \theta_{\bar{x}}(\bar{f}(s)) + \theta_{\bar{x}}(\epsilon k^0) = \theta_{\bar{x}}(\bar{f}(s)) + \epsilon.$$

This means that

$$\theta_{\bar{x}}(\bar{f}(\bar{x})) \leqq \inf_{s\in S} \theta_{\bar{x}}(\bar{f}(s)) + \epsilon, \qquad \epsilon > 0.$$

By Lemma 6.2, Theorem 1 of [12] (Scalar Ekeland's variational principle) and since f and $\bar{f}$ are (k^0, D, S)-lsc and bounded from below on S, there exists a $x_\epsilon \in S$ such that

1. $$\theta_{\bar{x}}(\bar{f}(x_\epsilon)) \leqq \theta_{\bar{x}}(\bar{f}(\bar{x})) \leqq \inf_{s\in S} \theta_{\bar{x}}(\bar{f}(s)) + \epsilon, \tag{21}$$
2. $\|x_\epsilon - \bar{x}\| \leqq \sqrt{\epsilon}$,
3. $$\text{for all } s \in S, \qquad \theta_{\bar{x}}(\bar{f}(x_\epsilon)) \leqq \theta_{\bar{x}}(\bar{f}(s)) + \sqrt{\epsilon}\,\|s - x_\epsilon\|\,. \tag{22}$$

By Lemma 6.4 and (21), for all $s \in S$ we have

$$\theta_{\bar{x}}(\bar{f}(x_\epsilon)) \leqq \inf_{s\in S} \theta_{\bar{x}}(\bar{f}(s)) + \epsilon \leqq \theta_{\bar{x}}(\bar{f}(s)) + \theta_{\bar{x}}(\epsilon k^0) = \theta_{\bar{x}}(\bar{f}(s)) - \theta_{\bar{x}}(-\epsilon k^0).$$

Now by Lemma 6.5 and $\bar{f} := f - f(\bar{x})$ and since $B(x_\epsilon) \subseteq D(x_\epsilon) \subseteq D(\bar{x})$, we can write

$$(\bar{f}(x_\epsilon) - \epsilon k^0 - \text{int } B(x_\epsilon)) \cap \bar{f}(S) = \emptyset.$$

This implies that $x_\epsilon \in \epsilon k^0$-$WM(S, f, B)$. Now we prove (20) and for this, suppose that there exists an element $s \in S$ such that

$$f(s) \in f(x_\epsilon) - \sqrt{\epsilon}\, \|s - x_\epsilon\|\, k^0 - \text{int } B(x_\epsilon) \Rightarrow$$

$$\Rightarrow \bar{f}(s) \in \bar{f}(x_\epsilon) - \sqrt{\epsilon}\, \|s - x_\epsilon\|\, k^0 - \text{int } B(x_\epsilon).$$

Since $\theta_{\bar{x}}$ is a strictly B-monotone continuous subadditive functional, then

$$\theta_{\bar{x}}(\bar{f}(s)) < \theta_{\bar{x}}(\bar{f}(x_\epsilon) - \sqrt{\epsilon}\, \|s - x_\epsilon\|\, k^0) \leqq \theta_{\bar{x}}(\bar{f}(x_\epsilon)) + \theta_{\bar{x}}(-\sqrt{\epsilon}\, \|s - x_\epsilon\|\, k^0).$$

Now by Lemma 6.4,

$$\theta_{\bar{x}}(-\sqrt{\epsilon}\, \|s - x_\epsilon\|\, k^0) = -\sqrt{\epsilon}\, \|s - x_\epsilon\| \Rightarrow \theta_{\bar{x}}(\bar{f}(x_\epsilon)) > \theta_{\bar{x}}(\bar{f}(s)) + \sqrt{\epsilon}\, \|s - x_\epsilon\|\,,$$

but this yields a contradiction because of (22). □

In the special case where $D(\omega_1) = D(\omega_2) = D$ is a fixed solid convex cone for all $\omega \in S$, we can have the following corollary.

Corollary 6.7. *Suppose that $D : X \rightrightarrows Y$ is a cone-valued map where $D(\omega)$ is a solid convex cone for all $\omega \in S$ and $k^0 \in \bigcap_{\omega \in S} \text{int } D(\omega)$. Consider the problem* (VVOP) *and assume that the objective function $f : X \to Y$ is (k^0, D, S)-lsc and bounded from below on $S \subseteq X$ and $\epsilon > 0$. If $\bar{x} \in \epsilon k^0$-$M(S, f, D)$ and $D(\omega) \subseteq D(\bar{x})$ for all $\omega \in S$, then there exists a point $x_\epsilon \in S$ such that the following holds,*

1. *$x_\epsilon \in \epsilon k^0$-$WM(S, f, D)$,*
2. *$\|\bar{x} - x_\epsilon\| \leqq \sqrt{\epsilon}$,*
3. *$x_\epsilon \in WM(S, f_{\epsilon k^0}, D)$ with $f_{\epsilon k^0}(x) = f(x) + \sqrt{\epsilon}\, \|x - x_\epsilon\|\, k^0$.*

In the special case, if $D(\omega_1) = D(\omega_2) = D$ is fixed solid convex cone, Corollary 6.7 covers Theorem 4.1 [37], Corollary 1 [2], Theorem 5.1 [3], Theorem 2 [4], Theorem 3.1 for vector valued map [7], Theorem 2.1 [8], Theorem 3.1 [23] and Theorem 10 [27]. For sure in the case $Y = \mathbb{R}$, we have the classical Ekeland variational principe [12].

7 Conclusions

Concepts for approximate nondominate, minimal and minimizer solutions [35, 36] of vector optimization problem with variable order structure will be used in order to derive variational principles and optimality conditions for problems with variable order structure. We have shown scalarization and variational principle for approximate minimal solutions. In the fourth coming paper, we will derive corresponding results for approximate minimizer and nondominate solutions and furthermore optimality conditions for these solution concepts.

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