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GENERAL COMPARISON PRINCIPLE FOR VARIATIONAL-HEMIVARIATIONAL INEQUALITIES

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ABSTRACT. We study quasilinear elliptic variational-hemivariational inequalities involving general Leray-Lions operators. The novelty of this paper is to provide existence and comparison results whereby only a local growth condition on Clarke's generalized gradient is required. Based on these results, in the second part the theory is extended to discontinuous variational-hemivariational inequalities. In the last section we give an example of the construction of sub- and supersolutions.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain with Lipschitz boundary $\partial\Omega$. By $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$, $1 < p < \infty$, we denote the usual Sobolev spaces with their dual spaces $(W^{1,p}(\Omega))^*$ and $W^{-1,q}(\Omega)$, respectively, where q is the Hölder conjugate satisfying $\frac{1}{p} + \frac{1}{q} = 1$. We consider the following elliptic variational-hemivariational inequality: Find $u \in K$ such that

$$\langle Au + F(u), v - u \rangle + \int_{\Omega} j_1^0(\cdot, u; v - u) dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u; \gamma v - \gamma u) d\sigma \geq 0, \quad \forall v \in K, \quad (1.1)$$

where $j_k^0(x, s; r)$, $k = 1, 2$ denotes the generalized directional derivative of the locally Lipschitz functions $s \mapsto j_k(x, s)$ at s in the direction r given by

$$j_k^0(x, s; r) = \limsup_{y \rightarrow s, t \downarrow 0} \frac{j_k(x, y + tr) - j_k(x, y)}{t}, \quad k = 1, 2, \quad (1.2)$$

(cf. [17, Chapter 2]). We denote by K a closed convex subset of $W^{1,p}(\Omega)$, and A is a second-order quasilinear differential operator in divergence form of Leray-Lions type given by

$$Au(x) = - \sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, u(x), \nabla u(x)).$$

The operator F stands for the Nemytskij operator associated with some Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ defined by

$$F(u)(x) = f(x, u(x), \nabla u(x)). \quad (1.3)$$

Furthermore, we denote the trace operator by $\gamma : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega)$ which is known to be linear, bounded and even compact.

The aim of this paper is to establish the method of sub- and supersolutions for problem (1.1). We prove the existence of solutions between a given pair of sub-supersolution assuming only a local growth condition of Clarke's generalized gradient, which extends results recently obtained by S.

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Carl in [7]. To complete our findings, we also give the proof for the existence of extremal solutions of problem (1.1) for a fixed ordered pair of sub- and supersolutions in case A has the form

$$Au(x) = - \sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, \nabla u(x)). \quad (1.4)$$

In the second part we consider (1.1) with a discontinuous Nemytskij operator F involved, which extends results in [36] and partly of [9]. Finally, we construct some sub- and supersolution of problem (1.1) where we make use of the special case $A = -\Delta_p$ which is the negative p -Laplacian. Let us consider some special cases of problem (1.1), where we suppose $A = -\Delta_p$, too.

- (1) If $K = W^{1,p}(\Omega)$ and j_k are smooth, problem (1.1) reduces to

$$\langle -\Delta_p u + F(u), v \rangle + \int_{\Omega} j'_1(\cdot, u) v dx + \int_{\partial\Omega} j'_2(\cdot, \gamma u) \gamma v d\sigma = 0, \quad \forall v \in W^{1,p}(\Omega),$$

which is equivalent to the weak formulation of the nonlinear boundary value problem

$$\begin{aligned} -\Delta_p u + F(u) + j'_1(u) &= 0 && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + j'_2(\gamma u) &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $\frac{\partial u}{\partial \nu}$ denotes the conormal derivative of u . The method of sub- and supersolution for this kind of problems is a special case of [6].

- (2) For $f \in V_0^*$, $K \subset W_0^{1,p}(\Omega)$ and $j_2 = 0$, (1.1) corresponds to the variational-hemivariational inequality given by

$$\langle -\Delta_p u + f, v - u \rangle + \int_{\Omega} j_1^0(\cdot, u; v - u) dx \geq 0, \quad \forall v \in K,$$

which has been discussed in detail in [5].

- (3) If $K \subset W_0^{1,p}(\Omega)$ and $j_k = 0$, then (1.1) is a classical variational inequality of the form

$$u \in K : \langle -\Delta_p u + F(u), v - u \rangle \geq 0, \quad \forall v \in K,$$

whose method of sub- and supersolution has been developed in [10, Chapter 5].

- (4) Let $K = W_0^{1,p}(\Omega)$ or $K = W^{1,p}(\Omega)$ and j_k not necessarily smooth. Then problem (1.1) is a hemivariational inequality, which contains for $K = W_0^{1,p}(\Omega)$ as a special case the following Dirichlet problem for the elliptic inclusion

$$\begin{aligned} -\Delta_p u + F(u) + \partial j_1(\cdot, u) &\ni 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (1.5)$$

and for $K = W^{1,p}(\Omega)$ the elliptic inclusion

$$\begin{aligned} -\Delta_p u + F(u) + \partial j_1(\cdot, u) &\ni 0 && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \partial j_2(\cdot, u) &\ni 0 && \text{on } \partial\Omega, \end{aligned} \quad (1.6)$$

where the multivalued functions $s \mapsto \partial j_k(x, s)$, $k = 1, 2$ stand for Clarke's generalized gradient of the locally Lipschitz function $s \mapsto j_k(x, s)$, $k = 1, 2$ given by

$$\partial j_k(x, s) = \{\xi \in \mathbb{R} : j_k^0(x, s; r) \geq \xi r, \forall r \in \mathbb{R}\}. \quad (1.7)$$

Problems of the form (1.5) and (1.6) have been studied in [15] and [6], respectively.

Existence results for variational-hemivariational inequalities with or without the method of sub- and supersolutions have been obtained under different structure and regularity conditions on the nonlinear functions by various authors. For example we refer to [4, 11, 12, 20, 22, 25, 29, 31]. In case that K is the whole space $W_0^{1,p}(\Omega)$ or $W^{1,p}(\Omega)$, respectively, problem (1.1) reduces to a hemivariational inequality which has been treated in [3, 16, 18, 19, 21, 24, 26, 30, 33].

Comparison principles for general elliptic operators A , including the negative p -Laplacian $-\Delta_p$, Clarke's generalized gradient $s \mapsto \partial j(x, s)$, satisfying a one-sided growth condition in the form

$$\xi_1 \leq \xi_2 + c_1(s_2 - s_1)^{p-1}, \quad (1.8)$$

for all $\xi_i \in \partial j(x, s_i), i = 1, 2$, for a.a. $x \in \Omega$, and for all s_1, s_2 with $s_1 < s_2$, can be found in [10]. Inspired by results recently obtained in [15] and [14], we prove the existence of (extremal) solutions for the variational-hemivariational inequality (1.1) within a sector of an ordered pair of sub- and supersolutions $\underline{u}, \bar{u}$ without assuming a one-sided growth condition on Clarke's gradient of the form (1.8).

2. NOTATION OF SUB- AND SUPERSOLUTION

For functions $u, v : \Omega \rightarrow \mathbb{R}$ we use the notation $u \wedge v = \min(u, v), u \vee v = \max(u, v), K \wedge K = \{u \wedge v : u, v \in K\}, K \vee K = \{u \vee v : u, v \in K\}$, and $u \wedge K = \{u\} \wedge K, u \vee K = \{u\} \vee K$ and introduce the following Definitions:

Definition 2.1. A function $\underline{u} \in W^{1,p}(\Omega)$ is said to be a subsolution of (1.1) if the following hold:

- (1) $F(\underline{u}) \in L^q(\Omega)$,
- (2) $\langle A\underline{u} + F(\underline{u}), w - \underline{u} \rangle + \int_{\Omega} j_1^0(\cdot, \underline{u}; w - \underline{u}) dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma \underline{u}; \gamma w - \gamma \underline{u}) d\sigma \geq 0, \forall w \in \underline{u} \wedge K$.

Definition 2.2. A function $\bar{u} \in W^{1,p}(\Omega)$ is said to be a supersolution of (1.1) if the following hold:

- (1) $F(\bar{u}) \in L^q(\Omega)$,
- (2) $\langle A\bar{u} + F(\bar{u}), w - \bar{u} \rangle + \int_{\Omega} j_1^0(\cdot, \bar{u}; w - \bar{u}) dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma \bar{u}; \gamma w - \gamma \bar{u}) d\sigma \geq 0, \forall w \in \bar{u} \vee K$.

In order to prove our main results, we additionally suppose the following assumptions

$$\underline{u} \vee K \subset K, \quad \bar{u} \wedge K \subset K. \quad (2.1)$$

3. PRELIMINARIES AND HYPOTHESES

Let $1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1$, and assume for the coefficients $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}, i = 1, \dots, N$ the following conditions.

- (A1) Each $a_i(x, s, \xi)$ satisfies Carathéodory conditions, i.e., is measurable in $x \in \Omega$ for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and continuous in (s, ξ) for a.e. $x \in \Omega$. Furthermore, a constant $c_0 > 0$ and a function $k_0 \in L^q(\Omega)$ exist so that

$$|a_i(x, s, \xi)| \leq k_0(x) + c_0(|s|^{p-1} + |\xi|^{p-1}),$$

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $|\xi|$ denotes the Euclidian norm of the vector ξ .

- (A2) The coefficients a_i satisfy a monotonicity condition with respect to ξ in the form

$$\sum_{i=1}^N (a_i(x, s, \xi) - a_i(x, s, \xi'))(\xi_i - \xi'_i) > 0,$$

for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$, and for all $\xi, \xi' \in \mathbb{R}^N$ with $\xi \neq \xi'$.

- (A3) A constant $c_1 > 0$ and a function $k_1 \in L^1(\Omega)$ exist such that

$$\sum_{i=1}^N a_i(x, s, \xi) \xi_i \geq c_1 |\xi|^p - k_1(x),$$

for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$, and for all $\xi \in \mathbb{R}^N$.

Condition (A1) implies that $A : W^{1,p}(\Omega) \rightarrow (W^{1,p}(\Omega))^*$ is bounded continuous and along with (A2) it holds that A is pseudomonotone. Due to (A1) the operator A generates a mapping from $W^{1,p}(\Omega)$ into its dual space defined by

$$\langle Au, \varphi \rangle = \int_{\Omega} \sum_{i=1}^N a_i(x, u, \nabla u) \frac{\partial \varphi}{\partial x_i} dx,$$

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $W^{1,p}(\Omega)$ and $(W^{1,p}(\Omega))^*$ and assumption (A3) is a coercivity type condition.

Let $[\underline{u}, \bar{u}]$ be an ordered pair of sub- and supersolutions of problem (1.1). We impose the following hypotheses on j_k and the nonlinearity f in problem (1.1):

- (j1) $x \mapsto j_1(x, s)$ and $x \mapsto j_2(x, s)$ are measurable in Ω and $\partial\Omega$, respectively, for all $s \in \mathbb{R}$.
- (j2) $s \mapsto j_1(x, s)$ and $s \mapsto j_2(x, s)$ are locally Lipschitz continuous in $\mathbb{R}$ for a.a. $x \in \Omega$ and for a.a. $x \in \partial\Omega$, respectively.
- (j3) There are functions $L_1 \in L_+^q(\Omega)$ and $L_2 \in L_+^q(\partial\Omega)$ such that for all $s \in [\underline{u}(x), \bar{u}(x)]$ the following local growth conditions hold:

$$\begin{aligned} \eta \in \partial j_1(x, s) : |\eta| &\leq L_1(x), \quad \text{for a.a. } x \in \Omega, \\ \xi \in \partial j_2(x, s) : |\xi| &\leq L_2(x), \quad \text{for a.a. } x \in \partial\Omega. \end{aligned} \quad (3.1)$$

- (F1) (i) $x \mapsto f(x, s, \xi)$ is measurable in Ω for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$.
- (ii) $(s, \xi) \mapsto f(x, s, \xi)$ is continuous in $\mathbb{R} \times \mathbb{R}^N$ for almost all $x \in \Omega$.
- (iii) There exist a constant $c_2 > 0$ and a function $k_3 \in L_+^q(\Omega)$ such that

$$|f(x, s, \xi)| \leq k_3(x) + c_2|\xi|^{p-1},$$

for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^N$, and for all $s \in [\underline{u}(x), \bar{u}(x)]$.

Note that the associated Nemytskij operator F defined by $F(u)(x) = f(x, u(x), \nabla u(x))$ is continuous and bounded from $[\underline{u}, \bar{u}] \subset W^{1,p}(\Omega)$ to $L^q(\Omega)$ (cf. [37]). We recall that the normed space $L^p(\Omega)$ is equipped with the natural partial ordering of functions defined by $u \leq v$ if and only if $v - u \in L_+^p(\Omega)$, where $L_+^p(\Omega)$ is the set of all nonnegative functions of $L^p(\Omega)$.

Based on an approach in [15], the main idea in our considerations is to modify the functions j_k . First we set for $k = 1, 2$

$$\alpha_k(x) := \min\{\xi : \xi \in \partial j_k(x, \underline{u}(x))\}, \quad \beta_k(x) := \max\{\xi : \xi \in \partial j_k(x, \bar{u}(x))\}. \quad (3.2)$$

By means of (3.2) we introduce the mappings $\tilde{j}_1 : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and $\tilde{j}_2 : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\tilde{j}_k(x, s) = \begin{cases} j_k(x, \underline{u}(x)) + \alpha_k(x)(s - \underline{u}(x)) & \text{if } s < \underline{u}(x), \\ j_k(x, s) & \text{if } \underline{u}(x) \leq s \leq \bar{u}(x), \\ j_k(x, \bar{u}(x)) + \beta_k(x)(s - \bar{u}(x)) & \text{if } s > \bar{u}(x). \end{cases} \quad (3.3)$$

The following lemma provides some properties of the functions $\tilde{j}_1$ and $\tilde{j}_2$.

Lemma 3.1. *Let the assumptions in (j1)–(j3) be satisfied. Then the modified functions $\tilde{j}_1 : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and $\tilde{j}_2 : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ have the following qualities:*

- ($\tilde{j}1$) $x \mapsto \tilde{j}_1(x, s)$ and $x \mapsto \tilde{j}_2(x, s)$ are measurable in Ω and $\partial\Omega$, respectively, for all $s \in \mathbb{R}$ and $s \mapsto \tilde{j}_1(x, s)$ and $s \mapsto \tilde{j}_2(x, s)$ are locally Lipschitz continuous in $\mathbb{R}$ for a.a. $x \in \Omega$ and for a.a. $x \in \partial\Omega$, respectively.
- ($\tilde{j}2$) Let $\partial\tilde{j}_k(x, s)$ be Clarke's generalized gradient of $s \mapsto \tilde{j}_k(x, s)$. Then for all $s \in \mathbb{R}$ the following estimates hold true:

$$\begin{aligned} \eta \in \partial\tilde{j}_1(x, s) : |\eta| &\leq L_1(x), \quad \text{for a.a. } x \in \Omega, \\ \xi \in \partial\tilde{j}_2(x, s) : |\xi| &\leq L_2(x), \quad \text{for a.a. } x \in \partial\Omega. \end{aligned} \quad (3.4)$$

- ($\tilde{j}3$) Clarke's generalized gradient of $s \mapsto \tilde{j}_1(x, s)$ and $s \mapsto \tilde{j}_2(x, s)$ are given by

$$\partial\tilde{j}_k(x, s) = \begin{cases} \alpha_k(x) & \text{if } s < \underline{u}(x), \\ \partial\tilde{j}_k(x, \underline{u}(x)) & \text{if } s = \underline{u}(x), \\ \partial j_k(x, s) & \text{if } \underline{u}(x) < s < \bar{u}(x), \\ \partial\tilde{j}_k(x, \bar{u}(x)) & \text{if } s = \bar{u}(x), \\ \beta_k(x) & \text{if } s > \bar{u}(x), \end{cases} \quad (3.5)$$

and the inclusions $\partial\tilde{j}_k(x, \underline{u}(x)) \subset \partial j_k(x, \underline{u}(x))$ and $\partial\tilde{j}_k(x, \bar{u}(x)) \subset \partial j_k(x, \bar{u}(x))$ are valid for $k = 1, 2$.

Proof. With a view to the assumptions (j1)–(j3) and the definition of $\tilde{j}_k$ in (3.3), one verifies the lemma in few steps. $\square$

With the aid of Lemma 3.1, we introduce the integral functionals J_1 and J_2 defined on $L^p(\Omega)$ and $L^p(\partial\Omega)$, respectively, given by

$$J_1(u) = \int_{\Omega} \tilde{j}_1(x, u(x)) dx, \quad u \in L^p(\Omega) \quad J_2(v) = \int_{\partial\Omega} \tilde{j}_2(x, v(x)) d\sigma, \quad v \in L^p(\partial\Omega). \quad (3.6)$$

Due to the conditions $(\tilde{j}1)$ – $(\tilde{j}2)$ and Lebourg's mean value theorem (see [17, Chapter 2]), the functionals $J_1 : L^p(\Omega) \rightarrow \mathbb{R}$ and $J_2 : L^p(\partial\Omega) \rightarrow \mathbb{R}$ are well-defined and Lipschitz continuous on bounded sets of $L^p(\Omega)$ and $L^p(\partial\Omega)$, respectively. This implies among others that Clarke's generalized gradients $\partial J_1 : L^p(\Omega) \rightarrow 2^{L^q(\Omega)}$ and $\partial J_2 : L^p(\partial\Omega) \rightarrow 2^{L^q(\partial\Omega)}$ are well-defined, too. Furthermore, by means of Aubin-Clarke's theorem (see [17]), for $u \in L^p(\Omega)$ and $v \in L^p(\partial\Omega)$ we get

$$\begin{aligned} \eta \in \partial J_1(u) &\implies \eta \in L^q(\Omega) \text{ with } \eta(x) \in \partial \tilde{j}_1(x, u(x)) \text{ for a.a. } x \in \Omega, \\ \xi \in \partial J_2(v) &\implies \xi \in L^q(\partial\Omega) \text{ with } \xi(x) \in \partial \tilde{j}_2(x, v(x)) \text{ for a.a. } x \in \partial\Omega. \end{aligned} \quad (3.7)$$

An important tool in our considerations is the following surjectivity result for multivalued pseudomonotone mappings perturbed by maximal monotone operators in reflexive Banach spaces.

Theorem 3.2. *Let X be a real reflexive Banach space with the dual space X^* , $\Phi : X \rightarrow 2^{X^*}$ a maximal monotone operator and $u_0 \in \text{dom}(\Phi)$. Let $A : X \rightarrow 2^{X^*}$ be a pseudomonotone operator, and assume that either A_{u_0} is quasi-bounded or Φ_{u_0} is strongly quasi-bounded. Assume further that $A : X \rightarrow 2^{X^*}$ is u_0 -coercive, that is, there exists a real-valued function $c : \mathbb{R}_+ \rightarrow \mathbb{R}$ with $c(r) \rightarrow +\infty$ as $r \rightarrow +\infty$ such that for all $(u, u^*) \in \text{graph}(A)$ one has $\langle u^*, u - u_0 \rangle \geq c(\|u\|_X) \|u\|_X$. Then $A + \Phi$ is surjective, that is, $\text{range}(A + \Phi) = X^*$.*

The proof of the theorem can be found for example in [34, Theorem 2.12]. The notation A_{u_0} and Φ_{u_0} stands for $A_{u_0}(u) := A(u_0 + u)$ and $\Phi_{u_0}(u) := \Phi(u_0 + u)$, respectively. Note that any bounded operator is, in particular, also quasi-bounded and strongly quasi-bounded. For more details we refer to [34]. The next proposition provides a sufficient condition to prove the pseudomonotonicity of multivalued operators and plays an important part in our argumentations. The proof is presented for example in [34, Chapter 2].

Proposition 3.3. *Let X be a reflexive Banach space, and assume that $A : X \rightarrow 2^{X^*}$ satisfies the following conditions:*

- (i) *For each $u \in X$ we have that $A(u)$ is a nonempty, closed and convex subset of X^* ;*
- (ii) *$A : X \rightarrow 2^{X^*}$ is bounded;*
- (iii) *If $u_n \rightharpoonup u$ in X and $u_n^* \rightharpoonup u^*$ in X^* with $u_n^* \in A(u_n)$ and if $\limsup \langle u_n^*, u_n - u \rangle \leq 0$, then $u^* \in A(u)$ and $\langle u_n^*, u_n \rangle \rightarrow \langle u^*, u \rangle$.*

Then the operator $A : X \rightarrow 2^{X^*}$ is pseudomonotone.

We denote by $i^* : L^q(\Omega) \rightarrow (W^{1,p}(\Omega))^*$ and $\gamma^* : L^q(\partial\Omega) \rightarrow (W^{1,p}(\Omega))^*$ the adjoint operators of the imbedding $i : W^{1,p}(\Omega) \rightarrow L^p(\Omega)$ and the trace operator $\gamma : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega)$, respectively, given by

$$\langle i^* \eta, \varphi \rangle = \int_{\Omega} \eta \varphi dx, \quad \forall \varphi \in W^{1,p}(\Omega), \quad \langle \gamma^* \xi, \varphi \rangle = \int_{\partial\Omega} \xi \gamma \varphi d\sigma, \quad \forall \varphi \in W^{1,p}(\Omega). \quad (3.8)$$

Next, we introduce the following multivalued operators,

$$\Phi_1(u) := (i^* \circ \partial J_1 \circ i)(u), \quad \Phi_2(u) := (\gamma^* \circ \partial J_2 \circ \gamma)(u), \quad (3.9)$$

where i, i^*, γ, γ^* are defined as mentioned above. The operators $\Phi_k, k = 1, 2$, have the following properties (see e.g. [6, Lemma 3.1 and Lemma 3.2]).

Lemma 3.4. *The multivalued operators $\Phi_1 : W^{1,p}(\Omega) \rightarrow 2^{(W^{1,p}(\Omega))^*}$ and $\Phi_2 : W^{1,p}(\Omega) \rightarrow 2^{(W^{1,p}(\Omega))^*}$ are bounded and pseudomonotone.*

Let $b : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be the cut-off function related to the given ordered pair $\underline{u}, \bar{u}$ of sub- and supersolutions defined by

$$b(x, s) = \begin{cases} (s - \bar{u}(x))^{p-1} & \text{if } s > \bar{u}(x), \\ 0 & \text{if } \underline{u}(x) \leq s \leq \bar{u}(x), \\ -(\underline{u}(x) - s)^{p-1} & \text{if } s < \underline{u}(x). \end{cases} \quad (3.10)$$

Clearly, the mapping b is a Carathéodory function satisfying the growth condition

$$|b(x, s)| \leq k_4(x) + c_3 |s|^{p-1}, \quad (3.11)$$

for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$, where $k_4 \in L^q_+(\Omega)$ and $c_3 > 0$. Furthermore, elementary calculations show the following estimate

$$\int_{\Omega} b(x, u(x))u(x)dx \geq c_4 \|u\|_{L^p(\Omega)}^p - c_5, \quad \forall u \in L^p(\Omega), \quad (3.12)$$

where c_4 and c_5 are some positive constants. Due to (3.11) the associated Nemytskij operator $B : L^p(\Omega) \rightarrow L^q(\Omega)$ defined by

$$Bu(x) = b(x, u(x)), \quad (3.13)$$

is bounded and continuous. Since the embedding $i : W^{1,p}(\Omega) \rightarrow L^p(\Omega)$ is compact, the composed operator $\widehat{B} := i^* \circ B \circ i : W^{1,p}(\Omega) \rightarrow (W^{1,p}(\Omega))^*$ is completely continuous.

For $u \in W^{1,p}(\Omega)$, we define the truncation operator T with respect to the functions $\underline{u}$ and $\bar{u}$ given by

$$Tu(x) = \begin{cases} \bar{u}(x) & \text{if } u(x) > \bar{u}(x), \\ u(x) & \text{if } \underline{u}(x) \leq u(x) \leq \bar{u}(x), \\ \underline{u}(x) & \text{if } u(x) < \underline{u}(x). \end{cases} \quad (3.14)$$

The mapping T is continuous and bounded from $W^{1,p}(\Omega)$ into $W^{1,p}(\Omega)$ which follows from the fact that the functions $\min(\cdot, \cdot)$ and $\max(\cdot, \cdot)$ are continuous from $W^{1,p}(\Omega)$ to itself and that T can be represented as $Tu = \max(u, \underline{u}) + \min(u, \bar{u}) - u$ (cf. [23]). Let $F \circ T$ be the composition of the Nemytskij operator F given by

$$(F \circ T)(u)(x) = f(x, Tu(x), \nabla Tu(x)).$$

Due to hypothesis (F1)(iii), the mapping $F \circ T : W^{1,p}(\Omega) \rightarrow L^q(\Omega)$ is bounded and continuous. We set $\widehat{F} : i^* \circ (F \circ T) : W^{1,p}(\Omega) \rightarrow (W^{1,p}(\Omega))^*$, and consider the multivalued operator

$$\widetilde{A} = A_T u + \widehat{F} + \lambda \widehat{B} + \Phi_1 + \Phi_2 : W^{1,p}(\Omega) \rightarrow 2^{(W^{1,p}(\Omega))^*}, \quad (3.15)$$

where λ is a constant specified later and the operator A_T is given by

$$\langle A_T u, \varphi \rangle = - \sum_{i=1}^N \int_{\Omega} a_i(x, Tu, \nabla u) \frac{\partial \varphi}{\partial x_i} dx.$$

We are going to prove the following properties for the operator $\widetilde{A}$.

Lemma 3.5. *The operator $\widetilde{A} : W^{1,p}(\Omega) \rightarrow 2^{(W^{1,p}(\Omega))^*}$ is bounded, pseudomonotone and coercive for λ sufficiently large.*

Proof. The boundedness of $\widetilde{A}$ follows directly from the boundedness of the specific operators A_T , $\widehat{F}$, $\widehat{B}$, Φ_1 and Φ_2 . As seen above, the operator $\widehat{B}$ is completely continuous and thus pseudomonotone. The elliptic operator $A_T + \widehat{F}$ is pseudomonotone because of hypotheses (A1), (A2), and (F1), and in view of Lemma 3.4 the operators Φ_1 and Φ_2 are bounded and pseudomonotone as well. Since pseudomonotonicity is invariant under addition, we conclude that $\widetilde{A} : W^{1,p}(\Omega) \rightarrow 2^{(W^{1,p}(\Omega))^*}$ is bounded and pseudomonotone. To prove the coercivity of $\widetilde{A}$, we have to find the existence of a real-valued function $c : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfying

$$\lim_{s \rightarrow +\infty} c(s) = +\infty, \quad (3.16)$$

such that for all $u \in W^{1,p}(\Omega)$ and $u^* \in \tilde{A}(u)$ the following holds

$$\langle u^*, u - u_0 \rangle \geq c(\|u\|_{W^{1,p}(\Omega)})\|u\|_{W^{1,p}(\Omega)}, \quad (3.17)$$

for some $u_0 \in K$. Let $u^* \in \tilde{A}(u)$, that is, u^* is of the form

$$u^* = (A_T + \hat{F} + \lambda \hat{B})(u) + i^* \eta + \gamma^* \xi,$$

where $\eta \in L^q(\Omega)$ with $\eta(x) \in \partial \tilde{j}_1(x, u(x))$ for a.a. $x \in \Omega$ and $\xi \in L^q(\partial\Omega)$ with $\xi(x) \in \partial \tilde{j}_2(x, u(x))$ for a.a. $x \in \partial\Omega$. Applying (A1), (A3), (F1)(iii), (3.12), $(\tilde{j}2)$, the trace operator $\gamma : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega)$ and Young's inequality yields

$$\begin{aligned} & \langle u^*, u - u_0 \rangle \\ &= \langle (A_T + \hat{F} + \lambda \hat{B})(u) + i^* \eta + \gamma^* \xi, u - u_0 \rangle \\ &= \int_{\Omega} \sum_{i=1}^N a_i(x, Tu, \nabla u) \frac{\partial u - \partial u_0}{\partial x_i} dx + \int_{\Omega} (f(\cdot, Tu, \nabla Tu)(u - u_0) + \lambda b(x, u)(u - u_0)) dx \\ &\quad + \int_{\Omega} \eta(u - u_0) dx + \int_{\partial\Omega} \xi \gamma(u - u_0) d\sigma \\ &\geq c_1 \|\nabla u\|_{L^p(\Omega)}^p - \|k_1\|_{L^1(\Omega)} - d_1 \|u\|_{L^p(\Omega)}^{p-1} - d_2 \|\nabla u\|_{L^p(\Omega)}^{p-1} - d_3 - \varepsilon \|\nabla u\|_{L^p(\Omega)}^p - c(\varepsilon) \|u\|_{L^p(\Omega)}^p \\ &\quad - d_5 \|u\|_{L^p(\Omega)} - d_6 \|\nabla u\|_{L^p(\Omega)}^{p-1} - d_7 + \lambda c_4 \|u\|_{L^p(\Omega)}^p - \lambda c_5 - d_8 - d_9 \|u\|_{L^p(\Omega)}^{p-1} \\ &\quad - d_{10} \|u\|_{L^p(\Omega)} - d_{11} - d_{12} \|u\|_{L^p(\partial\Omega)} - d_{13} \\ &= (c_1 - \varepsilon) \|\nabla u\|_{L^p(\Omega)}^p + (\lambda c_4 - c(\varepsilon)) \|u\|_{L^p(\Omega)}^p - d_{14} \|\nabla u\|_{L^p(\Omega)}^{p-1} - d_{15} \|u\|_{L^p(\Omega)}^{p-1} \\ &\quad - d_{16} \|u\|_{L^p(\Omega)} - d_{17} \end{aligned}$$

where d_j are some positive constants. Choosing $\varepsilon < c_1$ and λ such that $\lambda > \frac{c(\varepsilon)}{c_4}$ yields the estimate

$$\langle u^*, u - u_0 \rangle \geq d_{18} \|u\|_{W^{1,p}(\Omega)}^p - d_{19} \|u\|_{W^{1,p}(\Omega)}^{p-1} - d_{20} \|u\|_{W^{1,p}(\Omega)} - d_{21}.$$

Setting $c(s) = d_{18}s^{p-1} - d_{19}s^{p-2} - d_{20} - \frac{d_{21}}{s}$ for $s > 0$ and $c(0) = 0$ provides the estimate in (3.17) satisfying (3.16). This proves the coercivity of A and completes the proof of the lemma. $\square$

4. MAIN RESULTS

Theorem 4.1. *Let hypotheses (A1)–(A3), (j1)–(j3) and (F1) be satisfied and assume the existence of sub- and supersolutions $\underline{u}$ and $\bar{u}$, respectively, satisfying $\underline{u} \leq \bar{u}$ and (2.1). Then, there exists a solution of (1.1) in the order interval $[\underline{u}, \bar{u}]$.*

Proof. Let $I_K : W^{1,p}(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ be the indicator function corresponding to the closed convex set $K \neq \emptyset$ given by

$$I_K(u) = \begin{cases} 0 & \text{if } u \in K \\ +\infty & \text{if } u \notin K, \end{cases} \quad (4.1)$$

which is known to be proper, convex and lower semicontinuous. The variational-hemivariational inequality (1.1) can be rewritten as follows. Find $u \in K$ such that

$$\langle Au + F(u), v - u \rangle + I_K(v) - I_K(u) + \int_{\Omega} j_1^0(\cdot, u; v - u) dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u; \gamma v - \gamma u) d\sigma \geq 0, \quad (4.2)$$

for all $v \in W^{1,p}(\Omega)$. By using the operators $A_T, \hat{F}, \hat{B}$ and the functions $\tilde{j}_1, \tilde{j}_2$ introduced in Section 3, we consider the following auxiliary problem. Find $u \in K$ such that

$$\begin{aligned} & \langle A_T u + \hat{F}(u) + \lambda \hat{B}(u), v - u \rangle + I_K(v) - I_K(u) + \int_{\Omega} \tilde{j}_1^0(\cdot, u; v - u) dx \\ & + \int_{\partial\Omega} \tilde{j}_2^0(\cdot, \gamma u; \gamma v - \gamma u) d\sigma \geq 0, \end{aligned} \quad (4.3)$$

for all $v \in W^{1,p}(\Omega)$. Consider now the multivalued operator

$$\tilde{A} + \partial I_K : W^{1,p}(\Omega) \rightarrow 2^{(W^{1,p}(\Omega))^*}, \quad (4.4)$$

where $\tilde{A}$ is as in (3.15) and $\partial I_K : W^{1,p}(\Omega) \rightarrow 2^{(W^{1,p}(\Omega))^*}$ is the subdifferential of the indicator function I_K which is known to be a maximal monotone operator (cf. [34, Page 20]). Lemma 3.5 provides that $\tilde{A}$ is bounded, pseudomonotone and coercive. Applying Theorem 3.2 proves the surjectivity of $\tilde{A} + \partial I_K$ meaning that $\text{range}(\tilde{A} + \partial I_K) = (W^{1,p}(\Omega))^*$. Since $0 \in (W^{1,p}(\Omega))^*$, there exist a solution $u \in K$ of the inclusion

$$\tilde{A}(u) + \partial I_K(u) \ni 0. \quad (4.5)$$

This implies the existence of $\eta^* \in \Phi_1(u)$, $\xi^* \in \Phi_2(u)$, and $\theta^* \in \partial I_K(u)$ such that

$$A_T u + \hat{F}(u) + \lambda \hat{B}(u) + \eta^* + \xi^* + \theta^* = 0, \quad \text{in } (W^{1,p}(\Omega))^*, \quad (4.6)$$

where it holds in view of (3.7) and (3.9) that

$$\eta^* = i^* \eta \quad \text{and} \quad \xi^* = \gamma^* \xi$$

with

$$\eta \in L^q(\Omega) \quad \text{and} \quad \eta(x) \in \tilde{\partial} j_1(x, u(x)) \quad \text{as well as} \quad \xi \in L^q(\partial\Omega) \quad \text{and} \quad \xi(x) \in \tilde{\partial} j_2(x, \gamma u(x)).$$

Due to the Definition of Clarke's generalized gradient $\tilde{\partial} j_k(\cdot, u)$, $k = 1, 2$, one gets

$$\begin{aligned} \langle \eta^*, \varphi \rangle &= \int_{\Omega} \eta(x) \varphi(x) dx \leq \int_{\Omega} \tilde{j}_1^0(x, u(x); \varphi(x)) dx, \quad \forall \varphi \in W^{1,p}(\Omega), \\ \langle \xi^*, \varphi \rangle &= \int_{\partial\Omega} \xi(x) \gamma \varphi(x) d\sigma \leq \int_{\partial\Omega} \tilde{j}_2^0(x, \gamma u(x); \gamma \varphi(x)) d\sigma, \quad \forall \varphi \in W^{1,p}(\Omega). \end{aligned} \quad (4.7)$$

Moreover, we have the following estimate

$$\langle \theta^*, v - u \rangle \leq I_K(v) - I_K(u), \quad \forall v \in W^{1,p}(\Omega). \quad (4.8)$$

From (4.6) we conclude

$$\langle A_T u + \hat{F}(u) + \lambda \hat{B}(u) + \eta^* + \xi^* + \theta^*, \varphi \rangle = 0, \quad \forall \varphi \in W^{1,p}(\Omega). \quad (4.9)$$

Using the estimates in (4.7) and (4.8) to the equation above where φ is replaced by $v - u$, yields for all $v \in W^{1,p}(\Omega)$

$$\begin{aligned} 0 &= \langle A_T - \Delta_p u + \hat{F}(u) + \lambda \hat{B}(u) + \eta^* + \xi^* + \theta^*, v - u \rangle \\ &\leq \langle A_T u + \hat{F}(u) + \lambda \hat{B}(u), v - u \rangle + I_K(v) - I_K(u) \\ &\quad + \int_{\Omega} \tilde{j}_1^0(\cdot, u; v - u) dx + \int_{\partial\Omega} \tilde{j}_2^0(\cdot, \gamma u; \gamma v - \gamma u) d\sigma. \end{aligned}$$

Hence, we obtain a solution u of the auxiliary problem (4.3) which is equivalent to the problem. Find $u \in K$ such that

$$\langle A_T u + \hat{F}(u) + \lambda \hat{B}(u), v - u \rangle + \int_{\Omega} \tilde{j}_1^0(\cdot, u; v - u) dx + \int_{\partial\Omega} \tilde{j}_2^0(\cdot, \gamma u; \gamma v - \gamma u) d\sigma \geq 0, \quad \forall v \in K. \quad (4.10)$$

In the next step we have to show that any solution u of (4.10) belongs to $[\underline{u}, \bar{u}]$. By Definition 2.2 and by choosing $w = \bar{u} \vee u = \bar{u} + (u - \bar{u})^+ \in \bar{u} \vee K$, we obtain

$$\langle A\bar{u} + F(\bar{u}), (u - \bar{u})^+ \rangle + \int_{\Omega} j_1^0(\cdot, \bar{u}; (u - \bar{u})^+) dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma \bar{u}; \gamma(u - \bar{u})^+) d\sigma \geq 0,$$

and selecting $v = \bar{u} \wedge u = u - (u - \bar{u})^+ \in K$ in (4.10) provides

$$\langle A_T u + \hat{F}(u) + \lambda \hat{B}(u), -(u - \bar{u})^+ \rangle + \int_{\Omega} \tilde{j}_1^0(\cdot, u; -(u - \bar{u})^+) dx + \int_{\partial\Omega} \tilde{j}_2^0(\cdot, \gamma u; -\gamma(u - \bar{u})^+) d\sigma \geq 0.$$

Adding these inequalities yields

$$\begin{aligned} & \sum_{i=1}^N \int_{\Omega} (a_i(x, \bar{u}, \nabla \bar{u}) - a_i(x, Tu, \nabla u)) \frac{\partial(u - \bar{u})^+}{\partial x_i} dx + \int_{\Omega} (F(\bar{u}) - (F \circ T)(u))(u - \bar{u})^+ dx \\ & + \int_{\Omega} (j_1^0(\cdot, \bar{u}; 1) + \tilde{j}_1^0(\cdot, u; -1))(u - \bar{u})^+ dx + \int_{\partial\Omega} (j_2^0(\cdot, \gamma \bar{u}; 1) + \tilde{j}_2^0(\cdot, \gamma u; -1)) \gamma (u - \bar{u})^+ d\sigma \quad (4.11) \\ & \geq \lambda \int_{\Omega} B(u)(u - \bar{u})^+ dx. \end{aligned}$$

Let us analyze the specific integrals in (4.11). By using (A2) and the definition of the truncation operator, we obtain

$$\begin{aligned} & \int_{\Omega} (a_i(x, \bar{u}, \nabla \bar{u}) - a_i(x, Tu, \nabla u)) \frac{\partial(u - \bar{u})^+}{\partial x_i} dx \leq 0, \\ & \int_{\Omega} (F(\bar{u}) - (F \circ T)(u))(u - \bar{u})^+ dx = 0. \end{aligned} \quad (4.12)$$

Furthermore, we consider the third integral of (4.11) in case $u > \bar{u}$, otherwise it would be zero. Applying (1.7) and (3.3) proves

$$\begin{aligned} & \tilde{j}_1^0(x, u(x); -1) \\ & = \limsup_{s \rightarrow u(x), t \downarrow 0} \frac{\tilde{j}_1(x, s - t) - \tilde{j}_1(x, s)}{t} \\ & = \limsup_{s \rightarrow u(x), t \downarrow 0} \frac{j_1(x, \bar{u}(x)) + \beta_1(x)(s - t - \bar{u}(x)) - j_1(x, \bar{u}(x)) - \beta_1(x)(s - \bar{u}(x))}{t} \quad (4.13) \\ & = \limsup_{s \rightarrow u(x), t \downarrow 0} \frac{-\beta_1(x)t}{t} \\ & = -\beta_1(x). \end{aligned}$$

Proposition 2.1.2 in [17] along with (3.2) shows

$$j_1^0(x, \bar{u}(x); 1) = \max\{\xi : \xi \in \partial j_1(x, \bar{u}(x))\} = \beta_1(x). \quad (4.14)$$

In view of (4.13) and (4.14) we obtain

$$\int_{\Omega} (j_1^0(\cdot, \bar{u}; 1) + \tilde{j}_1^0(\cdot, u; -1))(u - \bar{u})^+ dx = \int_{\Omega} (\beta_1(x) - \beta_1(x))(u - \bar{u})^+ dx = 0, \quad (4.15)$$

and analog to this calculation

$$\int_{\partial\Omega} (j_2^0(\cdot, \gamma \bar{u}; 1) + \tilde{j}_2^0(\cdot, \gamma u; -1)) \gamma (u - \bar{u})^+ d\sigma = 0. \quad (4.16)$$

Due to (4.12), (4.15) and (4.16), we immediately realize that the left-hand-side in (4.11) is nonpositive. Thus, we have

$$\begin{aligned} 0 & \geq \lambda \int_{\Omega} B(u)(u - \bar{u})^+ dx \\ & = \lambda \int_{\Omega} b(\cdot, u)(u - \bar{u})^+ dx \\ & = \lambda \int_{\{x: u(x) > \bar{u}(x)\}} (u - \bar{u})^p dx \\ & = \lambda \int_{\Omega} ((u - \bar{u})^+)^p dx \\ & \geq 0, \end{aligned}$$

which implies $(u - \bar{u})^+ = 0$ and hence, $u \leq \bar{u}$. The proof for $\underline{u} \leq u$ is done in a similar way. So far we have shown that any solution of the inclusion (4.5) (which is a solution of (4.3) as well) belongs

to the interval $[\underline{u}, \bar{u}]$. The latter implies $A_T u = Au$, $B(u) = 0$ and $(F \circ T)(u) = F(u)$, and thus from (4.5) it follows

$$\langle Au + F(u) + i^* \eta + \gamma^* \xi, v - u \rangle \geq 0, \quad \forall v \in K,$$

where $\eta(x) \in \partial \tilde{j}_1(x, u(x)) \subset \partial j_1(x, u(x))$ and $\xi(x) \in \partial \tilde{j}_2(x, \gamma u(x)) \subset \partial j_2(x, \gamma u(x))$, which proves that $u \in [\underline{u}, \bar{u}]$ is also a solution of our original problem (1.1). This completes the proof of the theorem. $\square$

Let $\mathcal{S}$ denote the set of all solutions of (1.1) within the order interval $[\underline{u}, \bar{u}]$. In addition, we will assume that K has lattice structure, that is, K fulfills

$$K \vee K \subset K, \quad K \wedge K \subset K. \quad (4.17)$$

We are going to show that $\mathcal{S}$ possesses the smallest and greatest element with respect to the given partial ordering.

Theorem 4.2. *Let the hypothesis of Theorem 4.1 be satisfied. Then the solution set $\mathcal{S}$ is compact.*

Proof. First, we are going to show that $\mathcal{S}$ is bounded in $W^{1,p}(\Omega)$. Let $u \in \mathcal{S}$ be a solution of (4.2) and notice that $\mathcal{S}$ is $L^p(\Omega)$ -bounded because of $\underline{u} \leq u \leq \bar{u}$. This implies $\gamma \underline{u} \leq \gamma u \leq \gamma \bar{u}$ and thus, u is also bounded in $L^p(\partial\Omega)$. Choosing a fixed $v = u_0 \in K$ in (4.2) delivers

$$\langle Au + F(u), u_0 - u \rangle + \int_{\Omega} j_1^0(\cdot, u; u_0 - u) dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u; \gamma u_0 - \gamma u) d\sigma \geq 0.$$

Using (A1), (j3), (F1)(iii), Proposition 2.1.2 in [17] and Young's inequality yields

$$\begin{aligned} \langle Au, u \rangle &\leq \int_{\Omega} \sum_{i=1}^N |a_i(x, u, \nabla u)| \left| \frac{\partial u_0}{\partial x_i} \right| dx + \int_{\Omega} |f(x, u, \nabla u)| |u_0 - u| dx \\ &\quad + \int_{\Omega} \max\{\eta(u_0 - u) : \eta \in \partial j_1(x, u)\} dx + \int_{\partial\Omega} \max\{\xi(u_0 - u) : \xi \in \partial j_2(x, u)\} d\sigma \\ &\leq \int_{\Omega} \sum_{i=1}^N (k_0 + c_0 |u|^{p-1} + c_0 |\nabla u|^{p-1}) |\nabla u_0| dx + \int_{\Omega} (k_3 + c_2 |\nabla u|^{p-1}) |u_0 - u| dx \\ &\quad + \int_{\Omega} L_1 |u_0 - u| dx + \int_{\partial\Omega} L_2 |\gamma u_0 - \gamma u| d\sigma \\ &\leq e_1 + e_2 \|u\|_{L^p(\Omega)}^{p-1} + e_3 \|\nabla u\|_{L^p(\Omega)}^{p-1} + e_4 + e_5 \|u\|_{L^p(\Omega)} + e_6 \|\nabla u\|_{L^p(\Omega)}^{p-1} + \varepsilon \|\nabla u\|_{L^p(\Omega)}^p \\ &\quad + c(\varepsilon) \|u\|_{L^p(\Omega)}^p + e_7 + e_8 \|u\|_{L^p(\Omega)} + e_9 + e_{10} \|u\|_{L^p(\partial\Omega)} \\ &\leq \varepsilon \|\nabla u\|_{L^p(\Omega)}^p + e_{11} \|\nabla u\|_{L^p(\Omega)}^{p-1} + e_{12} \|\nabla u\|_{L^p(\Omega)} + e_{13}, \end{aligned}$$

where the left-hand side fulfills the estimate

$$\langle Au, u \rangle \geq c_1 \|\nabla u\|_{L^p(\Omega)}^p - k_1.$$

Thus, one has

$$(c_1 - \varepsilon) \|\nabla u\|_{L^p(\Omega)}^p \leq e_{11} \|\nabla u\|_{L^p(\Omega)}^{p-1} + e_{13},$$

where the choice $\varepsilon < c_1$ proves that $\|\nabla u\|_{L^p(\Omega)}$ is bounded. Hence, we obtain the boundedness of u in $W^{1,p}(\Omega)$. Let $(u_n) \subset \mathcal{S}$. Since $W^{1,p}(\Omega)$, $1 < p < \infty$, is reflexive, there exists a weak convergent subsequence, not relabelled, which yields along with the compact imbedding $i : W^{1,p}(\Omega) \rightarrow L^p(\Omega)$ and the trace operator $\gamma : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega)$

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } W^{1,p}(\Omega), \\ u_n &\rightarrow u \quad \text{in } L^p(\Omega) \quad \text{and a.e. pointwise in } \Omega, \\ \gamma u_n &\rightarrow \gamma u \quad \text{in } L^p(\partial\Omega) \quad \text{and a.e. pointwise in } \partial\Omega. \end{aligned} \quad (4.18)$$

As u_n solves (4.2), in particular, for $v = u \in K$, we obtain

$$\langle Au_n, u_n - u \rangle \leq \langle F(u_n), u - u_n \rangle + \int_{\Omega} j_1^0(\cdot, u_n; u - u_n) dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u_n; \gamma u - \gamma u_n) d\sigma. \quad (4.19)$$

Since $(s, r) \mapsto j_k^0(x, s; r)$, $k = 1, 2$, is upper semicontinuous and due to Fatou's Lemma, we get from (4.19)

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle Au_n, u_n - u \rangle &\leq \underbrace{\limsup_{n \rightarrow \infty} \langle F(u_n), u - u_n \rangle}_{\rightarrow 0} + \underbrace{\int_{\Omega} \limsup_{n \rightarrow \infty} j_1^0(\cdot, u_n; u - u_n) dx}_{\leq j_1^0(\cdot, u, 0) = 0} \\ &\quad + \underbrace{\int_{\partial\Omega} \limsup_{n \rightarrow \infty} j_2^0(\cdot, \gamma u_n; \gamma u - \gamma u_n) d\sigma}_{\leq j_2^0(\cdot, \gamma u, \gamma 0) = 0} \leq 0. \end{aligned} \quad (4.20)$$

The elliptic operator A satisfies the (S_+) -property, which due to and due to (4.20) and (4.18) implies

$$u_n \rightarrow u \text{ in } W^{1,p}(\Omega).$$

Replacing u by u_n in (1.1) yields the following inequality

$$\langle Au_n + F(u_n), v - u_n \rangle + \int_{\Omega} j_1^0(\cdot, u_n; v - u_n) dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u_n; \gamma v - \gamma u_n) d\sigma \geq 0, \quad \forall v \in K. \quad (4.21)$$

Passing to the limes superior in (4.21) and using Fatou's Lemma, the strong convergence of (u_n) in $W^{1,p}(\Omega)$ and the upper semicontinuity of $(s, r) \mapsto j_k^0(x, s; r)$, $k = 1, 2$, we have

$$\langle Au + F(u), v - u \rangle + \int_{\Omega} j_1^0(\cdot, u; v - u) dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u; \gamma v - \gamma u) d\sigma \geq 0, \quad \forall v \in K.$$

Hence, $u \in \mathcal{S}$. This shows the compactness of the solution set $\mathcal{S}$. $\square$

In order to prove the existence of extremal elements of the solution set $\mathcal{S}$, we drop the u -dependence of the operator A . Then, our assumptions read as follows.

- (A1') Each $a_i(x, \xi)$ satisfies Carathéodory conditions, i.e., is measurable in $x \in \Omega$ for all $\xi \in \mathbb{R}^N$ and continuous in ξ for a.e. $x \in \Omega$. Furthermore, a constant $c_0 > 0$ and a function $k_0 \in L^q(\Omega)$ exist so that

$$|a_i(x, \xi)| \leq k_0(x) + |\xi|^{p-1},$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$, where $|\xi|$ denotes the Euclidian norm of the vector ξ .

- (A2') The coefficients a_i satisfy a monotonicity condition with respect to ξ in the form

$$\sum_{i=1}^N (a_i(x, \xi) - a_i(x, \xi'))(\xi_i - \xi'_i) > 0,$$

for a.e. $x \in \Omega$, and for all $\xi, \xi' \in \mathbb{R}^N$ with $\xi \neq \xi'$.

- (A3') A constant $c_1 > 0$ and a function $k_1 \in L^1(\Omega)$ exist such that

$$\sum_{i=1}^N a_i(x, \xi) \xi_i \geq c_1 |\xi|^p - k_1(x),$$

for a.e. $x \in \Omega$, and for all $\xi \in \mathbb{R}^N$.

Then the operator $A : W^{1,p}(\Omega) \rightarrow (W^{1,p}(\Omega))^*$ acts in the following way

$$\langle Au, \varphi \rangle = \int_{\Omega} \sum_{i=1}^N a_i(x, \nabla u) \frac{\partial \varphi}{\partial x_i} dx.$$

Let us recall the definition of a directed set.

Definition 4.3. Let $(\mathcal{P}, \leq)$ be a partially ordered set. A subset $\mathcal{C}$ of $\mathcal{P}$ is said to be upward directed if for each pair $x, y \in \mathcal{C}$ there is a $z \in \mathcal{C}$ such that $x \leq z$ and $y \leq z$. Similarly, $\mathcal{C}$ is downward directed if for each pair $x, y \in \mathcal{C}$ there is a $w \in \mathcal{C}$ such that $w \leq x$ and $w \leq y$. If $\mathcal{C}$ is both upward and downward directed it is called directed.

Theorem 4.4. Let hypotheses (A1')–(A3') and (j1)–(j3) be fulfilled and assume that (F1) and (4.17) are valid. Then the solution set $\mathcal{S}$ of problem (1.1) is a directed set.

Proof. By Theorem 4.1, we have $\mathcal{S} \neq \emptyset$. Let $u_1, u_2 \in \mathcal{S}$ be given solutions of (1.1) and let $u_0 = \max\{u_1, u_2\}$. We have to show that there is an $u \in \mathcal{S}$ such that $u_0 \leq u$. Our proof is mainly based on an approach developed recently in [14] which relies on a properly constructed auxiliary problem. Let the operator $\widehat{B}$ be given basically as in (3.10)–(3.13) with the following slight change:

$$b(x, s) = \begin{cases} (s - \bar{u}(x))^{p-1} & \text{if } s > \bar{u}(x), \\ 0 & \text{if } \underline{u}(x) \leq s \leq \bar{u}(x), \\ -(u_0(x) - s)^{p-1} & \text{if } s < u_0(x). \end{cases} \quad (4.22)$$

We introduce truncation operators T_j related to u_j , and modify the truncation operator T as follows: For $j = 1, 2$, we define

$$T_j u(x) = \begin{cases} \bar{u}(x) & \text{if } u(x) > \bar{u}(x), \\ u(x) & \text{if } u_j(x) \leq u(x) \leq \bar{u}(x), \\ u_j(x) & \text{if } u(x) < u_j(x). \end{cases} \quad \text{and} \quad Tu(x) = \begin{cases} \bar{u}(x) & \text{if } u(x) > \bar{u}(x), \\ u(x) & \text{if } u_0(x) \leq u(x) \leq \bar{u}(x), \\ u_0(x) & \text{if } u(x) < u_0(x), \end{cases}$$

and we set

$$Gu(x) = f(x, Tu(x), \nabla Tu(x)) - \sum_{j=1}^2 |f(x, Tu(x), \nabla Tu(x)) - f(x, T_j u(x), \nabla T_j u(x))|,$$

as well as

$$\widehat{F} : i^* \circ G : W^{1,p}(\Omega) \rightarrow (W^{1,p}(\Omega))^*.$$

Moreover, we define

$$\alpha_{k,j}(x) := \min\{\xi : \xi \in \partial j_k(x, u_j(x))\}, \quad \beta_k(x) := \max\{\xi : \xi \in \partial j_k(x, \bar{u}(x))\},$$

and

$$\alpha_{k,0}(x) := \begin{cases} \alpha_{k,1}(x) & \text{if } x \in \{u_1 \geq u_2\}, \\ \alpha_{k,2}(x) & \text{if } x \in \{u_2 > u_1\}, \end{cases}$$

for $k, j = 1, 2$, and introduce the functions $\widetilde{j}_1 : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and $\widetilde{j}_2 : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\widetilde{j}_k(x, s) = \begin{cases} j_k(x, u_0(x)) + \alpha_{k,0}(x)(s - u_0(x)) & \text{if } s < u_0(x), \\ j_k(x, s) & \text{if } u_0(x) \leq s \leq \bar{u}(x), \\ j_k(x, \bar{u}(x)) + \beta_k(x)(s - \bar{u}(x)) & \text{if } s > \bar{u}(x). \end{cases} \quad (4.23)$$

Furthermore, we define the functions $h_{1,j} : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and $h_{2,j} : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ for $j = 0, 1, 2$ as follows:

$$h_{k,0}(x, s) = \begin{cases} \alpha_{k,0}(x) & \text{if } s \leq u_0(x), \\ \alpha_{k,0}(x) + \frac{\beta_k(x) - \alpha_{k,0}(x)}{\bar{u}(x) - u_0(x)}(s - u_0(x)) & \text{if } u_0(x) < s < \bar{u}(x), \\ \beta_k(x) & \text{if } s \geq \bar{u}(x), \end{cases}$$

and for $j = 1, 2$

$$h_{k,j}(x, s) = \begin{cases} \alpha_{k,j}(x) & \text{if } s \leq u_j(x), \\ \alpha_{k,j}(x) + \frac{\alpha_{k,0}(x) - \alpha_{k,j}(x)}{u_0(x) - u_k(x)}(s - u_j(x)) & \text{if } u_j(x) < s < u_0(x), \\ h_{k,0}(x, s) & \text{if } s \geq u_0(x), \end{cases}$$

where $k = 1, 2$. (Note that for $k = 2$ we understand the functions above being defined on $\partial\Omega$.) Apparently, the mappings $(x, s) \mapsto h_{k,j}(x, s)$ are Carathéodory functions which are piecewise

linear with respect to s . Let us introduce the Nemytskij operators $H_1 : L^p(\Omega) \rightarrow L^q(\Omega)$ and $H_2 : L^p(\partial\Omega) \rightarrow L^q(\partial\Omega)$ defined by

$$\begin{aligned} H_1 u(x) &= \sum_{j=1}^2 |h_{1,j}(x, u(x)) - h_{1,0}(x, u(x))|, \\ H_2 u(x) &= \sum_{j=1}^2 |h_{2,j}(x, \gamma(u(x))) - h_{2,0}(x, \gamma(u(x)))|. \end{aligned}$$

Due to the compact imbedding $W^{1,p}(\Omega) \rightarrow L^p(\Omega)$, and the compactness of the trace operator $\gamma : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega)$, the operators $\tilde{H}_1 = i^* \circ H_1 \circ i : W^{1,p}(\Omega) \rightarrow (W^{1,p}(\Omega))^*$ and $\tilde{H}_2 = \gamma^* \circ H_2 \circ \gamma : W^{1,p}(\Omega) \rightarrow (W^{1,p}(\Omega))^*$ are bounded and completely continuous, and thus pseudomonotone. Now, we consider the following auxiliary variational-hemivariational inequality: Find $u \in K$ such that

$$\begin{aligned} \langle Au + \hat{F}(u) + \lambda \hat{B}(u), v - u \rangle + \int_{\Omega} \tilde{j}_1^0(\cdot, u; v - u) dx - \langle \tilde{H}_1 u, v - u \rangle \\ + \int_{\partial\Omega} \tilde{j}_2^0(\cdot, \gamma u; \gamma v - \gamma u) d\sigma - \langle \tilde{H}_2 \gamma u, \gamma v - \gamma u \rangle \geq 0, \end{aligned} \quad (4.24)$$

for all $v \in K$. The construction of the auxiliary problem (4.24) including the functions H_k and G is inspired by a very recent approach introduced by Carl and Motreanu in [14]. The first part of the proof of Theorem 4.1 delivers the existence of a solution u of (4.24), since all calculations in Section 3 are still valid. In order to show that the solution set $\mathcal{S}$ of (1.1) is upward directed, we have to verify that a solution u of (4.24) satisfies $u_l \leq u \leq \bar{u}$, $l = 1, 2$. By assumption $u_l \in \mathcal{S}$, that is, u_l solves

$$u_l \in K : \langle Au_l + F(u_l), v - u_l \rangle + \int_{\Omega} j_1^0(\cdot, u_l; v - u_l) dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u_l; \gamma v - \gamma u_l) d\sigma \geq 0, \quad (4.25)$$

for all $v \in K$. Selecting $v = u \wedge u_l = u_l - (u_l - u)^+ \in K$ in the inequality above yields

$$\langle Au_l + F(u_l), -(u_l - u)^+ \rangle + \int_{\Omega} j_1^0(\cdot, u_l; -(u_l - u)^+) dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u_l; -\gamma(u_l - u)^+) d\sigma \geq 0. \quad (4.26)$$

Taking the special test function $v = u \vee u_l = u + (u_l - u)^+ \in K$ in (4.24), we get

$$\begin{aligned} \langle Au + \hat{F}(u) + \lambda \hat{B}(u), (u_l - u)^+ \rangle + \int_{\Omega} \tilde{j}_1^0(\cdot, u; (u_l - u)^+) dx - \langle \tilde{H}_1 u, (u_l - u)^+ \rangle \\ + \int_{\partial\Omega} \tilde{j}_2^0(\cdot, \gamma u; \gamma(u_l - u)^+) d\sigma - \langle \tilde{H}_2 \gamma u, \gamma(u_l - u)^+ \rangle \geq 0. \end{aligned} \quad (4.27)$$

Adding (4.26) and (4.27) yields

$$\begin{aligned} & \int_{\Omega} \sum_{i=1}^N (a_i(x, \nabla u) - a_i(x, \nabla u_l)) \frac{\partial(u_l - u)^+}{\partial x_i} dx \\ & + \int_{\Omega} \left(f(x, Tu), \nabla Tu \right) - f(x, u_l, \nabla u_l) - \sum_{j=1}^2 |f(x, Tu, \nabla Tu) - f(x, T_j u, \nabla T_j u)| (u_l - u)^+ dx \\ & + \int_{\Omega} \left(\tilde{j}_1^0(\cdot, u; 1) + j_1^0(\cdot, u_l; -1) - \sum_{j=1}^2 |h_{1,j}(x, u) - h_{1,0}(x, u)| \right) (u_l - u)^+ dx \\ & + \int_{\partial\Omega} \left(\tilde{j}_2^0(\cdot, \gamma u; 1) + j_2^0(\cdot, \gamma u_l; -1) - \sum_{j=1}^2 |h_{2,j}(x, \gamma u) - h_{2,0}(x, \gamma u)| \right) \gamma(u_l - u)^+ d\sigma \\ & \geq -\lambda \int_{\Omega} B(u)(u_l - u)^+ dx. \end{aligned} \quad (4.28)$$

The condition (A2') implies directly

$$\int_{\Omega} \sum_{i=1}^N (a_i(x, \nabla u) - a_i(x, \nabla u_l)) \frac{\partial(u_l - u)^+}{\partial x_i} dx \leq 0, \quad (4.29)$$

and the second integral can be estimated to obtain

$$\begin{aligned} & \int_{\Omega} \left(f(x, Tu, \nabla Tu) - f(x, u_l, \nabla u_l) - \sum_{j=1}^2 |f(x, Tu, \nabla Tu) - f(x, T_j u, \nabla T_j u)| \right) (u_l - u)^+ dx \\ & \leq \int_{\Omega} \left(f(x, Tu, \nabla Tu) - f(x, u_l, \nabla u_l) - |f(x, Tu, \nabla Tu) - f(x, T_l u, \nabla T_l u)| \right) (u_l - u)^+ dx \\ & = \int_{\{x \in \Omega: u_l(x) > u(x)\}} \left(f(x, Tu, \nabla Tu) - f(x, u_l, \nabla u_l) - |f(x, Tu, \nabla Tu) - f(x, u_l, \nabla u_l)| \right) (u_l - u) dx \\ & \leq 0. \end{aligned} \quad (4.30)$$

In order to investigate the third integral, we make use of some auxiliary calculation. In view of (4.23) we have for $u_l(x) > u(x)$

$$\begin{aligned} \tilde{j}_1^0(x, u(x); 1) &= \limsup_{s \rightarrow u(x), t \downarrow 0} \frac{\tilde{j}_1(x, s+t) - \tilde{j}_1(x, s)}{t} \\ &= \limsup_{s \rightarrow u(x), t \downarrow 0} \frac{j_1(x, u_0(x)) + \alpha_{1,0}(x)(s+t-u_0(x)) - j_1(x, u_0(x)) - \alpha_{1,0}(x)(s-u_0(x))}{t} \\ &= \limsup_{s \rightarrow u(x), t \downarrow 0} \frac{\alpha_{1,0}(x)t}{t} \\ &= \alpha_{1,0}(x). \end{aligned}$$

Applying Proposition 2.1.2 in [17] and (3.2) results in

$$\begin{aligned} j_1^0(x, u_l(x); -1) &= \max\{-\xi : \xi \in \partial j_1(x, u_l(x))\} \\ &= -\min\{\xi : \xi \in \partial j_1(x, u_l(x))\} \\ &= -\alpha_{1,l}(x). \end{aligned}$$

Furthermore, we have in case $u_l(x) > u(x)$

$$\begin{aligned} h_{1,l}(x, u(x)) &= \alpha_{1,l}(x), \\ h_{1,0}(x, u(x)) &= \alpha_{1,0}(x). \end{aligned}$$

Thus, we get

$$\begin{aligned} & \int_{\Omega} \left(\tilde{j}_1^0(\cdot, u; 1) + j_1^0(\cdot, u_l; -1) - \sum_{j=1}^2 |h_{1,j}(x, u) - h_{1,0}(x, u)| \right) (u_l - u)^+ dx \\ & \leq \int_{\Omega} \left(\tilde{j}_1^0(\cdot, u; 1) + j_1^0(\cdot, u_l; -1) - |h_{1,l}(x, u) - h_{1,0}(x, u)| \right) (u_l - u)^+ dx \\ & = \int_{\{x \in \Omega: u_l(x) > u(x)\}} (\alpha_{1,0}(x) - \alpha_{1,l}(x) - |\alpha_{1,l}(x) - \alpha_{1,0}(x)|) (u_l - u)^+ dx \\ & \leq 0. \end{aligned} \quad (4.31)$$

The same result can be proven for the boundary integral meaning

$$\int_{\partial\Omega} \left(\tilde{j}_2^0(\cdot, \gamma u; 1) + j_2^0(\cdot, \gamma u_l; -1) - \sum_{j=1}^2 |h_{2,j}(x, \gamma u) - h_{2,0}(x, \gamma u)| \right) \gamma (u_l - u)^+ d\sigma \leq 0. \quad (4.32)$$

Applying (4.29)–(4.32) to (4.28) yields

$$\begin{aligned}
0 &\geq -\lambda \int_{\Omega} B(u)(u_l - u)^+ dx \\
&= -\lambda \int_{\{x \in \Omega : u_l(x) > u(x)\}} -(u_0 - u)^{p-1}(u_l - u) dx \\
&\geq \lambda \int_{\Omega} ((u_l - u)^+)^p dx \\
&\geq 0,
\end{aligned}$$

and hence, $(u_l - u)^+ = 0$ meaning that $u_l \leq u$ for $l = 1, 2$. This proves $u_0 = \max\{u_1, u_2\} \leq u$. The proof for $u \leq \bar{u}$ can be shown in a similar way. More precisely, we obtain a solution $u \in K$ of (4.24) satisfying $\underline{u} \leq u_0 \leq u \leq \bar{u}$ which implies $\hat{F}(u) = f(\cdot, u, \nabla u)$, $\hat{B}(u) = 0$ and $H_1(u) = H_2(\gamma u) = 0$. The same arguments as at the end of the proof of Theorem 4.1 apply, which shows that u is in fact a solution of problem (1.1) belonging to the interval $[u_0, \bar{u}]$. Thus, the solution set $\mathcal{S}$ is upward directed. Analogously, one proves that $\mathcal{S}$ is downward directed. $\square$

Theorem 4.2 and Theorem 4.4 allow us to formulate the next theorem about the existence of extremal solutions.

Theorem 4.5. *Let the hypotheses of Theorem 4.4 be satisfied. Then the solution set $\mathcal{S}$ possesses extremal elements.*

Proof. Since $\mathcal{S} \subset W^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$ is separable, $\mathcal{S}$ is also separable, that is, there exists a countable, dense subset $Z = \{z_n : n \in \mathbb{N}\}$ of $\mathcal{S}$. We construct an increasing sequence $(u_n) \subset \mathcal{S}$ as follows. Let $u_1 = z_1$ and select $u_{n+1} \in \mathcal{S}$ such that

$$\max(z_n, u_n) \leq u_{n+1} \leq \bar{u}.$$

By Theorem 4.4, the element u_{n+1} exists because $\mathcal{S}$ is upward directed. Moreover, we can choose by Theorem 4.2 a convergent subsequence (denoted again by u_n) with $u_n \rightarrow u$ in $W^{1,p}(\Omega)$ and $u_n(x) \rightarrow u(x)$ a.e. in Ω . Since (u_n) is increasing, the entire sequence converges in $W^{1,p}(\Omega)$ and further, $u = \sup u_n$. One sees at once that $Z \subset [\underline{u}, u]$ which follows from

$$\max(z_1, \dots, z_n) \leq u_{n+1} \leq u, \quad \forall n,$$

and the fact that $[\underline{u}, u]$ is closed in $W^{1,p}(\Omega)$ implies

$$\mathcal{S} \subset \bar{Z} \subset [\underline{u}, u] = [\underline{u}, u].$$

Therefore, as $u \in \mathcal{S}$, we conclude that u is the greatest element in $\mathcal{S}$. The existence of the smallest solution of (1.1) in $[\underline{u}, \bar{u}]$ can be proven in a similar way. $\square$

Remark 4.6. *If A depends on s , we have to require additional assumptions. For example, if A satisfies in s a monotonicity condition, the existence of extremal solutions can be shown, too. In case $K = W^{1,p}(\Omega)$, a Lipschitz condition with respect to s is sufficient for proving extremal solutions. For more details we refer to [10].*

5. GENERALIZATION TO DISCONTINUOUS NEMYTSKIJ OPERATORS

In this section, we will extend our problem in (1.1) to include discontinuous nonlinearities f of the form $f : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$. We consider again the elliptic variational-hemivariational inequality

$$\langle Au + F(u), v - u \rangle + \int_{\Omega} j_1^0(\cdot, u; v - u) dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u; \gamma v - \gamma u) d\sigma \geq 0, \quad \forall v \in K, \quad (5.1)$$

where all denotations of Section 1 are valid. Here, F denotes the Nemytskij operator given by

$$F(u)(x) = f(x, u(x), u(x), \nabla u(x)). \quad (5.2)$$

where we will allow f to depend discontinuously on its third argument. The aim of this section is to deal with discontinuous Nemytskij operators $F : [\underline{u}, \bar{u}] \subset W^{1,p}(\Omega) \rightarrow L^q(\Omega)$ by combining the

results of Section 4 with an abstract fixed point result for not necessarily continuous operators, cf. [8, Theorem 1.1.1]. This will extend recent results obtained in [36]. Let us recall the Definitions of sub- and supersolutions.

Definition 5.1. A function $\underline{u} \in W^{1,p}(\Omega)$ is called a subsolution of (5.1) if the following hold:

- (1) $F(\underline{u}) \in L^q(\Omega)$,
- (2) $\langle A\underline{u} + F(\underline{u}), w - \underline{u} \rangle + \int_{\Omega} j_1^0(\cdot, \underline{u}; w - \underline{u})dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma\underline{u}; \gamma w - \gamma\underline{u})d\sigma \geq 0 \quad \forall w \in \underline{u} \wedge K$.

Definition 5.2. A function $\bar{u} \in W^{1,p}(\Omega)$ is called a supersolution of (5.1) if the following hold:

- (1) $F(\bar{u}) \in L^q(\Omega)$,
- (2) $\langle A\bar{u} + F(\bar{u}), w - \bar{u} \rangle + \int_{\Omega} j_1^0(\cdot, \bar{u}; w - \bar{u})dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma\bar{u}; \gamma w - \gamma\bar{u})d\sigma \geq 0, \quad \forall v \in \bar{u} \vee K$.

The conditions for Clarke's generalized gradient $s \mapsto \partial j_k(x, s)$ and the functions $j_k : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, $k = 1, 2$, are the same as in (j1)–(j3). We only change the property (F1) to the following:

- (F2) (i) $x \mapsto f(x, r, u(x), \xi)$ is measurable for all $r \in \mathbb{R}$, for all $\xi \in \mathbb{R}^N$, and for all measurable functions $u : \Omega \rightarrow \mathbb{R}$.
- (ii) $(r, \xi) \mapsto f(x, r, s, \xi)$ is continuous in $\mathbb{R} \times \mathbb{R}^N$ for all $s \in \mathbb{R}$ and for almost all $x \in \Omega$.
- (iii) $s \mapsto f(x, r, s, \xi)$ is decreasing for all $(r, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and for almost all $x \in \Omega$.
- (iv) There exist a constant $c_2 > 0$ and a function $k_2 \in L_+^q(\Omega)$ such that

$$|f(x, r, s, \xi)| \leq k_2(x) + c_0|\xi|^{p-1}, \quad (5.3)$$

for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^N$, and for all $r, s \in [\underline{u}(x), \bar{u}(x)]$.

By [2] the mapping $x \mapsto f(x, u(x), u(x), \nabla u(x))$ is measurable for $x \mapsto u(x)$ measurable, however, the associated Nemytskij operator $F : W^{1,p}(\Omega) \subset L^p(\Omega) \rightarrow L^q(\Omega)$ is not necessarily continuous. An important tool in extending the previous result to discontinuous Nemytskij operators is the next fixed point result. The proof of this lemma can be found in [8, Theorem 1.1.1].

Lemma 5.3. Let P be a subset of an ordered normed space, $G : P \rightarrow P$ an increasing mapping and $G[P] = \{Gx \mid x \in P\}$.

- (1) If $G[P]$ has a lower bound in P and the increasing sequences of $G[P]$ converge weakly in P , then G has the least fixed point x_* , and $x_* = \min\{x \mid Gx \leq x\}$.
- (2) If $G[P]$ has an upper bound in P and the decreasing sequences of $G[P]$ converge weakly in P , then G has the greatest fixed point x^* , and $x^* = \max\{x \mid x \leq Gx\}$.

One of our main results is the following Theorem.

Theorem 5.4. Let hypotheses (A1')–(A3'), (j1)–(j3), (F2) and (4.17) be satisfied and assume the existence of sub- and supersolutions $\underline{u}$ and $\bar{u}$ satisfying $\underline{u} \leq \bar{u}$ and (2.1). If f is right-continuous (respectively, left-continuous) in the third argument, then there exists a greatest solution u^* (respectively, a smallest solution u_*) of (5.1) in the order interval $[\underline{u}, \bar{u}]$.

Proof. We choose a fixed element $z \in [\underline{u}, \bar{u}]$ which is a supersolution of (5.1) satisfying $z \wedge K \subset K$ and consider the following auxiliary problem

$$u \in K : \langle Au + F_z(u), v - u \rangle + \int_{\Omega} j_1^0(\cdot, u; v - u)dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u; \gamma v - \gamma u)d\sigma \geq 0, \quad \forall v \in K, \quad (5.4)$$

where $F_z(u)(x) = f(x, u(x), z(x), \nabla u(x))$. It is readily seen that the mapping $(x, u, \nabla u) \mapsto f(x, u, z(x), \nabla u)$ is a Carathéodory function satisfying some growth condition as in (5.3). Since $F_z(z) = F(z)$, z is also a supersolution of (5.4). By Definition 5.1, we have for a given subsolution $\underline{u}$ of (5.1)

$$\langle A\underline{u} + F(\underline{u}), w - \underline{u} \rangle + \int_{\Omega} j_1^0(\cdot, \underline{u}; w - \underline{u})dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma\underline{u}; \gamma w - \gamma\underline{u})d\sigma \geq 0, \quad \forall w \in \underline{u} \wedge K.$$

Setting $w = \underline{u} - (\underline{u} - v)^+$ for all $v \in K$ and using the monotonicity of f with respect to s we get

$$\begin{aligned} 0 &\geq \langle A\underline{u} + F(\underline{u}), (\underline{u} - v)^+ \rangle - \int_{\Omega} j_1^0(\cdot, \underline{u}; -(\underline{u} - v)^+) dx - \int_{\partial\Omega} j_2^0(\cdot, \gamma\underline{u}; -\gamma(\underline{u} - v)^+) d\sigma \\ &\geq \langle A\underline{u} + F_z(\underline{u}), (\underline{u} - v)^+ \rangle - \int_{\Omega} j_1^0(\cdot, \underline{u}; -(\underline{u} - v)^+) dx - \int_{\partial\Omega} j_2^0(\cdot, \gamma\underline{u}; -\gamma(\underline{u} - v)^+) d\sigma, \quad \forall v \in K, \end{aligned}$$

which shows that $\underline{u}$ is also a subsolution of (5.4). Theorem 4.5 implies the existence of a greatest solution $u^* \in [\underline{u}, z]$ of (5.4). Now we introduce the set H given by $H := \{z \in W^{1,p}(\Omega) : z \in [\underline{u}, \bar{u}] \text{ and } z \text{ is a supersolution of (5.1) satisfying } z \wedge K \subset K\}$ and define the operator $L : H \rightarrow K$ by $z \mapsto u^* =: Lz$. This means that the operator L assigns to each $z \in H$ the greatest solution u^* of (5.4) in $[\underline{u}, z]$. Due to the lattice structure of the closed convex set K , Lz is also a supersolution of problem (5.1). In the next step we construct a decreasing sequence as follows:

$$\begin{aligned} u_0 &:= \bar{u} \\ u_1 &:= Lu_0 \quad \text{with } u_1 \in [\underline{u}, u_0] \\ u_2 &:= Lu_1 \quad \text{with } u_2 \in [\underline{u}, u_1] \\ &\vdots \\ u_n &:= Lu_{n-1} \quad \text{with } u_n \in [\underline{u}, u_{n-1}] \end{aligned} \tag{5.5}$$

As $u_n \in [\underline{u}, u_{n-1}]$, we get $u_n(x) \searrow u(x)$ a.e. $x \in \Omega$. Furthermore, the sequence u_n is bounded in $W^{1,p}(\Omega)$, that is, $\|u_n\|_{W^{1,p}(\Omega)} \leq C$ for all n (see the proof of Theorem 4.2). Due to the monotony of u_n and the compact embedding $i : W^{1,p}(\Omega) \rightarrow L^p(\Omega)$ and the trace operator $\gamma : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega)$ we obtain

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } W^{1,p}(\Omega), \\ u_n &\rightarrow u \quad \text{in } L^p(\Omega) \text{ and a.e. pointwise in } \Omega, \\ \gamma u_n &\rightarrow \gamma u \quad \text{in } L^p(\partial\Omega) \text{ and a.e. pointwise in } \partial\Omega. \end{aligned} \tag{5.6}$$

The fact that u_n is a solution of (5.4) with $z = u_{n-1}$ and $v = u \in K$ results in

$$\langle Au_n, u_n - u \rangle \leq \langle F_{u_{n-1}}(u_n), u - u_n \rangle + \int_{\Omega} j_1^0(\cdot, u_n; u - u_n) dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u_n; \gamma u - \gamma u_n) d\sigma.$$

Applying Fatou's Lemma, (5.6) and the upper semicontinuity of $(s, r) \rightarrow j_k^0(x, s; r)$, $k = 1, 2$, yields

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle Au_n, u_n - u \rangle &\leq \underbrace{\limsup_{n \rightarrow \infty} \langle F_{u_{n-1}}(u_n), u - u_n \rangle}_{\rightarrow 0} + \underbrace{\int_{\Omega} \limsup_{n \rightarrow \infty} j_1^0(\cdot, u_n; u - u_n) dx}_{\leq j_1^0(\cdot, u; 0) = 0} \\ &\quad + \underbrace{\int_{\partial\Omega} \limsup_{n \rightarrow \infty} j_2^0(\cdot, \gamma u_n; \gamma u - \gamma u_n) d\sigma}_{\leq j_2^0(\cdot, \gamma u; \gamma 0) = 0} \\ &\leq 0, \end{aligned}$$

which implies by the S_+ -property of A on $W^{1,p}(\Omega)$ along with (5.6)

$$u_n \rightarrow u \quad \text{in } W^{1,p}(\Omega).$$

The right-continuity of f and the strong convergence of the decreasing sequence (u_n) along with the upper semicontinuity of $j_k^0(x, \cdot; \cdot)$ allows us to pass to the lim sup in (5.4) where u (respectively,

z) is replaced by u_n (respectively, u_{n-1}). We have

$$\begin{aligned}
0 &\leq \limsup_{n \rightarrow \infty} \langle Au_n + F_{u_{n-1}}(u_n), v - u_n \rangle + \limsup_{n \rightarrow \infty} \int_{\Omega} j_1^0(\cdot, u_n; v - u_n) dx \\
&\quad + \limsup_{n \rightarrow \infty} \int_{\partial\Omega} j_2^0(\cdot, \gamma u_n; \gamma v - \gamma u_n) d\sigma \\
&\leq \lim_{n \rightarrow \infty} \langle Au_n + F_{u_{n-1}}(u_n), v - u_n \rangle + \int_{\Omega} \limsup_{n \rightarrow \infty} j_1^0(\cdot, u_n; v - u_n) dx \\
&\quad + \int_{\partial\Omega} \limsup_{n \rightarrow \infty} j_2^0(\cdot, \gamma u_n; \gamma v - \gamma u_n) d\sigma \\
&\leq \langle Au + F_u(u), v - u \rangle + \int_{\Omega} j_1^0(\cdot, u; v - u) dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u; \gamma v - \gamma u) d\sigma, \quad \forall v \in K.
\end{aligned}$$

This shows that u is a solution of (5.1) in the order interval $[\underline{u}, \bar{u}]$. Now, we still have to prove that u is the greatest solution of (5.1) in $[\underline{u}, \bar{u}]$. Let $\tilde{u}$ be any solution of (5.1) in $[\underline{u}, \bar{u}]$. Because of the fact that K has lattice structure, $\tilde{u}$ is also a subsolution of (5.1), respectively, a subsolution of (5.4). By the same construction as in (5.5) we obtain

$$\begin{aligned}
\tilde{u}_0 &:= \bar{u} \\
\tilde{u}_1 &:= Lu_0 \quad \text{with} \quad \tilde{u}_1 \in [\tilde{u}, u_0] \\
\tilde{u}_2 &:= Lu_1 \quad \text{with} \quad \tilde{u}_2 \in [\tilde{u}, u_1] \\
&\vdots \\
\tilde{u}_n &:= Lu_{n-1} \quad \text{with} \quad \tilde{u}_n \in [\tilde{u}, u_{n-1}].
\end{aligned} \tag{5.7}$$

Obviously, the sequences in (5.5) and (5.7) create the same extremal solutions u_n and $\tilde{u}_n$ which implies that $\tilde{u} \leq \tilde{u}_n = u_n$ for all n . Passing to the limit delivers the assertion. The existence of a smallest solution can be shown in a similar way. $\square$

In the next Theorem we will prove that only the monotony of f in the third argument is sufficient for the existence of extremal solutions. The function f needs neither be right-continuous nor left-continuous.

Theorem 5.5. *Assume that hypotheses (A1')–(A3'), (j1)–(j3), (F2) and (4.17) are valid and let $\underline{u}$ and $\bar{u}$ be sub- and supersolutions of (5.1) satisfying $\underline{u} \leq \bar{u}$ and (2.1). Then there exist extremal solutions u^* and u_* of (5.1) with $\underline{u} \leq u_* \leq u^* \leq \bar{u}$.*

Proof. As in the proof of Theorem 5.4 we consider the following auxiliary problem

$$u \in K : \langle Au + F_z(u), v - u \rangle + \int_{\Omega} j_1^0(\cdot, u; v - u) dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u; \gamma v - \gamma u) d\sigma \geq 0, \quad \forall v \in K, \tag{5.8}$$

where $F_z(u)(x) = f(x, u(x), z(x), \nabla u(x))$. We define again the set $H := \{z \in W^{1,p}(\Omega) : z \in [\underline{u}, \bar{u}] \text{ and } z \text{ is a supersolution of (5.1) satisfying } z \wedge K \subset K\}$ and introduce the fixed point operator $L : H \rightarrow K$ by $z \mapsto u^* =: Lz$. For a given supersolution $z \in H$, the element Lz is the greatest solution of (5.8) in $[\underline{u}, z]$ and thus, it holds $\underline{u} \leq Lz \leq z$ for all $z \in H$ which implies $L : H \rightarrow [\underline{u}, \bar{u}] \cap K$. Because of (4.17), Lz is also a supersolution of (5.8) satisfying

$$\langle ALz + F_z(Lz), w - Lz \rangle + \int_{\Omega} j_1^0(\cdot, Lz; w - Lz) dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma Lz; \gamma w - \gamma Lz) d\sigma \geq 0,$$

for all $w \in Lz \vee K$. By the monotonicity of f with respect to its third argument, $Lz \leq z$ and using the representation $w = Lz + (v - Lz)^+$ for any $v \in K$ we obtain

$$\begin{aligned}
0 &\leq \langle ALz + F_z(Lz), (v - Lz)^+ \rangle + \int_{\Omega} j_1^0(\cdot, Lz; (v - Lz)^+) dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma Lz; \gamma(v - Lz)^+) d\sigma \\
&\leq \langle ALz + F_{Lz}(Lz), (v - Lz)^+ \rangle + \int_{\Omega} j_1^0(\cdot, Lz; (v - Lz)^+) dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma Lz; \gamma(v - Lz)^+) d\sigma,
\end{aligned}$$

for all $v \in K$. Consequently, Lz is a supersolution of (5.1). This shows $L : H \rightarrow H$. Let $v_1, v_2 \in H$ and assume that $v_1 \leq v_2$. Then we have

$Lv_1 \in [\underline{u}, v_1]$ is the greatest solution of

$$\langle Au + F_{v_1}(u), v - u \rangle + \int_{\Omega} j_1^0(\cdot, u; v - u) dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u; \gamma v - \gamma u) d\sigma \geq 0, \quad \forall v \in K \quad (5.9)$$

and

$Lv_2 \in [\underline{u}, v_2]$ is the greatest solution of

$$\langle Au + F_{v_2}(u), v - u \rangle + \int_{\Omega} j_1^0(\cdot, u; v - u) dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u; \gamma v - \gamma u) d\sigma \geq 0, \quad \forall v \in K. \quad (5.10)$$

Since $v_1 \leq v_2$ it follows that $Lv_1 \leq v_2$, and due to (4.17), Lv_1 is also a subsolution of (5.9), that is, (5.9) holds, in particular, for $v \in Lv_1 \wedge K$, i.e.,

$$\begin{aligned} 0 &\geq \langle ALv_1 + F_{v_1}(Lv_1), (Lv_1 - v)^+ \rangle - \int_{\Omega} j_1^0(\cdot, Lv_1; -(Lv_1 - v)^+) dx \\ &\quad - \int_{\partial\Omega} j_2^0(\cdot, \gamma Lv_1; -\gamma(Lv_1 - v)^+) d\sigma, \end{aligned}$$

for all $v \in K$. Using the monotonicity of f with respect to its third argument s yields

$$\begin{aligned} 0 &\geq \langle ALv_1 + F_{v_1}(Lv_1), (Lv_1 - v)^+ \rangle - \int_{\Omega} j_1^0(\cdot, Lv_1; -(Lv_1 - v)^+) dx \\ &\quad - \int_{\partial\Omega} j_2^0(\cdot, \gamma Lv_1; -\gamma(Lv_1 - v)^+) d\sigma \\ &\geq \langle ALv_1 + F_{v_2}(Lv_1), (Lv_1 - v)^+ \rangle - \int_{\Omega} j_1^0(\cdot, Lv_1; -(Lv_1 - v)^+) dx \\ &\quad - \int_{\partial\Omega} j_2^0(\cdot, \gamma Lv_1; -\gamma(Lv_1 - v)^+) d\sigma, \end{aligned}$$

for all $v \in K$. Hence, Lv_1 is a subsolution of (5.10). By Theorem 4.5, we know there exists a greatest solution of (5.10) in $[Lv_1, v_2]$. But Lv_2 is the greatest solution of (5.10) in $[\underline{u}, v_2] \supseteq [Lv_1, v_2]$ and therefore, $Lv_1 \leq Lv_2$. This shows that L is increasing.

In the last step we have to prove that any decreasing sequence of $L(H)$ converges weakly in H . Let $(u_n) = (Lz_n) \subset L(H) \subset H$ be a decreasing sequence. Then $u_n(x) \searrow u(x)$ a.e. $x \in \Omega$ for some $u \in [\underline{u}, \bar{u}]$. The boundedness of u_n in $W^{1,p}(\Omega)$ can be shown similarly as in Section 4. Thus the compact imbedding $i : W^{1,p}(\Omega) \rightarrow L^p(\Omega)$ along with the monotony of u_n as well as the compactness of the trace operator $\gamma : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega)$ imply

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } W^{1,p}(\Omega), \\ u_n &\rightarrow u \quad \text{in } L^p(\Omega) \text{ and a.e. pointwise in } \Omega, \\ \gamma u_n &\rightarrow \gamma u \quad \text{in } L^p(\partial\Omega) \text{ and a.e. pointwise in } \partial\Omega. \end{aligned} \quad (5.11)$$

Since $u_n \in K$, it follows $u \in K$. From (5.8) with u replaced by u_n and v by u , and using the fact that $(s, r) \mapsto j_k^0(x, s; r)$, $k = 1, 2$, is upper semicontinuous, we obtain by applying Fatou's Lemma

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \langle Au_n, u_n - u \rangle \\ &\leq \limsup_{n \rightarrow \infty} \langle F_{z_n}(u_n), u - u_n \rangle + \limsup_{n \rightarrow \infty} \int_{\Omega} j_1^0(\cdot, u_n; u - u_n) dx + \limsup_{n \rightarrow \infty} \int_{\partial\Omega} j_2^0(\cdot, \gamma u_n; \gamma u - \gamma u_n) d\sigma \\ &\leq \underbrace{\limsup_{n \rightarrow \infty} \langle F_{z_n}(u_n), u - u_n \rangle}_{\rightarrow 0} + \underbrace{\int_{\Omega} \limsup_{n \rightarrow \infty} j_1^0(\cdot, u_n; u - u_n) dx}_{\leq j_1^0(\cdot, u; 0) = 0} + \underbrace{\int_{\partial\Omega} \limsup_{n \rightarrow \infty} j_2^0(\cdot, \gamma u_n; \gamma u - \gamma u_n) d\sigma}_{\leq j_2^0(\cdot, \gamma u; \gamma 0) = 0} \\ &\leq 0. \end{aligned}$$

The S_+ -property of A provides the strong convergence of (u_n) in $W^{1,p}(\Omega)$. As $Lz_n = u_n$ is also a supersolution of (5.8) Definition 5.2 yields

$$\langle Au_n + F_{z_n}(u_n), (v - u_n)^+ \rangle + \int_{\Omega} j_1^0(\cdot, u_n; (v - u_n)^+) dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u_n; \gamma(v - u_n)^+) d\sigma \geq 0,$$

for all $v \in K$. Due to $z_n \geq u_n \geq u$ and the monotonicity of f we get

$$\begin{aligned} 0 &\leq \langle Au_n + F_{z_n}(u_n), (v - u_n)^+ \rangle + \int_{\Omega} j_1^0(\cdot, u_n; (v - u_n)^+) dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u_n; \gamma(v - u_n)^+) d\sigma \\ &\leq \langle Au_n + F_u(u_n), (v - u_n)^+ \rangle + \int_{\Omega} j_1^0(\cdot, u_n; (v - u_n)^+) dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u_n; \gamma(v - u_n)^+) d\sigma, \end{aligned}$$

for all $v \in K$ and since the mapping $u \mapsto u^+ = \max(u, 0)$ is continuous from $W^{1,p}(\Omega)$ to itself (cf. [23]), we can pass to the upper limit on the right hand side for $n \rightarrow \infty$. This yields

$$\langle Au + F_u(u), (v - u)^+ \rangle + \int_{\Omega} j_1^0(\cdot, u; (v - u)^+) dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u; \gamma(v - u)^+) d\sigma \geq 0, \quad \forall v \in K,$$

which shows that u is a supersolution of (5.1), that is, $u \in H$. As $\bar{u}$ is an upper bound of $L(H)$, we can apply Lemma 5.3, which yields the existence of a greatest fixed point u^* of L in H . This implies that u^* must be the greatest solution of (5.1) in $[\underline{u}, \bar{u}]$. By analogous reasoning, one shows the existence of a smallest solution u_* of (5.1). This completes the proof of the theorem. $\square$

6. CONSTRUCTION OF SUB- AND SUPERSOLUTIONS

In this section we are going to construct sub- and supersolutions of problem (5.1) under the conditions (A1')–(A3'), (j1)–(j2) and (F2)(i)–(F2)(iii), where we drop the gradient dependence of f meaning $f(x, r, s) := f(x, r, s, \xi)$. Further, we set $A = -\Delta_p$ which is the negative p -Laplacian defined by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u) \quad \text{where} \quad \nabla u = (\partial u / \partial x_1, \dots, \partial u / \partial x_N). \quad (6.1)$$

The coefficients $a_i, i = 1, \dots, N$ are given by

$$a_i(x, s, \xi) = |\xi|^{p-2} \xi_i.$$

Thus, hypothesis (A1') is satisfied with $k_0 = 0$ and $c_0 = 1$. Hypothesis (A2') is a consequence of the inequalities from the vector-valued function $\xi \mapsto |\xi|^{p-2} \xi$ (see [10, Page 37]) and (A3') is satisfied with $c_1 = 1$ and $k_1 = 0$.

First, we suppose that $u = 0$ on $\partial\Omega$ and $j_2 = 0$ and denote for simplification $j := j_1$. Then, our variational-hemivariational inequality gets the form

$$\langle -\Delta_p u + F(u), v - u \rangle + \int_{\Omega} j^0(\cdot, u; v - u) dx \geq 0, \quad \forall v \in K, \quad (6.2)$$

where K is a closed convex subset of $W_0^{1,p}(\Omega)$. Furthermore, we denote by $\lambda_1 > 0$ the first eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$ and by φ_1 the first eigenfunction of $(-\Delta_p, W_0^{1,p}(\Omega))$ corresponding to λ_1 satisfying $\varphi_1 \in \operatorname{int}(C_0^1(\bar{\Omega})_+)$ and $\|\varphi_1\|_p = 1$ (cf. [1]). Here, $\operatorname{int}(C_0^1(\bar{\Omega})_+)$ describes the interior of the positive cone $C_0^1(\bar{\Omega})_+ = \{u \in C_0^1(\bar{\Omega}) : u(x) \geq 0, \forall x \in \Omega\}$ in the Banach space $C_0^1(\bar{\Omega})$ given by

$$\operatorname{int}(C_0^1(\bar{\Omega})_+) = \left\{ u \in C_0^1(\bar{\Omega}) : u(x) > 0, \forall x \in \Omega, \text{ and } \frac{\partial u}{\partial \nu}(x) < 0, \forall x \in \partial\Omega \right\}, \quad (6.3)$$

where $\frac{\partial u}{\partial \nu}(x)$ is the outer normal derivative of u in the point $x \in \partial\Omega$. The definitions of sub- and supersolution in this case are defined as follows.

Definition 6.1. A function $\underline{u} \in W^{1,p}(\Omega)$ is said to be a subsolution of (6.2) if the following hold:

- (1) $F(\underline{u}) \in L^q(\Omega)$, $\underline{u} \leq 0$ on $\partial\Omega$,
- (2) $\langle -\Delta_p \underline{u} + F(\underline{u}), w - \underline{u} \rangle + \int_{\Omega} j^0(\cdot, \underline{u}; w - \underline{u}) dx \geq 0, \quad \forall w \in \underline{u} \wedge K.$

Definition 6.2. A function $\bar{u} \in W^{1,p}(\Omega)$ is said to be a supersolution of (6.2) if the following hold:

- (1) $F(\bar{u}) \in L^q(\Omega)$, $\bar{u} \geq 0$ on $\partial\Omega$,
- (2) $\langle -\Delta_p \bar{u} + F(\bar{u}), w - \bar{u} \rangle + \int_{\Omega} j^0(\cdot, \bar{u}; w - \bar{u}) dx \geq 0$, $\forall w \in \bar{u} \vee K$.

We suppose the following conditions for f and Clarke's generalized gradient of $s \mapsto j(x, s)$, where $\lambda > \lambda_1$ is any fixed constant:

- (F3) (i) $\lim_{|s| \rightarrow \infty} \frac{f(x, s, s)}{|s|^{p-2}s} = +\infty$, uniformly with respect to a.a. $x \in \Omega$
- (ii) $\lim_{s \rightarrow 0} \frac{f(x, s, s)}{|s|^{p-2}s} = -\lambda$, uniformly with respect to a.a. $x \in \Omega$
- (iii) $\lim_{|s| \rightarrow \infty} \frac{\xi}{|s|^{p-2}s} = +\infty$, uniformly with respect to a.a. $x \in \Omega$, for all $\xi \in \partial j(x, s)$
- (iv) $\lim_{s \rightarrow 0} \frac{\xi}{|s|^{p-2}s} = 0$, uniformly with respect to a.a. $x \in \Omega$, for all $\xi \in \partial j(x, s)$

Proposition 6.3. Assume hypotheses (j1)–(j2), (F2)(i)–(F2)(iii) and (F3). Then there exists a constant a_λ such that $a_\lambda e$ and $-a_\lambda e$ are supersolution and subsolution of problem (6.2), where $e \in \text{int}(C_0^1(\bar{\Omega})_+)$ is the unique solution of $-\Delta_p u = 1$ in $W_0^{1,p}(\Omega)$. Moreover, $-\varepsilon \varphi_1$ is a supersolution and $\varepsilon \varphi_1$ is a subsolution of (6.2) provided that $\varepsilon > 0$ is sufficiently small.

Proof. A sufficient condition for a subsolution $\underline{u} \in W^{1,p}(\Omega)$ of problem (6.2) is $\underline{u} \leq 0$ on $\partial\Omega$, $F(\underline{u}) \in L^q(\Omega)$ and

$$-\Delta_p \underline{u} + F(\underline{u}) + \xi \leq 0 \text{ in } W^{-1,q}(\Omega), \text{ for all } \xi \in \partial j(\cdot, \underline{u}). \quad (6.4)$$

Multiplying (6.4) with $(\underline{u} - v)^+ \in W_0^{1,p}(\Omega) \cap L_+^p(\Omega)$ and using the fact $j^0(\cdot, \underline{u}; -1) \geq -\xi$, for all $\xi \in \partial j(\cdot, \underline{u})$, yields

$$\begin{aligned} 0 &\geq \langle -\Delta_p \underline{u} + F(\underline{u}) + \xi, (\underline{u} - v)^+ \rangle \\ &= \langle -\Delta_p \underline{u} + F(\underline{u}), (\underline{u} - v)^+ \rangle + \int_{\Omega} \xi (\underline{u} - v)^+ dx \\ &\geq \langle -\Delta_p \underline{u} + F(\underline{u}), (\underline{u} - v)^+ \rangle - \int_{\Omega} j^0(\cdot, \underline{u}; -1) (\underline{u} - v)^+ dx \\ &= \langle -\Delta_p \underline{u} + F(\underline{u}), (\underline{u} - v)^+ \rangle - \int_{\Omega} j^0(\cdot, \underline{u}; -(\underline{u} - v)^+) dx, \quad \forall v \in K, \end{aligned}$$

and thus, $\underline{u}$ is a subsolution of (6.2). Analogously, $\bar{u} \in W^{1,p}(\Omega)$ is a supersolution of problem (6.2) if $\bar{u} \geq 0$ on $\partial\Omega$, $F(\bar{u}) \in L^q(\Omega)$ and if the following inequality is satisfied

$$-\Delta_p \bar{u} + F(\bar{u}) + \xi \geq 0 \text{ in } W^{-1,q}(\Omega), \text{ for all } \xi \in \partial j(\cdot, \bar{u}).$$

The main idea of this proof is to show the applicability of [13, Lemmas 2.1–2.3]. We put $g(x, s) = f(x, s, s) + \xi + \lambda |s|^{p-2}s$ for $\xi \in \partial j(x, s)$ and notice that in our considerations the nonlinearity g does not need to be a continuous function. The condition (F3) yields the following limit values

$$\lim_{|s| \rightarrow \infty} \frac{g(x, s)}{|s|^{p-2}s} = +\infty, \quad \text{and} \quad \lim_{s \rightarrow 0} \frac{g(x, s)}{|s|^{p-2}s} = 0.$$

By [13, Lemmas 2.1–2.3] we obtain a pair of positive sub- and supersolutions given by $\underline{u} = \varepsilon \varphi_1$ and $\bar{u} = a_\lambda e$, respectively, a pair of negative sub- and supersolutions given by $\underline{u} = -a_\lambda e$ and $\bar{u} = -\varepsilon \varphi_1$. $\square$

With the aid of these constructed sub- and supersolutions, we see at once that the assumptions (j3) and (F2)(iv) are satisfied, too.

Example 6.4. The function $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x, r, s) = \begin{cases} -(\lambda + 1)|s|^{p-2}s + e^{|r|+|x|}|r|^{p-1}r & \text{for } s < -1, \\ -\lambda|s|^{p-2}s + e^{|r|+|x|}|r|^{p-1}r & \text{for } -1 \leq s \leq 1, \\ -(\lambda + 1)|s|^{p-2}s + e^{|r|+|x|}|r|^{p-1}r & \text{for } s > 1, \end{cases} \quad (6.5)$$

fulfills the assumption (F2) and (F3), where $\lambda > \lambda_1$ is fixed. Clarke's generalized gradient can be given by

$$\partial j(x, s) = \begin{cases} (e^{-(s+1)} - 2)|s|^{p-2}s & \text{if } s < -1, \\ [-1, 1] & \text{if } s = -1, \\ -|s|^p & \text{if } -1 < s < 1, \\ [-1, 1] & \text{if } s = 1, \\ e^{(s-1)(|x|+1)}s^{p-1} & \text{if } s > 1, \end{cases} \quad (6.6)$$

where all conditions in (j1)-(j3) and (F3) are satisfied.

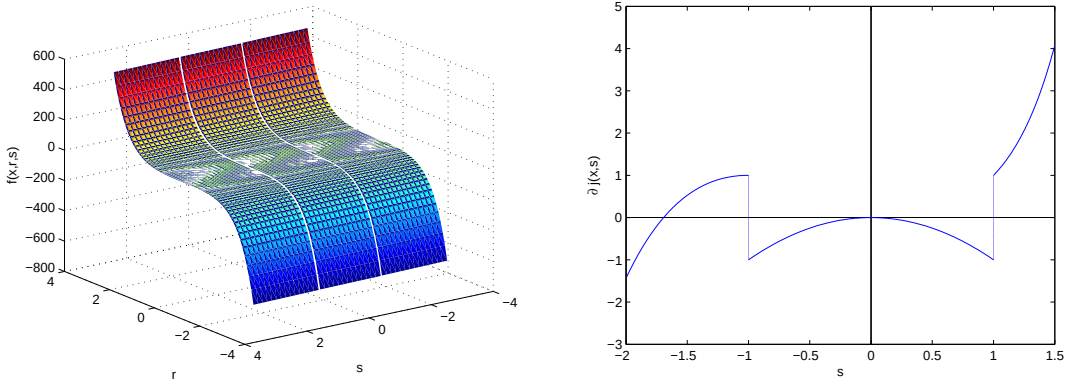


FIGURE 1. The function f and Clarke's generalized gradient $s \mapsto j(x, s)$ with respect to $\Omega = (0, \pi)$, $\varphi_1(x) = \sin(x)$, $\lambda_1 = 1$, $\lambda = 2$, $p = 2$ and $x = 1$.

Remark 6.5. In order to apply Theorem 5.5 we need to satisfy the assumptions

$$\underline{u} \vee K \subset K, \quad \bar{u} \wedge K \subset K, \quad K \vee K \subset K, \quad K \wedge K \subset K, \quad (6.7)$$

which depends on the specific K . For example the obstacle problem is given by

$$K = \{v \in W_0^{1,p}(\Omega) : v(x) \leq \psi(x) \text{ for a.e. } x \in \Omega\}, \quad \psi \in L^\infty(\Omega), \psi \geq C > 0, \quad (6.8)$$

where C is a positive constant. One can show that for the positive pair of sub- and supersolutions in Proposition 6.3 all these conditions in (6.7) with respect to the closed convex set K defined in (6.8) can be satisfied.

Next, we are going to construct sub- and supersolutions for our main problem (5.1) meaning

$$\langle -\Delta_p u + F(u), v - u \rangle + \int_{\Omega} j_1^0(\cdot, u; v - u) dx + \int_{\partial\Omega} j_2^0(\cdot, \gamma u; \gamma v - \gamma u) d\sigma \geq 0, \quad \forall v \in K. \quad (6.9)$$

To this end, we study some auxiliary problems in form of differential equations with Neumann boundary values. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the quasilinear elliptic equation

$$\begin{aligned} -\Delta_p u &= h_1(x, u) - \beta|u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} &= \lambda|u|^{p-2}u + h_2(x, u) & \text{on } \partial\Omega, \end{aligned} \quad (6.10)$$

where $\frac{\partial u}{\partial \nu}$ means the outer normal derivative of u with respect to $\partial\Omega$, β, λ are real parameters and the nonlinearities $h_1 : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and $h_2 : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ are some Carathéodory functions which satisfy the following conditions:

- (H') (a) $\lim_{s \rightarrow 0} \frac{h_1(x, s)}{|s|^{p-2}s} = 0$, uniformly with respect to a.a. $x \in \Omega$.
 (b) $\lim_{|s| \rightarrow \infty} \frac{h_1(x, s)}{|s|^{p-2}s} = -\infty$, uniformly with respect to a.a. $x \in \Omega$.
 (c) h_1 is bounded on bounded sets.
 (d) $\lim_{s \rightarrow 0} \frac{h_2(x, s)}{|s|^{p-2}s} = 0$, uniformly with respect to a.a. $x \in \partial\Omega$.
 (e) $\lim_{|s| \rightarrow \infty} \frac{h_2(x, s)}{|s|^{p-2}s} = -\infty$, uniformly with respect to a.a. $x \in \partial\Omega$.
 (f) h_2 is bounded on bounded sets.

In [32] the authors study the Steklov eigenvalue problem

$$\begin{aligned} -\Delta_p u &= -|u|^{p-2}u && \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} &= \lambda |u|^{p-2}u && \text{on } \partial\Omega, \end{aligned} \quad (6.11)$$

where they proved that the first eigenvalue $\lambda_1 > 0$ is isolated and simple and the corresponding eigenfunction φ_1 is strictly positive in $\bar{\Omega}$ and belongs to $L^\infty(\Omega)$ (cf. [27, Lemma 5.6 and Theorem 4.3]). This implies with a view to the results of Lieberman in [28] that $\varphi_1 \in C^{1,\alpha}(\bar{\Omega})$ and thus, $\varphi_1 \in \text{int}(C^1(\bar{\Omega})_+)$, where $\text{int}(C^1(\bar{\Omega})_+)$ denotes the interior of the positive cone $C^1(\bar{\Omega})_+ = \{u \in C^1(\bar{\Omega}) : u(x) \geq 0, \forall x \in \Omega\}$ in the Banach space $C^1(\bar{\Omega})$, given by

$$\text{int}(C^1(\bar{\Omega})_+) = \{u \in C^1(\bar{\Omega}) : u(x) > 0, \forall x \in \bar{\Omega}\}.$$

Now we consider solutions of the Neumann boundary value problem

$$\begin{aligned} -\Delta_p u &= -\varsigma |u|^{p-2}u + 1 && \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} &= 1 && \text{on } \partial\Omega, \end{aligned} \quad (6.12)$$

where $\varsigma > 1$ is a constant. As in [35], one can show that problem (6.12) has a unique solution $e \in \text{int}(C^1(\bar{\Omega})_+)$. Next, we recall the notations of sub- and supersolutions of problem (6.10).

Definition 6.6. A function $\underline{u} \in W^{1,p}(\Omega)$ is called a subsolution of (6.10) if the following holds

$$\int_{\Omega} |\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla \varphi dx \leq \int_{\Omega} (h_1(x, \underline{u}) - \beta |\underline{u}|^{p-2} \underline{u}) \varphi dx + \int_{\partial\Omega} (\lambda |\underline{u}|^{p-2} \underline{u} + h_2(x, \underline{u})) \varphi d\sigma,$$

for all $\varphi \in W^{1,p}(\Omega)_+$.

Definition 6.7. A function $\bar{u} \in W^{1,p}(\Omega)$ is called a supersolution of (6.10) if the following holds

$$\int_{\Omega} |\nabla \bar{u}|^{p-2} \nabla \bar{u} \nabla \varphi dx \geq \int_{\Omega} (h_1(x, \bar{u}) - \beta |\bar{u}|^{p-2} \bar{u}) \varphi dx + \int_{\partial\Omega} (\lambda |\bar{u}|^{p-2} \bar{u} + h_2(x, \bar{u})) \varphi d\sigma,$$

for all $\varphi \in W^{1,p}(\Omega)_+$.

With the aid of (6.11) and (6.12) we start by generating two ordered pairs of sub- and supersolutions of problem (6.10) having constant signs.

Lemma 6.8. Assume (H'), $\lambda > \lambda_1$, $\beta \in (0, 1)$ and let e be the unique solution of problem (6.12). Then there exists a constant $\vartheta > 0$ such that ϑe and $-\vartheta e$ are supersolution and subsolution, respectively, of problem (6.10). In addition, $\varepsilon \varphi_1$ is a subsolution and $-\varepsilon \varphi_1$ is a supersolution of problem (6.10) provided the number $\varepsilon > 0$ is sufficiently small.

Proof. First, we prove that $\underline{u} = \varepsilon\varphi_1$ is a positive subsolution, where the positive constant ε is stated later. Thanks to the auxiliary eigenvalue problem (6.11), we get

$$\int_{\Omega} |\nabla(\varepsilon\varphi_1)|^{p-2} \nabla(\varepsilon\varphi_1) \nabla \varphi dx = - \int_{\Omega} (\varepsilon\varphi_1)^{p-1} \varphi dx + \int_{\partial\Omega} \lambda_1 (\varepsilon\varphi_1)^{p-1} \varphi d\sigma, \forall \varphi \in W^{1,p}(\Omega). \quad (6.13)$$

In order to show that $\underline{u}$ is a subsolution, we have to indicate the validity of Definition 6.6 for $\underline{u}$ which means that the inequality

$$\begin{aligned} & \int_{\Omega} |\nabla(\varepsilon\varphi_1)|^{p-2} \nabla(\varepsilon\varphi_1) \nabla \varphi dx \\ & \leq \int_{\Omega} (h_1(x, \varepsilon\varphi_1) - \beta(\varepsilon\varphi_1)^{p-1}) \varphi dx + \int_{\partial\Omega} (\lambda(\varepsilon\varphi_1)^{p-1} + h_2(x, \varepsilon\varphi_1)) \varphi d\sigma, \end{aligned} \quad (6.14)$$

is satisfied for all $\varphi \in W^{1,p}(\Omega)_+$. In view of (6.13), the inequality (6.14) is complied if the following holds

$$\int_{\Omega} ((\beta - 1)(\varepsilon\varphi_1)^{p-1} - h_1(x, \varepsilon\varphi_1)) \varphi dx + \int_{\partial\Omega} ((\lambda_1 - \lambda)(\varepsilon\varphi_1)^{p-1} - h_2(x, \varepsilon\varphi_1)) \varphi d\sigma \leq 0, \quad (6.15)$$

for all $\varphi \in W^{1,p}(\Omega)_+$, where $0 < \beta < 1$ and $\lambda > \lambda_1$. Because of (H')(a) and (H')(d) there are numbers $\delta_\beta, \delta_\lambda > 0$ such that

$$\begin{aligned} \frac{|h_1(x, s)|}{|s|^{p-1}} &< 1 - \beta \quad \text{for a.a. } x \in \Omega \text{ and all } 0 < |s| \leq \delta_\beta, \\ \frac{|h_2(x, s)|}{|s|^{p-1}} &< \lambda - \lambda_1 \quad \text{for a.a. } x \in \partial\Omega \text{ and all } 0 < |s| \leq \delta_\lambda. \end{aligned} \quad (6.16)$$

We select $0 < \varepsilon < \min\{\delta_\beta/\|\varphi_1\|_\infty, \delta_\lambda/\|\varphi_1\|_\infty\}$, where $\|\cdot\|_\infty$ stands for the supremum norm, to get

$$\begin{aligned} \int_{\Omega} ((\beta - 1)(\varepsilon\varphi_1)^{p-1} - h_1(x, \varepsilon\varphi_1)) \varphi dx &\leq \int_{\Omega} \left(\beta - 1 + \frac{|h_1(x, \varepsilon\varphi)|}{(\varepsilon\varphi_1)^{p-1}} \right) (\varepsilon\varphi_1)^{p-1} \varphi dx \\ &< \int_{\Omega} (\beta - 1 + 1 - \beta) (\varepsilon\varphi_1)^{p-1} \varphi dx = 0, \end{aligned} \quad (6.17)$$

respectively,

$$\begin{aligned} \int_{\partial\Omega} ((\lambda_1 - \lambda)(\varepsilon\varphi_1)^{p-1} - h_2(x, \varepsilon\varphi_1)) \varphi d\sigma &\leq \int_{\partial\Omega} \left(\lambda_1 - \lambda + \frac{|h_2(x, \varepsilon\varphi)|}{(\varepsilon\varphi_1)^{p-1}} \right) (\varepsilon\varphi_1)^{p-1} \varphi d\sigma \\ &< \int_{\partial\Omega} (\lambda_1 - \lambda + \lambda - \lambda_1) (\varepsilon\varphi_1)^{p-1} \varphi d\sigma = 0. \end{aligned} \quad (6.18)$$

Applying (6.17) and (6.18) to (6.14) implies directly that $\underline{u} = \varepsilon\varphi_1$ is a positive subsolution. In order to prove that $\bar{u} = -\varepsilon\varphi_1$ is a negative supersolution, we argue in much the same manner.

Let $\bar{u} = \vartheta e$ with a positive constant ϑ . Due to the auxiliary problem (6.12) we obtain

$$\int_{\Omega} |\nabla(\vartheta e)|^{p-2} \nabla(\vartheta e) \nabla \varphi dx = -\varsigma \int_{\Omega} (\vartheta e)^{p-1} \varphi dx + \int_{\Omega} \vartheta^{p-1} \varphi dx + \int_{\partial\Omega} \vartheta^{p-1} \varphi d\sigma, \forall \varphi \in W^{1,p}(\Omega). \quad (6.19)$$

Definition 6.7 is fulfilled for $\bar{u} = \vartheta e$ if the following inequality is satisfied

$$\int_{\Omega} |\nabla(\vartheta e)|^{p-2} \nabla(\vartheta e) \nabla \varphi dx \geq \int_{\Omega} (h_1(x, \vartheta e) - \beta(\vartheta e)^{p-1}) \varphi dx + \int_{\partial\Omega} (\lambda(\vartheta e)^{p-1} + h_2(x, \vartheta e)) \varphi d\sigma, \quad (6.20)$$

for all $\varphi \in W^{1,p}(\Omega)_+$. Using (6.19) to (6.20) yields an equivalent formulation to satisfy Definition 6.7 resulting in

$$\int_{\Omega} (\vartheta^{p-1} - \tilde{c}(\vartheta e)^{p-1} - h_1(x, \vartheta e)) \varphi dx + \int_{\partial\Omega} (\vartheta^{p-1} - \lambda(\vartheta e)^{p-1} - h_2(x, \vartheta e)) \varphi d\sigma \geq 0, \quad (6.21)$$

where $\tilde{c} = \varsigma - \beta$ with $\tilde{c} > 0$. Because of (H')(b) there exists $s_\varsigma > 0$ such that

$$\frac{h_1(x, s)}{s^{p-1}} < -\tilde{c}, \quad \text{for a.a. } x \in \Omega \text{ and all } s > s_\varsigma, \quad (6.22)$$

and by (H')(c) we get

$$|-h_1(x, s) - \tilde{c}s^{p-1}| \leq |h_1(x, s)| + \tilde{c}s^{p-1} \leq c_\varsigma, \text{ for a.a. } x \in \Omega \text{ and all } s \in [0, s_\varsigma]. \quad (6.23)$$

Therefore, we obtain

$$h_1(x, s) \leq -\tilde{c}s^{p-1} + c_\varsigma, \text{ for a.a. } x \in \Omega \text{ and all } s \geq 0. \quad (6.24)$$

Applying (6.24) to the first integral in (6.21) provides

$$\begin{aligned} \int_{\Omega} (\vartheta^{p-1} - \tilde{c}(\vartheta e)^{p-1} - h_1(x, \vartheta e)) \varphi dx &\geq \int_{\Omega} (\vartheta^{p-1} - \tilde{c}(\vartheta e)^{p-1} + \tilde{c}(\vartheta e)^{p-1} - c_\varsigma) \varphi dx \\ &= \int_{\Omega} (\vartheta^{p-1} - c_\varsigma) \varphi dx, \end{aligned} \quad (6.25)$$

which proves that for $\vartheta \geq c_\varsigma^{\frac{1}{p-1}}$ the integral is nonnegative. Hypothesis (H')(e) implies the existence of a $s_\lambda > 0$ such that

$$\frac{h_2(x, s)}{s^{p-1}} < -\lambda, \text{ for a.a. } x \in \Omega \text{ and all } s > s_\lambda. \quad (6.26)$$

Because of (H')(f) there exists a constant $c_\lambda > 0$ such that

$$|-h_2(x, s) - \lambda s^{p-1}| \leq |h_2(x, s)| + \lambda s^{p-1} \leq c_\lambda, \text{ for a.a. } x \in \Omega \text{ and all } s \in [0, s_\lambda]. \quad (6.27)$$

Finally, we obtain

$$h_2(x, s) \leq -\lambda s^{p-1} + c_\lambda, \text{ for a.a. } x \in \partial\Omega \text{ and all } s \geq 0. \quad (6.28)$$

Applying (6.28) to the second integral in (6.21) yields

$$\begin{aligned} \int_{\partial\Omega} (\vartheta^{p-1} - \lambda(\vartheta e)^{p-1} - h_2(x, \vartheta e)) \varphi dx &\geq \int_{\partial\Omega} (\vartheta^{p-1} - \lambda(\vartheta e)^{p-1} + \lambda(\vartheta e)^{p-1} - c_\lambda) \varphi dx \\ &\geq \int_{\partial\Omega} (\vartheta^{p-1} - c_\lambda) \varphi dx. \end{aligned} \quad (6.29)$$

Choosing $\vartheta \geq \max \left\{ c_\varsigma^{\frac{1}{p-1}}, c_\lambda^{\frac{1}{p-1}} \right\}$ proves that both integrals in (6.21) are nonnegative and thus, $\bar{u} = \vartheta e$ is a positive supersolution of problem (6.10). In order to prove that $\underline{u} = -\vartheta e$ is a negative subsolution we make use of the following estimates

$$\begin{aligned} h_1(x, s) &\geq -\tilde{c}s^{p-1} - c_\varsigma, \text{ for a.a. } x \in \Omega \text{ and all } s \leq 0, \\ h_2(x, s) &\geq -\lambda s^{p-1} - c_\lambda, \text{ for a.a. } x \in \partial\Omega \text{ and all } s \leq 0, \end{aligned} \quad (6.30)$$

which can be derivated as stated above. With the aid of (6.30) one verifies that $\underline{u} = -\vartheta e$ is a negative subsolution of problem (6.10). $\square$

According to Lemma 6.8, we obtain a positive pair $[\varepsilon\varphi_1, \vartheta e]$ and a negative pair $[-\vartheta e, -\varepsilon\varphi_1]$ of sub- and supersolutions of problem (6.10) assumed $\varepsilon > 0$ is sufficiently small.

Now, we will use this result to our variational-hemivariational inequality in (6.9). Due to this, we suppose the following conditions for f and Clarke's generalized gradients of $s \mapsto j_k(x, s)$, $k = 1, 2$, where $\lambda > \lambda_1$ and $\beta \in (0, 1)$ are some fixed constants:

- (F4) (i) $\lim_{|s| \rightarrow \infty} \frac{f(x, s, s)}{|s|^{p-2}s} = +\infty$, uniformly with respect to a.a. $x \in \Omega$
- (ii) $\lim_{s \rightarrow 0} \frac{f(x, s, s)}{|s|^{p-2}s} = \beta$, uniformly with respect to a.a. $x \in \Omega$
- (iii) $\lim_{|s| \rightarrow \infty} \frac{\xi}{|s|^{p-2}s} = +\infty$, uniformly with respect to a.a. $x \in \Omega$, for all $\xi \in \partial j_1(x, s)$
- (iv) $\lim_{s \rightarrow 0} \frac{\xi}{|s|^{p-2}s} = 0$, uniformly with respect to a.a. $x \in \Omega$, for all $\xi \in \partial j_1(x, s)$
- (v) $\lim_{|s| \rightarrow \infty} \frac{\eta}{|s|^{p-2}s} = +\infty$, uniformly with respect to a.a. $x \in \partial\Omega$, for all $\eta \in \partial j_2(x, s)$

$$(vi) \lim_{s \rightarrow 0} \frac{\eta}{|s|^{p-2}s} = \lambda, \text{ uniformly with respect to a.a. } x \in \partial\Omega, \text{ for all } \eta \in \partial j_2(x, s)$$

Proposition 6.9. *Let the conditions (j1)–(j2), (F2)(i)–(F2)(iii) and (F4) be satisfied. Then there is a constant ϑ such that ϑe and $-\vartheta e$ are supersolution and subsolution of problem (6.9), where $e \in \text{int}(C^1(\bar{\Omega})_+)$ is the unique solution of (6.12). Moreover, $-\varepsilon\varphi_1$ is a supersolution and $\varepsilon\varphi_1$ is a subsolution of (6.9) provided that $\varepsilon > 0$ is sufficiently small.*

Proof. As in the proof of Proposition 6.3, we first give a sufficient condition for a subsolution of (6.9). A function $\underline{u} \in W^{1,p}(\Omega)$ is a subsolution of problem (6.9) if $F(\underline{u}) \in L^q(\Omega)$ and if the inequality

$$-\Delta_p \underline{u} + F(\underline{u}) + \xi + \eta \leq 0 \text{ in } (W^{1,p}(\Omega))^*, \text{ for all } \xi \in \partial j_1(\cdot, \underline{u}) \text{ and all } \eta \in \partial j_2(\cdot, \underline{u}), \quad (6.31)$$

is fulfilled. To prove this, we multiply (6.31) with $(\underline{u} - v)^+ \in W^{1,p}(\Omega) \cap L_+^p(\Omega)$ and we use the fact $j_1^0(\cdot, \underline{u}; -1) \geq -\xi$, for all $\xi \in \partial j_1(\cdot, \underline{u})$ and $j_2^0(\cdot, \gamma \underline{u}; -\gamma 1) \geq -\eta$, for all $\eta \in \partial j_2(\cdot, \underline{u})$, to obtain

$$\begin{aligned} 0 &\geq \langle -\Delta_p \underline{u} + F(\underline{u}) + \xi + \eta, (\underline{u} - v)^+ \rangle \\ &= \langle -\Delta_p \underline{u} + F(\underline{u}), (\underline{u} - v)^+ \rangle + \int_{\Omega} \xi (\underline{u} - v)^+ dx + \int_{\partial\Omega} \xi \gamma (\underline{u} - v)^+ d\sigma \\ &\geq \langle -\Delta_p \underline{u} + F(\underline{u}), (\underline{u} - v)^+ \rangle - \int_{\Omega} j_1^0(\cdot, \underline{u}; -1) (\underline{u} - v)^+ dx - \int_{\partial\Omega} j_2^0(\cdot, \gamma \underline{u}; -\gamma 1) \gamma (\underline{u} - v)^+ d\sigma \\ &= \langle -\Delta_p \underline{u} + F(\underline{u}), (\underline{u} - v)^+ \rangle - \int_{\Omega} j_1^0(\cdot, \underline{u}; -(\underline{u} - v)^+) dx - \int_{\partial\Omega} j_2^0(\cdot, \gamma \underline{u}; -\gamma (\underline{u} - v)^+) d\sigma, \quad \forall v \in K, \end{aligned}$$

which shows that $\underline{u}$ is a subsolution of (6.9). By the same calculation, we obtain that $\bar{u} \in W^{1,p}(\Omega)$ is a supersolution of problem (6.9) if $F(\bar{u}) \in L^q(\Omega)$ and if the following inequality is satisfied

$$-\Delta_p \bar{u} + F(\bar{u}) + \xi + \eta \geq 0 \text{ in } (W^{1,p}(\Omega))^*, \text{ for all } \xi \in \partial j_1(\cdot, \bar{u}) \text{ and for all } \eta \in \partial j_2(\cdot, \bar{u}).$$

Now, we set $h_1(x, s) = \beta|s|^{p-2}s - f(x, s, s) - \xi$ for $\xi \in \partial j_1(x, s)$ and $h_2(x, s) = -\lambda|s|^{p-2}s - \eta$ for $\eta \in \partial j_2(x, s)$ and notice that in our considerations the nonlinearities h_1 and h_2 do not need to be continuous functions. Applying assumption (F3) provides

$$\begin{aligned} \lim_{|s| \rightarrow \infty} \frac{h_1(x, s)}{|s|^{p-2}s} &= -\infty, \quad \text{and} \quad \lim_{s \rightarrow 0} \frac{h_1(x, s)}{|s|^{p-2}s} = 0, \\ \lim_{|s| \rightarrow \infty} \frac{h_2(x, s)}{|s|^{p-2}s} &= -\infty, \quad \text{and} \quad \lim_{s \rightarrow 0} \frac{h_2(x, s)}{|s|^{p-2}s} = 0. \end{aligned}$$

With a view to Lemma 6.8, we see that the suppositions therein are satisfied. This yields an ordered pair of positive sub- and supersolutions given by $\underline{u} = \varepsilon\varphi_1$ and $\bar{u} = \vartheta e$, respectively, a pair of negative sub- and supersolutions given by $\underline{u} = -\vartheta e$ and $\bar{u} = -\varepsilon\varphi_1$ of problem (6.9). $\square$

Note again that in order to apply the existence and comparison result the constructed sub-supersolutions have to satisfy additional conditions related to K meaning that

$$\underline{u} \vee K \subset K, \quad \bar{u} \wedge K \subset K$$

are fulfilled, too.

Example 6.10. *Let the function $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be given by*

$$f(x, r, s) = \begin{cases} (\beta - 2s)|s|^{p-2}s + e^{|r|+|x|}|r|^{p-1}r & \text{for } s < -1, \\ (\beta - s)|s|^{p-2}s + e^{|r|+|x|}|r|^{p-1}r & \text{for } -1 \leq s \leq 1, \\ (-\beta - s)|s|^{p-2}s + e^{|r|+|x|}|r|^{p-1}r & \text{for } s > 1. \end{cases} \quad (6.32)$$

One easily verifies the validity of the assumptions (F2) and (F4), where $0 < \beta < 1$ is fixed. Moreover, Clarke's gradient $s \mapsto \partial j(x, s)$ from Example 6.4 can also be used as example for

$\partial j_1(x, \cdot)$. The function

$$\partial j_2(x, s) = \begin{cases} (e^{-(s+1)} - 2)|s|^{p-2}s & \text{if } s < -1, \\ [-\lambda, 1] & \text{if } s = -1, \\ \lambda|s|^{p-2}s & \text{if } -1 < s < 1, \\ [-1, \lambda] & \text{if } s = 1, \\ (e^{(s-1)(|x|+1)} + s - 3)s^{p-1} & \text{if } s > 1, \end{cases} \quad (6.33)$$

satisfies the assumptions (j1)–(j3) for some fixed $\lambda > \lambda_1$.

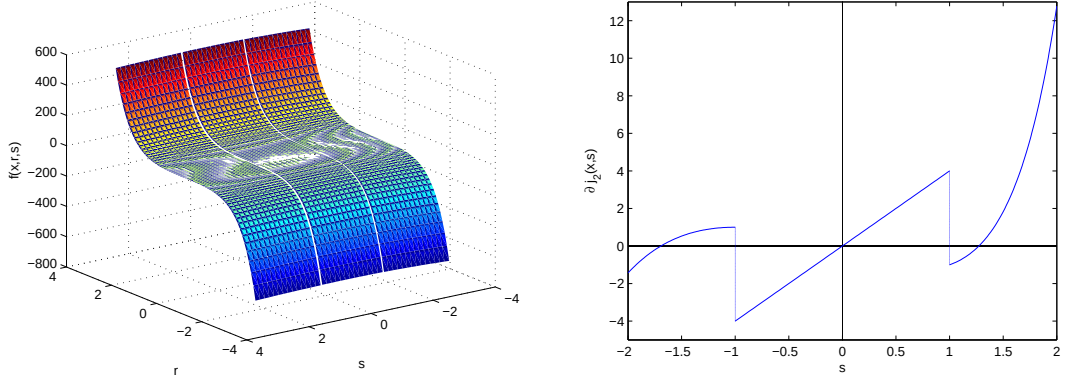


FIGURE 2. The function f and Clarke's generalized gradient $s \mapsto j_2(x, s)$ with respect to $\Omega = (0, \pi)$, $\varphi_1(x) = e^x$, $\lambda_1 = 1$, $\lambda = 4$, $p = 2$, $\beta = 0.5$ and $x = 1$.

To obtain extremal solutions of problem (1.1), it is required that the given closed convex set K fulfills the lattice structure conditions as stated in (4.17). In Remark 6.5 the one-sided obstacle problem is presented as a closed convex set in $W_0^{1,p}(\Omega)$ satisfying these assumptions. The same holds true in $W^{1,p}(\Omega)$ meaning that

$$K = \{v \in W^{1,p}(\Omega) : v(x) \leq \psi(x) \text{ for a.e. } x \in \Omega\}, \quad \psi \in L^\infty(\Omega), \psi \geq C > 0, \quad (6.34)$$

is a closed convex set in $W^{1,p}(\Omega)$ having lattice structure. Other interest closed convex sets in $W^{1,p}(\Omega)$ are the following

$$\begin{aligned} K_1 &= \{v \in W^{1,p}(\Omega) : \alpha(x) \leq v(x) \text{ a.e. } x \in \Omega\}, \\ K_2 &= \{v \in W^{1,p}(\Omega) : \beta_1(x) \leq v(x) \leq \beta_2(x) \text{ a.e. } x \in \Omega\}, \\ K_3 &= \{v \in W^{1,p}(\Omega) : |\nabla v(x)| \leq C \text{ a.e. } x \in \Omega\}, \\ K_4 &= \{v \in W^{1,p}(\Omega) : \int_{\Omega} v(x) dx \geq a_1\}, \\ K_5 &= \{v \in W^{1,p}(\Omega) : \int_{\Omega} v(x) dx \leq a_2\}, \end{aligned} \quad (6.35)$$

where β_1, β_2 and α are given functions and a_1, a_2 and C are some constants. One sees at once that K_1, K_2 and K_3 fulfill the assumptions in (4.17), however, the sets K_4 and K_5 just satisfy $K \wedge K \subset K$ and $K \vee K \subset K$, respectively.

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- 01-09.** Jan Prüß, Vicente Vergara, Rico Zacher, *Well-Posedness and Long-Time Behaviour for the Non-Isothermal Cahn-Hilliard Equation with Memory*