Geometric Duality in Multiple Objective Linear Programming

Frank Heyde* Andreas Löhne†

November 9, 2006

Abstract

We develop in this article a geometric approach to duality in Multiple Objective Linear Programming. This approach is based on a very old idea, the duality of polytopes, which can be traced back to the old Greeks. We show that there is an inclusion reversing one-to-one map between the minimal faces of the image of the primal objective and the maximal faces of the image of the dual objective map.

1 Introduction

Duality for multiple objective linear programs seems to have its origin in 70th, see e.g. Kornbluth [13], Roedder [17], Isermann [9, 10] and Brumelle [2]. More recent expositions are Jahn [11, 12], Luc [15] and Göpfert and Nehse [4], where also nonlinear problems are considered.

As noticed in [4, p. 64], the practical relevance of vectorial duality theory is quite low in comparison with the relevance of duality in scalar optimization. Moreover, in the linear case there occured some difficulties, such as a duality gap in the case $b = 0$ (where b is the right-hand side of the inequality constraints). In [6], this duality gap could be closed by using a set-valued approach. In [14, 7, 8], this set-valued approach is revisited from a lattice theoretic point of view. The aim of these papers is to work in an appropriate complete lattice in order to have a duality theory which can be formulated along the lines of the scalar duality theory. In particular, the infimum and supremum can be used to define solutions. Another goal (especially in [8]) is to have a "simple" dual problem. This means, the dual problem should be at least not more complicated than the primal problem.

Nevertheless, in all the mentioned references there is a basic difference to the present article. Instead of speaking about strong duality if the optimal values of a pair of dual optimization problems are equal, we deal with a duality relation between the polyhedral image set of the primal problem and the polyhedral image of the dual problem, which is similar to duality of polytopes (see Figure 1).

It is well-known from the theory of convex polytopes (see e.g. [5]) that two polytopes $\mathcal{P}$ and $\mathcal{P}^*$ in $\mathbb{R}^q$ are said to be dual to each other provided there exists a one-to-one mapping Ψ between the set of all faces of $\mathcal{P}$ and the set of all faces of $\mathcal{P}^*$ such that Ψ is inclusion-reversing, i.e., faces $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{P}$ satisfy $\mathcal{F}_1 \subseteq \mathcal{F}_2$ if and only if the faces $\Psi(\mathcal{F}_1)$ and $\Psi(\mathcal{F}_2)$ satisfy $\Psi(\mathcal{F}_1) \supseteq \Psi(\mathcal{F}_2)$ [5].

Denoting by $\mathcal{P}$ and $\mathcal{D}$ the images of the objective functions of our given problem (P) and its dual problem (D), respectively, we show that there is an inclusion reversing one-to-one map Ψ between the set of all K-maximal proper faces of $\mathcal{D}$ and the set of all weakly C-minimal proper faces of $\mathcal{P}$, where K and C are appropriate ordering cones. With the aid of such a map Ψ we

*Martin-Luther-Universität Halle–Wittenberg, NWF III, Institut für Mathematik, D-06099 Halle, Germany, email: frank.heyde@mathematik.uni-halle.de.

†Martin-Luther-Universität Halle–Wittenberg, NWF III, Institut für Mathematik, D-06099 Halle, Germany, email: andreas.loehne@mathematik.uni-halle.de.

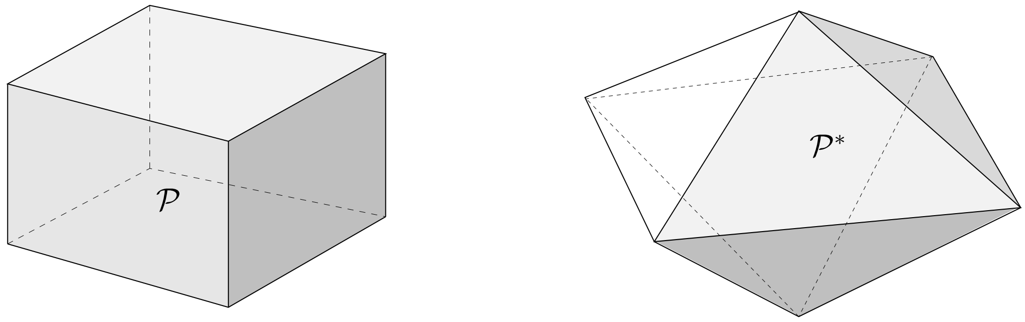

Figure 1: Example of a pair of dual polytopes in $\mathbb{R}^3$.

can compute the weakly C-minimal faces of $\mathcal{P}$ whenever we know the K-maximal faces of $\mathcal{D}$ and vice versa. In particular, we are given by Ψ a one-to-one correspondence between weakly C-minimal vertices (facets) of $\mathcal{P}$ and K-maximal facets (vertices) of $\mathcal{D}$. It is worth to mention that there is a connection between the lattice theoretic duality in [8] and the geometric duality in the present article. This is shortly discussed in the end of Section 3.

In a forthcoming paper [3] we give an application of geometric duality, a dual variant of Benson's outer approximation algorithm [1].

2 Preliminaries

Let $\mathcal{A} \subseteq \mathbb{R}^q$ and let $\mathcal{C} \subseteq \mathbb{R}^q$ be a closed convex cone. Denoting by $\operatorname{ri}\mathcal{C}$ the relative interior of $\mathcal{C}$, we set

$$\operatorname{Min}_{\mathcal{C}}\mathcal{A} := \{y \in \mathcal{A} \mid (\{y\} - \operatorname{ri}\mathcal{C}) \cap \mathcal{A} = \emptyset\} \qquad \text{and} \qquad \operatorname{Max}_{\mathcal{C}}\mathcal{A} := \operatorname{Min}_{(-\mathcal{C})}\mathcal{A}.$$

In the following we consider two special ordering cones, namely

$$C := \mathbb{R}^q_+ \qquad \text{and} \qquad K := \mathbb{R}_+ \cdot (0, 0, \dots, 0, 1)^T = \{y \in \mathbb{R}^q \mid y_1 = \dots = y_{q-1} = 0, y_q \geq 0\}.$$

For the choice $\mathcal{C} = C$ we obtain the set of weakly C-minimal elements of $\mathcal{A}$, given by

$$\operatorname{Min}_C\mathcal{A} := \left\{y \in \mathcal{A} \mid (\{y\} - \operatorname{int}\mathbb{R}^q_+) \cap \mathcal{A} = \emptyset\right\}.$$

In case of $\mathcal{C} = K$ we get the set of K-maximal elements of $\mathcal{A}$, namely

$$\operatorname{Max}_K\mathcal{A} := \{y \in \mathcal{A} \mid (\{y\} + K \setminus \{0\}) \cap \mathcal{A} = \emptyset\}.$$

For the convenience of the reader, we recall some facts concerning the facial structure of polyhedral sets [18]. Let $\mathcal{A} \subseteq \mathbb{R}^q$ be a convex set. A convex subset $\mathcal{F} \subseteq \mathcal{A}$ is called a *face* of $\mathcal{A}$ if

$$\left(y^1, y^2 \in \mathcal{A}, \quad \lambda \in (0,1), \quad \lambda y^1 + (1-\lambda) y^2 \in \mathcal{F}\right) \quad \Rightarrow \quad y^1, y^2 \in \mathcal{F}.$$

A face $\mathcal{F}$ of $\mathcal{A}$ is called *proper* if $\emptyset \neq \mathcal{F} \neq \mathcal{A}$. A set $\mathcal{E} \subseteq \mathcal{A}$ is called an *exposed face* of $\mathcal{A}$ if there are $c \in \mathbb{R}^q$ and $\gamma \in \mathbb{R}$ such that $\mathcal{A} \subseteq \left\{y \in \mathbb{R}^q \mid c^T y \geq \gamma\right\}$ and $\mathcal{E} = \left\{y \in \mathbb{R}^q \mid c^T y = \gamma\right\} \cap \mathcal{A}$. The proper $(r-1)$-dimensional faces of an r-dimensional polyhedral set $\mathcal{A}$ are called *facets* of $\mathcal{A}$. A point $y \in \mathcal{A}$ is called a vertex of $\mathcal{A}$ if $\{y\}$ is a face of $\mathcal{A}$. Let $\mathcal{A}$ be a polyhedral set in $\mathbb{R}^q$. Then $\mathcal{A}$ has a finite number of faces, each of which is exposed and a polyhedral set. Every proper face of $\mathcal{A}$ is the intersection of those facets of $\mathcal{A}$ that contain it, and the relative boundary of $\mathcal{A}$ is the union of all the facets of $\mathcal{A}$. If $\mathcal{A}$ has a nonempty face of dimension s, then $\mathcal{A}$ has faces of all dimensions from s to $\dim\mathcal{A}$ (see [18], Theorem 3.2.2).

If $\operatorname{int} \mathcal{A} \neq \emptyset$ then $\mathcal{A}$ is a q-dimensional polyhedral set, hence the facets of $\mathcal{A}$ are the $(q-1)$-dimensional faces of $\mathcal{A}$, i.e., the maximal (w.r.t. inclusion) proper faces. A subset $\mathcal{F} \subseteq \mathcal{A}$ is a proper face if and only if it is a proper exposed face, i.e., there is a supporting hyperplane $\mathcal{H}$ to $\mathcal{A}$ such that $\mathcal{F} = \mathcal{H} \cap \mathcal{A}$. We call a hyperplane $\mathcal{H} := \left\{y \in \mathbb{R}^q \mid c^T y = \gamma\right\}$ (i.e., $c \neq 0$) supporting to $\mathcal{A}$ if

$$\forall y \in \mathcal{A} : c^T y \geq \gamma \qquad \wedge \qquad \exists y^0 \in \mathcal{A} : c^T y^0 = \gamma.$$

3 Main result

Throughout the article, let $m, n, q \in \mathbb{N}$ and $A \in \mathbb{R}^{m \times n}, M \in \mathbb{R}^{q \times n}, b \in \mathbb{R}^m$ be given and let the ordering cones C and K be defined as above. Further we set $k = (1, \ldots, 1)^T \in \mathbb{R}^q$. We consider the following vector optimization problem

$$\text{(P)} \qquad \operatorname{Min}_C M[\mathcal{X}], \qquad \mathcal{X} := \{x \in \mathbb{R}^n \mid Ax \geq b\},$$

We define a dual linear objective function by $D : \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^q,\ D(u, c) := \left(c_1, ..., c_{q-1}, b^T u\right)^T$ and consider the following dual vector optimization problem

$$\text{(D)} \qquad \operatorname{Max}_K D[\mathcal{U}], \qquad \mathcal{U} := \left\{(u, c) \in \mathbb{R}^m \times \mathbb{R}^q \mid (u, c) \geq 0,\ A^T u = M^T c,\ k^T c = 1\right\}.$$

It is our goal to show a duality relation between the sets

$$\begin{array}{lclcll} \mathcal{P} & := & M[\mathcal{X}] + C & = & \{y \in \mathbb{R}^q \mid \quad \exists x \in \mathcal{X} : \ y \in \{Mx\} + C \quad \} & \text{and} \\ \mathcal{D} & := & D[\mathcal{U}] - K & = & \{y \in \mathbb{R}^q \mid \exists (u, c) \in \mathcal{U} : \ y \in \{D(u, c)\} - K \ \}. & \end{array}$$

To this end we construct an inclusion reversing one-to-one map Ψ between the K-maximal proper faces of $\mathcal{D}$ and the weakly C-minimal proper faces of $\mathcal{P}$.

Consider the coupling function $\varphi : \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}$, defined by

$$\varphi(y, v) := \sum_{i=1}^{q-1} y_i v_i + y_q \Big(1 - \sum_{i=1}^{q-1} v_i\Big) - v_q.$$

Note that $\varphi(\cdot, v)$ and $\varphi(y, \cdot)$ are affine. Choosing the values of the primal and dual objective function as arguments, we just get

$$\varphi(Mx, D(u, c)) = c^T Mx - b^T u. \tag{1}$$

The coupling function φ is used to define the following two set-valued maps

$$\mathcal{H} : \mathbb{R}^q \rightrightarrows \mathbb{R}^q, \quad \mathcal{H}(v) := \{y \in \mathbb{R}^q \mid \varphi(y, v) = 0\},$$

$$\mathcal{H}^* : \mathbb{R}^q \rightrightarrows \mathbb{R}^q, \quad \mathcal{H}^*(y) := \{v \in \mathbb{R}^q \mid \varphi(y, v) = 0\}.$$

Of course, $\mathcal{H}(v)$ and $\mathcal{H}^*(y)$ are hyperplanes in $\mathbb{R}^q$ for all $v, y \in \mathbb{R}^q$. Using the notation

$$c(v) := \left(v_1, \ldots, v_{q-1}, 1 - \sum_{i=1}^{q-1} v_i\right)^T \qquad \text{and} \qquad c^*(y) := \left(y_1 - y_q, \ldots, y_{q-1} - y_q, -1\right)^T$$

it is easy to see that

$$\mathcal{H}(v) = \left\{y \in \mathbb{R}^q \mid c(v)^T y = v_q\right\} \qquad \text{and} \qquad \mathcal{H}^*(y) = \left\{v \in \mathbb{R}^q \mid c^*(y)^T v = -y_q\right\}.$$

Obviously, the set-valued maps $\mathcal{H}$ and $\mathcal{H}^*$ are injective. The map $\mathcal{H}$ is now used to define the function $\Psi : 2^{\mathbb{R}^q} \to 2^{\mathbb{R}^q}$,

$$\Psi(\mathcal{F}^*) := \bigcap_{v \in \mathcal{F}^*} \mathcal{H}(v) \cap \mathcal{P}.$$

It follows the main result which shows that Ψ is a duality map between $\mathcal{P}$ and $\mathcal{D}$.

Theorem 1. *Ψ is an inclusion reversing one-to-one map between the set of all K-maximal proper faces of $\mathcal{D}$ and the set of all weakly C-minimal proper faces of $\mathcal{P}$ and the inverse map is given by*

$$\Psi^{-1}(\mathcal{F}) = \bigcap_{y \in \mathcal{F}} \mathcal{H}^*(y) \cap \mathcal{D}. \tag{2}$$

Moreover, for every K-maximal proper face $\mathcal{F}^*$ of $\mathcal{D}$ it holds $\dim \mathcal{F}^* + \dim \Psi(\mathcal{F}^*) = q - 1$.

The proof of this theorem is given in the last section.

Let us consider an important special case. Vertices as well as facets are actually the most important faces from the point of view of applications. Therefore we extract some corresponding conclusions from the above theorem.

Corollary 1. *The following statements are equivalent.*

(i) *v is a K-maximal vertex of $\mathcal{D}$.*

(ii) *$\mathcal{H}(v) \cap \mathcal{P}$ is a weakly C-minimal $(q-1)$-dimensional facet of $\mathcal{P}$.*

Moreover, if $\mathcal{F}$ is a weakly C-minimal $(q-1)$-dimensional facet of $\mathcal{P}$, there is some uniquely defined point $v \in \mathbb{R}^q$ such that $\mathcal{F} = \mathcal{H}(v) \cap \mathcal{P}$.

Proof. (i) $\Rightarrow$ (ii). Since $\mathcal{H}(v) \cap \mathcal{P} = \Psi(\{v\})$, Theorem 1 implies that $\mathcal{H}(v) \cap \mathcal{P}$ is a weakly C-minimal proper face of $\mathcal{P}$. Theorem 1 also implies that $\dim(\mathcal{H}(v) \cap \mathcal{P}) = q - 1 - \dim\{v\} = q - 1$.

(ii) $\Rightarrow$ (i). Let $\mathcal{H}(v) \cap \mathcal{P}$ be a weakly C-minimal $(q-1)$-dimensional facet of $\mathcal{P}$. By Theorem 1, $\Psi^{-1}(\mathcal{H}(v) \cap \mathcal{P})$ is a K-maximal vertex of $\mathcal{D}$, denoted by $\bar{v}$. It follows that $\Psi \circ \Psi^{-1}(\mathcal{H}(v) \cap \mathcal{P}) = \Psi(\{\bar{v}\})$ and hence $\mathcal{H}(v) \cap \mathcal{P} = \mathcal{H}(\bar{v}) \cap \mathcal{P}$ implying $\mathcal{H}(v) = \mathcal{H}(\bar{v})$ as $\dim(\mathcal{H}(v) \cap \mathcal{P}) = q - 1$. The mapping $\mathcal{H}$ being injective implies $v = \bar{v}$.

To show the last statement, let $\mathcal{F}$ be a weakly C-minimal $(q-1)$-dimensional facet of $\mathcal{P}$. Hence $\Psi^{-1}(\mathcal{F})$ is a K-maximal vertex of $\mathcal{D}$, denoted by v. It follows that $\mathcal{F} = \Psi \circ \Psi^{-1}(\mathcal{F}) = \Psi(\{v\}) = \mathcal{H}(v) \cap \mathcal{P}$. By $\dim(\mathcal{H}(v) \cap \mathcal{P}) = q - 1$ and $\mathcal{H}$ being injective, v is uniquely defined. $\square$

Corollary 2. *The following statements are equivalent.*

(i) *y is a weakly C-minimal vertex of $\mathcal{P}$.*

(ii) *$\mathcal{H}^*(y) \cap \mathcal{D}$ is a K-maximal $(q-1)$-dimensional facet of $\mathcal{D}$.*

Moreover, if $\mathcal{F}^*$ is a K-maximal $(q-1)$-dimensional facet of $\mathcal{D}$, there is some uniquely defined point $y \in \mathbb{R}^q$ such that $\mathcal{F}^* = \mathcal{H}^*(y) \cap \mathcal{D}$.

Proof. (i) $\Rightarrow$ (ii). Let y be a weakly C-minimal vertex of $\mathcal{P}$. By Theorem 1, the set $\mathcal{F}^* := \Psi^{-1}(\{y\}) = \mathcal{H}^*(y) \cap \mathcal{D}$ is a K-maximal face of $\mathcal{D}$. From Theorem 1 we also conclude that $\dim \mathcal{F}^* = q - 1 - \dim\{y\} = q - 1$. Thus $\mathcal{F}^*$ is a facet of $\mathcal{D}$.

(ii) $\Rightarrow$ (i). Let $\mathcal{H}^*(y) \cap \mathcal{D}$ be a K-maximal $(q-1)$-dimensional facet of $\mathcal{D}$. By Theorem 1 $\Psi(\mathcal{H}^*(y) \cap \mathcal{D})$ is a weakly C-minimal vertex of $\mathcal{P}$, denoted by $\bar{y}$. It follows that $\Psi^{-1} \circ \Psi(\mathcal{H}^*(y) \cap \mathcal{D}) = \Psi^{-1}(\{\bar{y}\})$ and hence $\mathcal{H}^*(y) \cap \mathcal{D} = \mathcal{H}^*(\bar{y}) \cap \mathcal{D}$. Since $\dim(\mathcal{H}^*(y) \cap \mathcal{D}) = q - 1$ and $\mathcal{H}^*$ is injective, we get $y = \bar{y}$.

To show the last statement, let $\mathcal{F}^*$ be a K-maximal $(q-1)$-dimensional facet of $\mathcal{D}$. Hence $\Psi(\mathcal{F}^*)$ is a C-minimal vertex of $\mathcal{P}$, denoted by y. It follows that $\mathcal{F}^* = \Psi^{-1} \circ \Psi(\mathcal{F}^*) = \Psi^{-1}(\{y\}) = \mathcal{H}^*(y) \cap \mathcal{D}$. By $\dim(\mathcal{H}^*(y) \cap \mathcal{D}) = q - 1$ and $\mathcal{H}^*$ being injective, y is uniquely defined. $\square$

Remark. In [8] we developed a duality theory based on a lattice theoretic approach. The dual problem (D) in the present article is related to the set-valued dual problem in [8]. Indeed, both problems have the same constraints, given by $\mathcal{U}$. The set-valued objective map of the dual problem in [8] can be expressed by the objective function of (D) as $(u, c) \mapsto \mathcal{H}(D(u, c))$. Moreover, (u, c) being a weakly efficient solution for the dual problem (LD) in [8] is equivalent to $D(u, c)$ being a K-maximal point of $\mathcal{D}$.

4 Examples

The geometric duality is illustrated by the following two examples.

Example 1. Consider problem (P) with the following data

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & -1 \\ 2 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}, \qquad b = \begin{pmatrix} -4 \\ 4 \\ 3 \\ 4 \end{pmatrix}.$$

The set $\mathcal{D}$ can be easily calculated as $\mathcal{D} = \text{co}\,\left\{(\frac{1}{3},\frac{4}{3})^T,(\frac{1}{2},\frac{3}{2})^T,(\frac{2}{3},\frac{4}{3})^T,(1,0)^T\right\} - K$, where co $\mathcal{A}$ denotes the convex hull of a set $\mathcal{A}$ (see Figure 2).

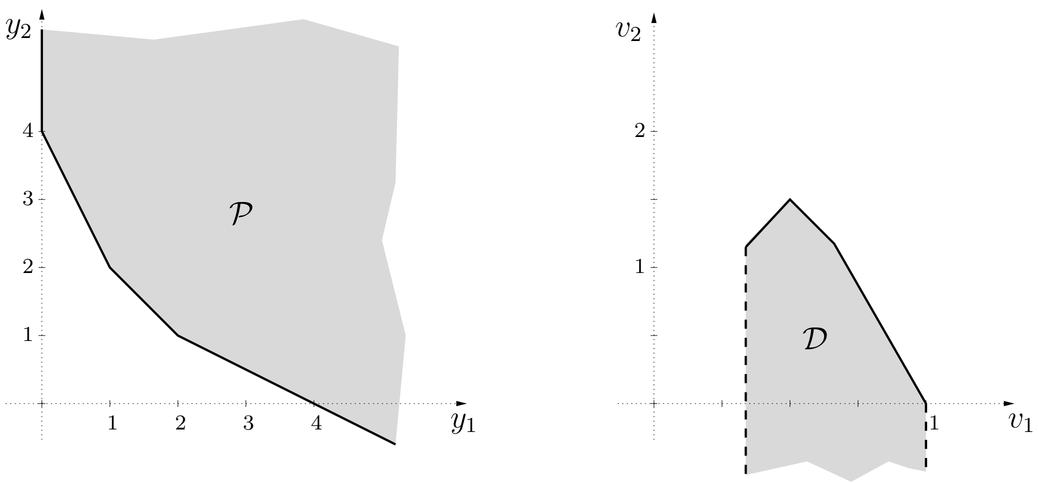

Figure 2: The three weakly C-minimal vertices of $\mathcal{P}$ correspond to the three K-maximal facets of $\mathcal{D}$ and the four weakly C-minimal facets of $\mathcal{P}$ correspond to the four K-maximal vertices of $\mathcal{D}$.

Example 2. Consider problem (P) with the following data

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}.$$

An easy computation shows that $\mathcal{D} = \text{co}\,\left\{(0,0,0)^T,(1,0,0)^T,(0,1,0)^T,(\frac{1}{3},\frac{1}{3},\frac{1}{3})^T\right\} - K$ (see Figure 3).

5 Proof of the main result

The proof of the main result is based on several auxiliary assertions, which are given below. The following pairs of dual scalar linear optimization problems, depending on parameters $v, y \in \mathbb{R}^q$, play an important role in the following considerations.

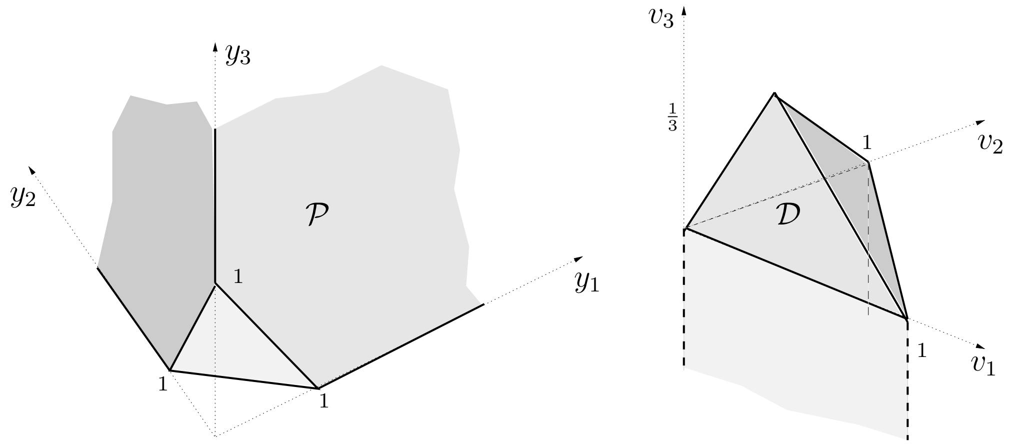

Figure 3: The three weakly C-minimal vertices of $\mathcal{P}$ correspond to the three K-maximal facets of $\mathcal{D}$, the six weakly C-minimal edges of $\mathcal{P}$ correspond to the six K-maximal edges of $\mathcal{D}$ and the four weakly C-minimal facets of $\mathcal{P}$ correspond to the four K-maximal vertices of $\mathcal{D}$.

$$(\mathrm{P}_1(v)) \qquad \min_{x\in\mathcal{X}} c(v)^T Mx \qquad \mathcal{X} := \{x \in \mathbb{R}^n \mid Ax \geq b\}$$

$$(\mathrm{D}_1(v)) \qquad \max_{u\in\mathcal{T}(v)} b^T u \qquad \mathcal{T}(v) := \left\{u \in \mathbb{R}^m \mid u \geq 0, \;\; A^T u = M^T c(v)\right\}$$

$$(\mathrm{P}_2(y)) \qquad \min_{x\in\mathcal{S}(y)} z \qquad \mathcal{S}(y) := \{(x,z) \in \mathbb{R}^n \times \mathbb{R} \mid Ax \geq b, \;\; Mx - kz \leq y\}$$

$$(\mathrm{D}_2(y)) \qquad \max_{(u,c)\in\mathcal{U}} (b^T u - y^T c) \qquad \mathcal{U} := \left\{(u,c) \in \mathbb{R}^m \times \mathbb{R}^q \mid (u,c) \geq 0, \;\; A^T u = M^T c, \;\; k^T c = 1\right\}$$

Note that with the above notation it holds

$$\mathcal{D} = \left\{v \in \mathbb{R}^q \mid c(v) \geq 0, \exists u \in \mathcal{T}(v) : b^T u \geq v_q\right\}. \tag{3}$$

We start with a characterization of weakly C-minimal points of $\mathcal{P}$.

Lemma 1. *The following three statements are equivalent.*

(i) $y^0 \in \operatorname{Min}_C \mathcal{P}$.

(ii) *there is some* $x^0 \in \mathbb{R}^n$ *such that* $(x^0, 0)$ *solves* $(\mathrm{P}_2(y^0))$.

(iii) *there is some* $(u^0, c^0) \in \mathcal{U}$ *with* $b^T u^0 = {y^0}^T c^0$ *solving* $(\mathrm{D}_2(y^0))$.

Proof. (ii)⇒(i). If $(x^0, 0)$ solves $(\mathrm{P}_2(y^0))$ then $x^0 \in \mathcal{X}$ and $Mx^0 \leq y^0$ hence $y^0 \in \mathcal{P}$. Assume that there is some $y \in \mathcal{P}$ (i.e., there is some $x \in \mathcal{X}$ with $Mx \leq y$) with $y < y^0$ then there is some $z < 0$ such that $y \leq y^0 + kz$, whence $Mx - kz \leq y - kz \leq y^0$. Thus we have $(x, z) \in \mathcal{S}(y^0)$ where $z < 0$. This contradicts the optimality of $(x^0, 0)$.

(i)⇒(ii). If $y^0 \in \operatorname{Min}_C \mathcal{P}$ then there exists some $x^0 \in \mathcal{X}$ with $Mx^0 \leq y^0$, i.e., $(x^0, 0) \in \mathcal{S}(y^0)$. Assume that there is some $(x, z) \in \mathcal{S}(y^0)$ with $z < 0$. Let $y := y^0 + zk$ then $y < y^0$ and $Mx \leq y^0 + kz = y$, i.e., $y \in \mathcal{P}$. This contradicts y^0 being weakly C-minimal.

(ii)⇔(iii). By duality of $(\mathrm{P}_2(y^0))$ and $(\mathrm{D}_2(y^0))$.

□

Lemma 2. *Every K-maximal proper face of $\mathcal{D}$ contains a vertex.*

Proof. Let $\mathcal{F}^*$ be a K-maximal proper face of $\mathcal{D}$. It suffices to show that $\mathcal{F}^*$ contains no lines ([16], Cor. 18.5.3.). Assume on the contrary that $\mathcal{F}^*$ contains a line, i.e., there are $\bar{v} \in \mathcal{F}^*$ and $\psi \in \mathbb{R}^q \setminus \{0\}$ such that $\bar{v} + \lambda\psi \in \mathcal{F}^*$ for all $\lambda \in \mathbb{R}$. Since for every $v \in \mathcal{F}^* \subseteq \mathcal{D}$ it holds $v_1 \geq 0, \dots, v_{q-1} \geq 0$ we have $\psi_1 = \dots = \psi_{q-1} = 0$. Thus, $\psi \neq 0$ implies $K \subseteq \{\lambda\psi \mid \lambda \in \mathbb{R}\}$. We get $\{\bar{v}\} + K \subseteq \mathcal{F}^*$, contradicting the K-maximality of $\mathcal{F}^*$. $\square$

Lemma 3. *Consider a hyperplane $\mathcal{H}^* := \left\{v \in \mathbb{R}^q \mid {c^*}^T v = \gamma\right\}$. Then the following statements are equivalent.*

(i) *$\mathcal{H}^*$ is a supporting hyperplane to $\mathcal{D}$ such that $\mathcal{H}^* \cap \mathcal{D}$ is K-maximal.*

(ii) *$\mathcal{H}^*$ is a supporting hyperplane to $D[\mathcal{U}]$ and $c^*_q < 0$.*

Proof. (i) $\Rightarrow$ (ii). If $\mathcal{H}^*$ is a supporting hyperplane to $\mathcal{D}$, then there is some $v^0 \in \mathcal{D}$ with ${c^*}^T v^0 = \gamma$ and for $v \in \mathcal{D}$ it holds ${c^*}^T v \geq \gamma$. By definition of $\mathcal{D}$ we have $\bar{v} := v^0 - e_q \in \mathcal{D}$ ($e_q = (0, ..., 0, 1)^T$), implying that $c^*_q \leq 0$. Since $c^*_q = 0$ would imply $\bar{v} \in \mathcal{H}^* \cap \mathcal{D}$ and $v^0 \in (\bar{v} + K \setminus \{0\}) \cap \mathcal{D}$, contradicting the maximality of $\mathcal{H}^* \cap \mathcal{D}$, we conclude $c^*_q < 0$. As $v^0 \in \mathcal{D}$, there are $v^1 \in D[\mathcal{U}] \subseteq \mathcal{D}$ and $z \geq 0$ such that $v^0 = v^1 - e_q z$. Hence ${c^*}^T v^1 = {c^*}^T v^0 + c^*_q z \leq \gamma$. This implies ${c^*}^T v^1 = \gamma$. Therefore $\mathcal{H}^*$ is a supporting hyperplane to $D[\mathcal{U}]$.

(ii) $\Rightarrow$ (i). If $\mathcal{H}^*$ is a supporting hyperplane to $D[\mathcal{U}]$ then there is some $v^0 \in D[\mathcal{U}]$ with ${c^*}^T v^0 = \gamma$ and for all $v \in D[\mathcal{U}]$ it holds ${c^*}^T v \geq \gamma$. Since $c^*_q < 0$, it follows that ${c^*}^T v \geq \gamma$ for all $v \in D[\mathcal{U}] - K = \mathcal{D}$. By $v^0 \in \mathcal{D}$ and ${c^*}^T v^0 = \gamma$ we conclude that $\mathcal{H}^*$ is a supporting hyperplane to $\mathcal{D}$.

In order to show that $\mathcal{H}^* \cap \mathcal{D}$ is K-maximal, let $v^0 \in \mathcal{H}^* \cap \mathcal{D}$ be given. Hence, ${c^*}^T v^0 = \gamma$. For every $v \in v^0 + K \setminus \{0\}$ it holds ${c^*}^T v < \gamma$, because of $c^*_q < 0$. Since ${c^*}^T v \geq \gamma$ for all $v \in \mathcal{D}$, we obtain $(v^0 + K \setminus \{0\}) \cap \mathcal{D} = \emptyset$. $\square$

Lemma 4. *Let $y \in \mathbb{R}^q$. The following statements are equivalent.*

(i) *y is a weakly C-minimal point of $\mathcal{P}$.*

(ii) *$\mathcal{H}^*(y) \cap \mathcal{D}$ is a K-maximal proper face of $\mathcal{D}$.*

Moreover, for every K-maximal proper face $\mathcal{F}^*$ of $\mathcal{D}$ there exists some $y \in \mathbb{R}^q$ such that $\mathcal{F}^* = \mathcal{H}^*(y) \cap \mathcal{D}$.

Proof. By Lemma 1, (i) is equivalent to

(iii) There exists some $(u^0, c^0) \in \mathcal{U}$ with $y^T c^0 = b^T u^0$ solving ($\mathrm{D}_2(y)$).

Taking into account (1), we see that (iii) is equivalent to

(iv) $\varphi(y, v) \geq 0$ for all $v \in D[\mathcal{U}]$ and there exists some $v^0 \in D[\mathcal{U}]$ with $\varphi(y, v^0) = 0$.

Statement (iv) is equivalent to

(v) $\mathcal{H}^*(y)$ is a supporting hyperplane to $D[\mathcal{U}]$.

Regarding the fact that $\mathcal{H}^*(y) = \left\{v \in \mathbb{R}^q \mid c^*(y)^T v = -y_q\right\}$ with $c^*(y)_q = -1 < 0$, (v) is equivalent to (ii) by Lemma 3.

Let $\mathcal{F}^*$ be a K-maximal proper face of $\mathcal{D}$. Then there exists a supporting hyperplane $\mathcal{H}^* := \left\{v \in \mathbb{R}^q \mid {c^*}^T v = \gamma\right\}$ (i.e., $c^* \neq 0$) to $\mathcal{D}$ such that $\mathcal{F}^* = \mathcal{H}^* \cap \mathcal{D}$. By Lemma 3, we have $c^*_q < 0$. Setting

$$y := \left(\frac{\gamma - c^*_1}{c^*_q}, \dots, \frac{\gamma - c^*_{q-1}}{c^*_q}, \frac{\gamma}{c^*_q}\right)^T$$

we obtain $\mathcal{H}^* = \mathcal{H}^*(y)$. Hence $\mathcal{F}^* = \mathcal{H}^*(y) \cap \mathcal{D}$. $\square$

Lemma 5. *Consider a hyperplane $\mathcal{H} := \{y \in \mathbb{R}^q \mid c^T y = \gamma\}$. The following statements are equivalent.*

(i) *$\mathcal{H}$ is a supporting hyperplane to $\mathcal{P}$.*

(ii) *$c \geq 0$ and $\mathcal{H}$ is a supporting hyperplane to $M[\mathcal{X}]$.*

Proof. (i) $\Rightarrow$ (ii). If $\mathcal{H}$ is a supporting hyperplane to $\mathcal{P}$ then there is some $y^0 \in \mathcal{P}$ with $c^T y^0 = \gamma$ and for all $y \in \mathcal{P}$ it holds $c^T y \geq \gamma$. By the definition of $\mathcal{P}$ we have $y^0 + w \in \mathcal{P}$ for all $w \in C = \mathbb{R}^q_+$, hence $c^T w \geq 0$ for all $w \in \mathbb{R}^q_+$. This implies $c \geq 0$. Since $y^0 \in \mathcal{P}$, there is $y^1 \in M[\mathcal{X}] \subseteq \mathcal{P}$ and $w \in C$ such that $y^0 = y^1 + w$. Hence $c^T y^1 = c^T y^0 - c^T w \leq \gamma$. This implies $c^T y^1 = \gamma$. Therefore $\mathcal{H}$ is a supporting hyperplane to $M[\mathcal{X}]$.

(ii) $\Rightarrow$ (i). If $\mathcal{H}$ is a supporting hyperplane to $M[\mathcal{X}]$ then there is some $y^0 \in M[\mathcal{X}]$ with $c^T y^0 = \gamma$ and for all $y \in M[\mathcal{X}]$ it holds $c^T y \geq \gamma$. Since $c \geq 0$, it follows that $c^T y \geq \gamma$ for all $y \in M[\mathcal{X}] + \mathbb{R}^q_+$. By $y^0 \in \mathcal{P}$ and $c^T y^0 = \gamma$ we conclude that $\mathcal{H}$ is a supporting hyperplane to $\mathcal{P}$. $\square$

Lemma 6. *Every proper face of $\mathcal{P}$ is weakly C-minimal.*

Proof. Let $\mathcal{F}$ be a proper face of $\mathcal{P}$. There is a supporting hyperplane $\mathcal{H} := \{y \in \mathbb{R}^q \mid c^T y = \gamma\}$ (i.e., $c \neq 0$) to $\mathcal{P}$ such that $\mathcal{F} = \mathcal{H} \cap \mathcal{P}$. By Lemma 5 we have $c \geq 0$. Let $y \in \mathcal{F}$, then $y \in \mathcal{P}$ implying the existence of $x^0 \in \mathcal{X}$ such that $Mx^0 \leq y$, i.e., $(x^0, 0) \in \mathcal{S}(y)$ and $c^T y = \gamma$. Suppose there are $x \in \mathcal{X}$ and $z < 0$ such that $Mx - kz \leq y$, i.e., $Mx < y$. Since $\mathcal{H} = \{y \in \mathbb{R}^q \mid c^T y = \gamma\}$ is a supporting hyperplane to $\mathcal{P}$ and $Mx \in \mathcal{P}$, we have $\gamma \leq c^T Mx < c^T y = \gamma$, a contradiction. Hence $(x, 0)$ solves $(\mathrm{P}_2(y))$. By Lemma 1 this implies that $y \in \operatorname{Min}_C \mathcal{P}$. $\square$

Lemma 7. *Let $v \in \mathbb{R}^q$. The following statements are equivalent.*

(i) *v is a K-maximal point of $\mathcal{D}$.*

(ii) *$\mathcal{H}(v) \cap \mathcal{P}$ is a weakly C-minimal proper face of $\mathcal{P}$.*

Moreover, for every proper face $\mathcal{F}$ of $\mathcal{P}$ there exists some $v \in \mathbb{R}^q$ such that $\mathcal{F} = \mathcal{H}(v) \cap \mathcal{P}$.

Proof. Taking into account (3), we conclude that (i) is equivalent to

(iii) $c(v) \geq 0$ and there exists some $u^0 \in \mathbb{R}^m$ solving $(\mathrm{D}_1(v))$ such that $v_q = b^T u^0$.

By duality between $(\mathrm{P}_1(v))$ and $(\mathrm{D}_1(v))$, (iii) is equivalent to

(iv) $c(v) \geq 0$ and there exists some $x^0 \in \mathbb{R}^n$ solving $(\mathrm{P}_1(v))$ such that $v_q = c(v)^T M x^0$.

Statement (iv) is equivalent to

(v) $c(v) \geq 0$ and $\mathcal{H}(v)$ is a supporting hyperplane to $M[\mathcal{X}]$.

By Lemma 5 and Lemma 6, (v) is equivalent to (ii).

To show the last conclusion, let $\mathcal{F}$ be a proper face of $\mathcal{P}$. Hence there exists some supporting hyperplane $\mathcal{H} := \{y \in \mathbb{R}^q \mid c^T y = \gamma\}$ (i.e., $c \neq 0$) to $\mathcal{P}$ such that $\mathcal{F} = \mathcal{H} \cap \mathcal{P}$. By Lemma 5, we have $c \geq 0$. Without loss of generality we can assume that $k^T c = 1$ ($k = (1, \dots, 1)^T$). Setting $v_i := c_i$ for $i = 1, \dots, q-1$ and $v_q := \gamma$, we have $\mathcal{H} = \mathcal{H}(v)$. Hence $\mathcal{F} = \mathcal{H}(v) \cap \mathcal{P}$. $\square$

Now we are able to give the proof of our main result.

Proof of Theorem 1. (a) We show that, if $\mathcal{F}^*$ is a K-maximal proper face of $\mathcal{D}$, then $\Psi(\mathcal{F}^*)$ is a weakly C-minimal proper face of $\mathcal{P}$. By Lemma 7, $\mathcal{H}(v) \cap \mathcal{P}$ is a weakly C-minimal proper face of $\mathcal{P}$ for each $v \in \mathcal{F}^*$, hence $\Psi(\mathcal{F}^*)$ is a weakly C-minimal face of $\mathcal{P}$. It remains to show that $\Psi(\mathcal{F}^*)$ is nonempty. By Lemma 4 there is some $y^0 \in \operatorname{Min}_C \mathcal{P}$ such that $\mathcal{F}^* = \mathcal{H}^*(y^0) \cap \mathcal{D}$, hence $y^0 \in \Psi(\mathcal{F}^*)$.

(b) We prove that $\Psi^*(\mathcal{F}) := \bigcap_{y\in\mathcal{F}} \mathcal{H}^*(y) \cap \mathcal{D}$ is a K-maximal proper face of $\mathcal{D}$ if $\mathcal{F}$ is a weakly C-minimal proper face of $\mathcal{P}$. By Lemma 4, $\mathcal{H}^*(y) \cap \mathcal{D}$ is a K-maximal proper face of $\mathcal{D}$ for each $y \in \mathcal{F}$. Hence $\Psi^*(\mathcal{F})$ is a K-maximal proper face of $\mathcal{D}$ if this set is nonempty. Indeed, by Lemma 7, there is some $v^0 \in \operatorname{Max}_K \mathcal{D}$ such that $\mathcal{F} = \mathcal{H}(v^0) \cap \mathcal{P}$ implying $v^0 \in \Psi^*(\mathcal{F})$.

(c) In order to show that Ψ is a bijection and that $\Psi^{-1}(\mathcal{F}) = \bigcap_{y\in\mathcal{F}} \mathcal{H}^*(y) \cap \mathcal{D} =: \Psi^*(\mathcal{F})$, we have to show the following two statements: (c_1) $\Psi^*(\Psi(\mathcal{F}^*)) = \mathcal{F}^*$ for all K-maximal proper faces $\mathcal{F}^*$ of $\mathcal{D}$ and (c_2) $\Psi(\Psi^*(\mathcal{F})) = \mathcal{F}$ for all weakly C-minimal proper faces $\mathcal{F}$ of $\mathcal{P}$.

(c_1) First we show that $\mathcal{F}^* \subseteq \Psi^*(\Psi(\mathcal{F}^*))$. Assume the contrary, i.e., there is some $v^0 \in \mathcal{F}^*$ such that $v^0 \notin \Psi^*(\Psi(\mathcal{F}^*))$. Hence there exists some $y^0 \in \Psi(\mathcal{F}^*)$ such that $v^0 \notin \mathcal{H}^*(y^0) \cap \mathcal{D}$. This implies $v^0 \notin \mathcal{H}^*(y^0)$ since $v^0 \in \mathcal{D}$. It follows that $y^0 \notin \mathcal{H}(v^0)$, whence $y^0 \notin \Psi(\mathcal{F}^*)$, a contradiction. To show the opposite inclusion, let $y^0 \in \operatorname{Min}_C \mathcal{P}$ such that $\mathcal{F}^* = \mathcal{H}^*(y^0) \cap \mathcal{D}$. The existence of such a point y^0 is ensured by Lemma 4. It follows that $y^0 \in \Psi(\mathcal{F}^*)$. Hence $\Psi^*(\Psi(\mathcal{F}^*)) \subseteq \mathcal{H}^*(y^0) \cap \mathcal{D} = \mathcal{F}^*$.

(c_2) The proof works analogously using Lemma 7 instead of Lemma 4.

(d) Obviously, Ψ is inclusion reversing.

(e) It remains to prove that $\dim \mathcal{F}^* + \dim \Psi(\mathcal{F}^*) = q - 1$ for all K-maximal proper faces $\mathcal{F}^*$ of $\mathcal{D}$. Consider some fixed $\mathcal{F}^*$ and set $r := \dim \mathcal{F}^*$ and $s := \dim \Psi(\mathcal{F}^*)$. By the first part of the proof, $\mathcal{F} := \Psi(\mathcal{F}^*)$ is a weakly C-minimal face of $\mathcal{P}$. Hence there exist proper faces $\mathcal{F} \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \cdots \subsetneq \mathcal{F}_{q-1-s}$ (all of them being weakly C-minimal by Lemma 6) such that $\dim \mathcal{F}_{q-1-s} = q - 1$. From the properties of Ψ, we conclude that $0 \leq \dim \Psi^{-1}(\mathcal{F}_{q-1-s}) \leq r - (q - 1 - s)$. Hence $r + s \geq q - 1$. Since every K-maximal face of $\mathcal{D}$ has a vertex (Lemma 2), there are K-maximal faces $\mathcal{F}^* \supsetneq \mathcal{F}_1^* \supsetneq \mathcal{F}_2^* \supsetneq \cdots \supsetneq \mathcal{F}_r^*$ such that $\dim \mathcal{F}_r^* = 0$. It follows that $s + r \leq \dim \Psi(\mathcal{F}_r^*) \leq q - 1$. Together we have $s + r = q - 1$.

References

[1] Benson, H. P.: An outer approximation algorithm for generating all efficient extreme points in the outcome set of a multiple objective linear programming problem, *Journal of Global Optimization* **13**, (1998), 1-24

[2] Brumelle, S., Duality for Multiple objective convex programs, *Math. Oper. Res.* **6** (2), (1981), 159-172

[3] Ehrgott, M.; Löhne, A.; Shao, L.: A dual variant of Benson's outer approximation algorithm, draft, 2006

[4] Göpfert, A., Nehse, R., *Vektoroptimierung*, BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1990

[5] Grünbaum, B.: *Convex Polytopes*, prepared by Kaibel,V.; Klee, V.; Ziegler, G. M.; 2nd ed., Graduate Texts in Mathematics 221. New York, Springer, 2003

[6] Hamel, A.; Heyde, F.; Löhne, A.; Tammer, C.; Winkler, K.: Closing the duality gap in linear vector optimization, *Journal of Convex Analysis* **11** (1), (2004), 163-178

[7] Heyde, F.; Löhne, A.; Tammer, C: The attainment of the solution of the dual program in vertices for vectorial linear programs, *Proceedings of the 7th International Conference on Multi-Objective Programming and Goal Programming, Tours, France, June 12-14, 2006.*

[8] Heyde, F.; Löhne, A.; Tammer, C: On duality in multi-objective linear programming, draft, 2006

[9] Isermann, H., On some relations between a dual pair of multiple objective linear programs, *Z. Oper. Res., Ser. A* **22**, (1978), 33-41

[10] Isermann, H., Duality in multiple objective linear programming, Multiple Criteria Probl. Solving, Proc. Conf. Buffalo 1977, LN Econ. Math. Syst. 155, (1978), 274-285

[11] Jahn, J.: *Mathematical Vector Optimization in Partially Ordered Linear Spaces*, Verlag Peter Lang, Frankfurt am Main-Bern-New York, 1986

[12] Jahn, J.: *Vector Optimization. Theory, Applications, and Extensions*, Springer-Verlag, Berlin, 2004

[13] Kornbluth, J. S. H., Duality, indifference and sensitivity analysis in multiple objective linear programming, *Operations Research Quarterly* **25**, (1974), 599-614

[14] Löhne, A.: Tammer, C.: A new approach to duality in linear vector optimization, *Optimization*, to appear

[15] Luc, D. T.: *Theory of Vector Optimization*, Lecture Notes in Economics and Mathematical Sciences, 319, Springer-Verlag, Berlin, 1988

[16] Rockafellar, R.T.: *Convex Analysis*, Princeton University Press, Princeton, 1972

[17] Rödder, W., A generalized saddlepoint theory. Its application to duality theory for linear vector optimum problems, *European Journal of Operations Research* **1**, (1977), 55-59

[18] Webster, R.: *Convexity*, Oxford Science Publications, Oxford, Oxford University Press, 1994