The Kolmogorov complexity of infinite words*

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Abstract

We present a brief survey of results on relations between the Kolmogorov complexity of infinite strings and several measures of information content (dimensions) known from dimension theory, information theory or fractal geometry.

Special emphasis is laid on bounds on the complexity of strings in constructively given subsets of the Cantor space. Finally, we compare the Kolmogorov complexity to the subword complexity of infinite strings.

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Contents

The aim of this paper is to briefly survey several results on the Kolmogorov complexity of infinite strings. We focus on those results which can be derived by elementary methods from the Kolmogorov complexity of finite strings (words) and counting arguments for sets of finite strings (languages) as e.g. structure functions and the concept of entropy of languages.

The concept of Kolmogorov or program size complexity was introduced in the papers by Solomonoff [So64], Kolmogorov [Ko65] and Chaitin [Ch66] in the sixties (for the complete history see the textbooks by Calude [Ca02] or Li and Vitányi [LV97]). It measures the information content of a (finite) string as the size of the smallest program that computes the string, that is, the complexity of a string is the amount of information necessary to print the string.

The original intention of Kolmogorov was to give an alternative approach to information theory not depending on probability theory. A first fact proving evidence of this intention was P. Martin-Löf's [ML66] characterisation of infinite random strings. Roughly speaking, if an infinite string is random then most of its initial words have maximum Kolmogorov complexity, that is, have a complexity which is close to their length. To put it into the context of information, the amount of information which must provided in order to specify a particular symbol of a random sequence is one unit of information (e.g. one bit if we consider binary sequences).

Although Kolmogorov was interested mainly in the complexity of finite strings, Kolmogorov complexity was also applied to infinite strings. Here it was compared to information-theoretic size measures (or dimensions). These dimensions are also known from fractal geometry (see [Fa90]). It turned out that some of them are closely related to Kolmogorov complexity. Whereas the papers [Da74, Da75] give an account on the Kolmogorov complexity of single infinite strings, the papers [Br74, CH94, Ry84, Ry86, Ry94, St81, St93] set Kolmogorov complexity of individual infinite strings in relation to the dimension (topological entropy, Hausdorff dimension) of sets containing these strings.

The first one of those dimensions is called *Minkowski* or *box-counting dimension*. It is also known under several other names (cf. [Fa90]). The other measures are the *Hausdorff dimension* and the *packing* or *modified box-counting dimension*.

Another way to approach Kolmogorov complexity of infinite strings is to further pursue the investigation of randomness (see [Ca02, LV97, Sc71, vL87]) and partial randomness [CS06, St93, Ta02]. Here we have several characterisations of random strings combining complexity or martingales and order functions as initiated by Schnorr [Sc71]. Recently the constructivisation of dimension as in [AH04, Lu00, Lu03a, Lu03b] gave new insight into these problems. These papers use the concept of so-called s-gales, a combination of martingales and exponential order functions. Their relation to Kolmogorov complexity is based on Levin's [ZL70] construction of a universal semi-measure. The coincidence of Lutz's [AH04, Lu00, Lu03a, Lu03b] constructive dimension and Kolmogorov complexity of infinite words is immediate from Theorem 3.4 of [ZL70] and Theorem 3.6 [Lu03b]. For a detailed explanation see also [St05].

In this paper we focus on results linking Kolmogorov complexity of sets of infinite strings to their dimensions. A major point is that we show how to derive these results utilising simple bounds on the Kolmogorov complexity of finite strings and transfer them from languages (sets of finite strings) to sets of infinite strings by means of limit concepts. This is done in an elementary manner using structural (combinatorial) properties as the entropy of languages. Similar approaches were already pursued in part in the papers [Hi05] and [St93, St98]. In contrast to Hitchcock's paper [Hi05] which uses the concept of s-gales the present paper is based on elementary properties of Kolmogorov complexity of finite strings.

We start with a brief account of Kolmogorov complexity of finite strings and the entropy of languages in Section 2. Then, in Section 3, we proceed to the derivation of results linking the entropy of languages to dimensions of sets of infinite strings. Section 4 gives general bounds on Kolmogorov complexity of sets of infinite strings by their dimensions.

More precise bounds on complexity via dimension for certain classes of sets of infinite strings are obtained utilising structural properties. In Section 5 we present results for sets having self-similarity properties or defined by computability constraints.

In the final Section 6 we introduce another version of complexity of infinite strings and present its relation to dimension and Kolmogorov complexity.

1 Notation

In this section we introduce the notation used throughout the paper. By $\mathbb{N} = \{0, 1, 2, \ldots\}$ we denote the set of natural numbers. Let X be an alphabet of cardinality $|X| = r \geq 2$. By X^* we denote the set of finite words on X, including the *empty word* e, and X^ω is the set of infinite strings (ω-words) over X. Subsets of X^* will be referred to as *languages* and subsets of X^ω as *ω-languages*.

For $w \in X^*$ and $\eta \in X^* \cup X^\omega$ let $w \cdot \eta$ be their *concatenation*. This concatenation product extends in an obvious way to subsets $W \subseteq X^*$ and $B \subseteq X^* \cup X^\omega$. For a language W let $W^* := \bigcup_{i \in \mathbb{N}} W^i$, and by $W^\omega := \{w_1 \cdots w_i \cdots : w_i \in W \setminus \{e\}\}$ we denote the set of infinite strings formed by concatenating words in W. Furthermore $|w|$ is the *length* of the word $w \in X^*$ and $\mathbf{pref}(B)$ is the set of all finite prefixes of strings in $B \subseteq X^* \cup X^\omega$. We shall abbreviate $w \in \mathbf{pref}(\eta)$ $(\eta \in X^* \cup X^\omega)$ by $w \sqsubseteq \eta$, and $\eta[0..n]$ is the n-length prefix of η provided $|\eta| \geq n$. A language $W \subseteq X^*$ is referred to as *prefix-free* provided $w \sqsubseteq v$ and $w, v \in W$ imply $w = v$.

We denote by $B/w := \{\eta : w \cdot \eta \in B\}$ the *left derivative* of the set $B \subseteq X^* \cup X^\omega$. As usual a language $W \subseteq X^*$ is *regular* provided its set of left derivatives $\{W/w : w \in X^*\}$ is finite. In the sequel we assume the reader to be familiar with basic facts of language theory. As usual, the class of recursively enumerable languages is denoted by Σ_1, the class containing their complements by Π_1. Thus, $\Sigma_1 \cap \Pi_1$ is the class of recursive languages.

We consider the set X^ω as a metric space (Cantor space) (X^ω, ρ) of all ω-words over the alphabet X where the metric ρ is defined as follows.

$$\rho(\xi, \eta) := \inf\{r^{-|w|} : w \sqsubset \xi \wedge w \sqsubset \eta\} .$$

This space is a compact, and $\mathcal{C}(F) := \{\xi : \mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)\}$ turns out to be the *closure* of the set F (smallest closed subset containing F) in (X^ω, ρ).

Besides the ω-power W^ω we define still two more mappings transforming languages to ω-languages, the δ- or i.o.-limit $\overrightarrow{W} := \{\xi : |\mathbf{pref}(\xi) \cap W| = \infty\}$ and the a.e.-limit $W^\uparrow := \{\xi : |\mathbf{pref}(\xi) \setminus W| < \infty\}$. It is immediate that $W^\uparrow \subseteq \overrightarrow{W} \subseteq \overrightarrow{\mathbf{pref}(W)} = \mathbf{pref}(W)^\uparrow$ and $\mathcal{C}(F) = \mathbf{pref}(F)^\uparrow = \overrightarrow{\mathbf{pref}(F)}$. Moreover, we have

$$\overrightarrow{\mathbf{pref}(\overrightarrow{V}) \cap V} = \overrightarrow{V} \text{ and } (\mathbf{pref}(W^\uparrow) \cap W)^\uparrow = W^\uparrow . \tag{1}$$

2 Kolmogorov complexity of finite words and the entropy of languages

In this section we briefly recall the concept of Kolmogorov complexity of finite words. For a more comprehensive introduction see the textbooks [Ca02] and [LV97]. To this end let $\varphi : X^* \to X^*$ be a partial-recursive function. The complexity of a word $w \in X^*$ with respect to φ is defined as

$$K_\varphi(w) := \inf\{|\pi| : \pi \in X^* \wedge \varphi(\pi) = w\}. \tag{2}$$

It is well known that there is an optimal partial-recursive function $\mathfrak{U} : X^* \to X^*$, that is, a function satisfying that for every partial-recursive function φ

$$\exists c_\varphi \forall w (w \in X^* \to K_{\mathfrak{U}}(w) \leq K_\varphi(w) + c_\varphi) \tag{3}$$

If one considers only partial-recursive functions φ with prefix-free domain $dom(\varphi) \subseteq X^*$ we obtain in the same way an optimal partial-recursive function $\mathfrak{C}$.

Proposition 2.1 *There is a partial recursive function $\mathfrak{C} : X^* \to X^*$ with prefix-free domain $dom(\mathfrak{C})$ such that for every partial-recursive functions φ with prefix-free domain $dom(\varphi)$ there is a constant c_φ such that*

$$\forall w (w \in X^* \to K_{\mathfrak{C}}(w) \leq K_\varphi(w) + c_\varphi) .$$

Following [LV97] the complexity $K_{\mathfrak{C}}$ will be called *prefix complexity*.

A third version useful for our considerations is the conditional complexity. Let $\mathcal{A} \in \{\mathbb{N}, X^*\}$, consider a partial-recursive function $\psi : X^* \times \mathcal{A} \to X^*$ and set

$$K_\psi(w \mid a) := \inf\{|\pi| : \pi \in X^* \wedge \psi(\pi, a) = w\}. \tag{4}$$

Again we have an optimal partial-recursive function $\mathfrak{A} : X^* \times \mathcal{A} \to X^*$ satisfying that for every partial-recursive function ψ

$$\exists c_\psi \forall w \forall a (w \in X^* \wedge a \in \mathcal{A} \to K_{\mathfrak{A}}(w \mid a) \leq K_\psi(w \mid a) + c_\psi) .$$

For this conditional complexity we have the following two properties. The first one is Theorem 1.2 in [ZL70] (see also Theorem 2.1.3 in [LV97]).

Theorem 2.2 (Kolmogorov) *Let $M \subseteq X^* \times \mathcal{A}$ be a recursively enumerable set such that each section $M_a := \{w : (w, a) \in M\}$ is finite. Then there is a $c \in \mathbb{N}$ such that $K(w \mid a) \leq \log_r |M_a| + c$ for all $a \in \mathcal{A}$ and $w \in M_a$.*

The next theorem is a slight extension of Theorem 2.9 of [St93].

Theorem 2.3 *Let* $M, M' \subseteq X^* \times \mathcal{A}$ *be recursively enumerable sets such that each section* $M'_a := \{w : (w, a) \in M\}$ *is finite, and let* $s : \mathcal{A} \to \mathbb{N}$ *be a recursive function satisfying* $|M'_a| \leq s(a)$.

Then there is a $c \in \mathbb{N}$ *such that* $K(w \mid a) \leq \log_r(\max\{s(a) - |M_a|, 1\}) + c$ *for all* $a \in \mathcal{A}$ *and* $w \in M'_a \setminus M_a$.

Proof. We construct a function $\psi : X^* \times \mathcal{A} \to X^*$ such that $K_\psi(w \mid a) \leq \log_r(\max\{s(a) - |M_a|, 1\})$ for all $a \in \mathcal{A}$ and $w \in M'_a \setminus M_a$.

Let $M_a^{(t)} \subseteq M \cap M', t \leq |M_a \cap M'_a|$ be the set of the first t elements of the form (v, a) in the enumeration of $M \cap M'$.

For input (π, a) we enumerate M until we get $t_0 := \max\{s(a) - r^{|\pi|}, 1\}$ elements of the form (w, a). Let π be the qth element of $X^{|\pi|}$ in lexicographical order. Now enumerate M' until q elements in $M'_a \setminus M_a^{(t_0)}$ appear and put $\psi(\pi, a) := v$ when (v, a) is this qth element. ❏

A special case of the conditional complexity is the *length-conditional complexity.* Here we have a partial-recursive function $\psi : X^* \times \mathbb{N} \to X^*$ and set

$$K_\psi(w \mid n) := \inf\{|\pi| : \psi(\pi, n) = w \wedge |w| = n\}. \tag{5}$$

Again we have an optimal partial-recursive function $\mathfrak{L} : X^* \times \mathbb{N} \to X^*$ satisfying that for every partial-recursive function ψ

$$\exists c_\psi \forall w \forall n (w \in X^* \wedge n \in \mathbb{N} \to K_{\mathfrak{L}}(w \mid n) \leq K_\psi(w \mid n) + c_\psi)\,.$$

The following relation between the optimal functions is obvious.

$$K_{\mathfrak{L}}(w \mid |w|) \leq K_{\mathfrak{U}}(w) + c_1 \leq K_{\mathfrak{C}}(w) + c_2 \tag{6}$$

holds for all $w \in X^*$ and constants c_1, c_2 depending only on $\mathfrak{L}, \mathfrak{U}$ and $\mathfrak{C}$. In the sequel we shall fix these optimal functions and denote the corresponding complexities by $K(\cdot \mid n)$, K and H, respectively.[1]

The inequalities in Eq. (6) can be, to some extent, reversed (see [Ca02, LV97]).

$$\begin{array}{rcl} K(w \mid |w|) + 2 \cdot \log_r |w| + c_1 & \geq & K(w) \quad \text{and} \\ K(w) + 2 \cdot \log_r |w| + c_2 & \geq & H(w) \end{array} \tag{7}$$

[1] This follows the notation of [Ca02] whereas [LV97] uses C for the usual complexity and K for the prefix complexity.

for all $w \in X^*$ and suitable constants $c_1, c_2 \in \mathbb{R}$.

For a language $W \subseteq X^*$ define its *length-structure function*[2] $s_W : \mathbb{N} \to \mathbb{N}$ by $s_W(n) := |W \cap X^n|$ and its *entropy* as (cf. [CM58, HP94, St93]),

$$H_W = \limsup_{n\to\infty} \frac{\log_r(1 + s_W(n))}{n} .$$

Then $\alpha > H_W$ implies $\sum\limits_{w\in W} r^{-\alpha\cdot|w|} < \infty$, and $\sum\limits_{w\in W} r^{-\alpha\cdot|w|} < \infty$ implies $\alpha \geq H_W$.

We have the following connection between the length-structure function or the entropy of a language $W \subseteq X^*$ and the Kolmogorov complexity of words $w \in W$. The first one is a simple counting argument.

Corollary 2.4 *If* $W \subseteq X^*$ *and* $W \cap X^n \neq \emptyset$ *then* $K(w_n \mid n) \geq \log_r s_W(n)$ *for some* $w_n \in W \cap X^n$.

The next one is an easy consequence of Theorems 2.2 and 2.3.

Corollary 2.5 *If* $W \in \Sigma_1 \cup \Pi_1$ *then there is a* $c > 0$ *such that*

$$\forall w(w \in W \to K(w \mid |w|) \leq \log_r s_W(|w|) + c) .$$

In a similar way one can prove

Corollary 2.6 *If* $W \in \Sigma_1 \cap \Pi_1$ *then there is a* $c > 0$ *such that*

$$\forall w(w \in W \to K(w) \leq \log_r \sum\nolimits_{i=0}^{|w|} s_W(i) + c) .$$

We conclude this introductory section with the consideration of the sets $W_\alpha := \{w : K(w) < \alpha \cdot |w|\}$. A simple counting argument shows that

$$H_{W_\alpha} \leq \alpha \text{ for } 0 \leq \alpha \leq 1 . \tag{8}$$

For certain $\alpha \in [0, 1]$ the sets W_α can be effectively described. To this end we mention that a real number $\alpha \in [0, 1]$ is called *left-computable*[3] provided $\{(p, q) : p, q \in \mathbb{N} \wedge \frac{p}{q} < \alpha\} \in \Sigma_1$.

Lemma 2.7 *If* $\alpha \in [0, 1]$ *is left-computable then* $W_\alpha \in \Sigma_1$.

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[2]This function is not to be confused with the *Kolmogorov structure function* defined e.g. in [LV97, Section 2.2.2].

[3]These numbers are also called *semi-computable from below.*

The proof is by standard methods of computable analysis (see e.g. [We00]). It uses the well-known fact (see e.g. [Ca02, LV97]) that the set $\{(w,k) : k \in \mathbb{N} \wedge k \geq K(w)\}$ is recursively enumerable.

It should be mentioned that Eq. (8) and Lemma 2.7 hold also if we replace K by one of the complexities $K(\cdot \mid n)$ and H.

3 Sets of infinite words and their dimensions

In this section we define the various size measures (dimensions) related to Kolmogorov complexity, and we show how these dimensions are related to the entropy of languages. We start with the simplest one, the box counting dimension.

Definition 3.1 *Let $F \subseteq X^\omega$. The quantities*

$$\begin{array}{rclcl} \underline{\dim}_B F & = & \liminf\limits_{n\to\infty} \frac{\log_r(s_{\mathbf{pref}(F)}(n)+1)}{n} & & \textit{and} \\ \overline{\dim}_B F & = & \limsup\limits_{n\to\infty} \frac{\log_r(s_{\mathbf{pref}(F)}(n)+1)}{n} & = & H_{\mathbf{pref}(F)} \end{array}$$

are called the lower and upper Minkowski (or box counting) dimension of F, respectively.

Since $\mathbf{pref}(F) = \mathbf{pref}(\mathcal{C}(F))$, we have $\underline{\dim}_B F = \underline{\dim}_B \mathcal{C}(F)$ and $\overline{\dim}_B F = \overline{\dim}_B \mathcal{C}(F)$.

Next we recall the definition of the Hausdorff measure and Hausdorff dimension of a subset of (X^ω, ρ) (see e.g. [Fa90]). In the setting of languages this can be read as follows (see [St93, St98]). For $F \subseteq X^\omega$ and $0 \leq \alpha \leq 1$ the equation

$$\mathbb{L}_\alpha(F) := \lim_{l\to\infty} \inf\Big\{ \sum_{w\in W} r^{-\alpha\cdot|w|} : F \subseteq W \cdot X^\omega \wedge \forall w (w \in W \to |w| \geq l)\Big\} \quad (9)$$

defines the *α-dimensional metric outer measure* on X^ω. $\mathbb{L}_\alpha$ satisfies the following (For a typical plot of $\mathbb{L}_\alpha(F)$ as a function of α see Figure 1 below.).

Corollary 3.2 *If $\mathbb{L}_\alpha(F) < \infty$ then $\mathbb{L}_{\alpha+\varepsilon}(F) = 0$ for all $\varepsilon > 0$.*

Then the *Hausdorff dimension* of F is defined as

$$\dim_H F := \sup\{\alpha : \alpha = 0 \vee \mathbb{L}_\alpha(F) = \infty\} = \inf\{\alpha : \mathbb{L}_\alpha(F) = 0\}.$$

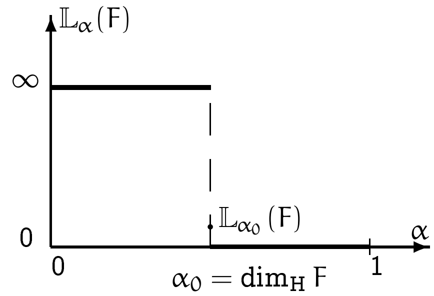

Figure 1: A typical plot of $\mathbb{L}_\alpha(F)$ as a function of α when $|F| = \infty$

For the definition of the *packing dimension* $\dim_P$ we use its characterisation as modified upper box counting dimension $\overline{\dim}_{MB}$ (see [Fa90]).

$$\dim_P F := \overline{\dim}_{MB} F = \inf\{\sup_{i\in\mathbb{N}} \overline{\dim}_B F_i : \bigcup_{i\in\mathbb{N}} F_i \supseteq F\}$$

The following properties of the just introduced dimensions should be mentioned. First,

$$\dim_H F \le \underline{\dim}_B F \le \overline{\dim}_B F \text{ and } \dim_H F \le \dim_P F \le \overline{\dim}_B F\,.$$

Every dimension is monotone, that is, $E \subseteq F$ implies $\dim E \le \dim F$ and shift invariant, that is, $\dim w\cdot F = \dim F$. Moreover $\overline{\dim}_B$, $\dim_P$, $\dim_H$ have the following properties.

$$\begin{aligned} \overline{\dim}_B (E \cup F) &= \max\{\overline{\dim}_B E, \overline{\dim}_B F\} \text{ and} \\ \dim(\bigcup_{i\in\mathbb{N}} F_i) &= \sup_{i\in\mathbb{N}} \dim F_i \quad \text{for } \dim \in \{\dim_P, \dim_H\} \end{aligned}$$

That is, the upper box counting dimension is stable, and Hausdorff and packing dimension are countably stable.

Finally, we give a connection to the entropy of languages (see [St93]). We start with some simple inequalities.

$$\dim_H \overrightarrow{V} \le H_V \tag{10}$$
$$\dim_P W^{\uparrow} \le H_W \tag{11}$$

Proof. In order to prove Eq. (10) we show that $\alpha > H_V$ implies $\mathbb{L}_\alpha(\overrightarrow{V}) = 0$.

If $\alpha > H_V$ then $\sum_{v\in V} r^{-\alpha\cdot|v|} < \infty$. Define $V^{(i)} := \{v : v \in V \wedge |\mathbf{pref}(v) \cap V| = i+1\}$, that is, $V^{(i)}$ is the set of words in V having exactly i proper prefixes in V. By construction $V^{(i)} \cdot X^\omega \supseteq \overrightarrow{V}$, and $V = \bigcup_{i\in\mathbb{N}} V^{(i)}$ is a disjoint union. Now, Eq. (9) shows that $\mathbb{L}_\alpha(\overrightarrow{V}) \leq \sum_{v\in V^{(i)}} r^{-\alpha\cdot|v|}$. The latter tends to zero as i approaches infinity.

To show Eq. (11) we observe that $W^\uparrow = \bigcup_{w\in X^*} E_w$ where $E_w := \{\xi : \mathbf{pref}(\xi) \subseteq \mathbf{pref}(w) \cup (W \cap w \cdot X^*)\}$. It is readily seen that $\overline{\dim}_B E_w \leq H_W$. Thus $\dim_P W^\uparrow \leq H_W$. ❑

These inequalities can, to some extent, be reversed (cf. [Ro70, St93]).

Lemma 3.3 *Let* $F \subseteq X^\omega$*. Then*

$$\dim_H F = \inf\{H_V : F \subseteq \overrightarrow{V}\} \quad \textit{and} \quad \dim_P F = \inf\{H_W : F \subseteq W^\uparrow\}.$$

For the sake of completeness we give a short proof.

Proof. In view of Eq. (10) and (11) the inequalities "$\leq$" are obvious.

Let $\alpha > \dim_H F$. Then $\mathbb{L}_\alpha(F) = 0$. Consequently, for every $i \in \mathbb{N}$ there is a $V_i \subseteq X^*$ such that $F \subseteq V_i \cdot X^\omega$ and $\sum_{w\in V_i} r^{-\alpha\cdot|w|} \leq 2^{-i}$. Then $\sum_{w\in V} r^{-\alpha\cdot|w|} < \infty$ for $V := \bigcup_{i\in\mathbb{N}} V_i$. Thus $H_V \leq \alpha$ and, moreover, $F \subseteq \overrightarrow{V}$, since $w \in V_i$ implies $|w| \geq \frac{\log_r 2}{\alpha}$. This completes the proof of the first identity.

For the proof of the second assertion it suffices to show that $\dim_P F \geq \inf\{H_W : F \subseteq W^\uparrow\}$. We start with a covering $\bigcup_{i\in\mathbb{N}} F_i \supseteq F$ satisfying $\sup_{i\in\mathbb{N}} \overline{\dim}_B F_i \leq \dim_P F + \varepsilon$. Since $\overline{\dim}_B \bigcup_{i=0}^n F_i = \max_{0\leq i\leq n} \overline{\dim}_B F_i$, we may assume that $F_i \subseteq F_{i+1}$. This implies $\sup_{i\in\mathbb{N}} \overline{\dim}_B F_i = \lim_{i\to\infty} \overline{\dim}_B F_i$. Define a family of natural numbers $(n_i)_{i\in\mathbb{N}}$ as follows:

1. $n_i < n_{i+1}$, and
2. $\frac{\log_r s_{\mathbf{pref}(F_i)}(n)}{n} \leq \overline{\dim}_B F_i + 2^{-i}$ for all $n \geq n_i$.

This is possible, because $\overline{\dim}_B F_i = \limsup_{n\to\infty} \frac{\log_r s_{F_i}(n)}{n}$.

Now define W in the following way: $W \cap X^n := \mathbf{pref}(F_i) \cap X^n$ for $n_i \leq n < n_{i+1}$. Since $F_i \subseteq F_{i+1}$, $\xi \in F_i$ implies $|\mathbf{pref}(\xi) \setminus W| \leq n_i$, whence $\xi \in W^\uparrow$.

Then $\frac{\log_r s_W(n)}{n} \leq \frac{\log_r s_{F_i}(n)}{n} \leq \overline{\dim}_B F_i + 2^{-i}$ whenever $n_{i+1} > n \geq n_i$, and, consequently, $H_W \leq \sup_{i\in\mathbb{N}} \overline{\dim}_B F_i \leq \dim_P F + \varepsilon$. ❑

Lemma 3.3 proves close connections between the Hausdorff dimension of an ω-language and the limit $\overrightarrow{V}$, on the one hand, and the packing dimension and the limit $W^\uparrow$, on the other hand. In the next section we shall prove similar connections between two version of Kolmogorov complexity and the limit-operations $\overrightarrow{V}$ and $W^{\uparrow}$.

4 General bounds on Kolmogorov complexity by dimensions

The Kolmogorov complexity of an infinite word $\xi \in X^{\omega}$ is a function $k : \mathbb{N} \to \mathbb{N}$ mapping the n-length prefix $\xi[0..n]$ of ξ to its corresponding complexity $K(\xi[0..n])$ (or $H(\xi[0..n])$ or $K(\xi[0..n] \mid n)$, respectively) (see [Da74]). A large part of investigations deals with the following first order approximations of the complexity of individual ω-words and sets of ω-words:

$$\begin{aligned} \kappa(\xi) &:= \limsup_{n\to\infty} \frac{K(\xi[0..n])}{n}, \quad \kappa(F) := \sup_{\xi\in F} \kappa(\xi) \quad \text{and} \\ \underline{\kappa}(\xi) &:= \liminf_{n\to\infty} \frac{K(\xi[0..n])}{n}, \quad \underline{\kappa}(F) := \sup_{\xi\in F} \underline{\kappa}(\xi) \end{aligned}$$

which will be referred to as the *upper* and *lower Kolmogorov complexity* of ξ or F, respectively. Observe that κ and $\underline{\kappa}$ are independent of the chosen word complexity K, H or $K(\cdot \mid n)$.

We obtain the following connection to the languages W_α.

$$\{\xi : \underline{\kappa}(\xi) \leq \alpha\} = \bigcap_{\gamma>\alpha} \overrightarrow{W_\gamma} \text{ and } \{\xi : \kappa(\xi) \leq \alpha\} = \bigcap_{\gamma>\alpha} W_\gamma^{\uparrow} \tag{12}$$

Proof. The inequalities $\{\xi : \underline{\kappa}(\xi) \leq \alpha\} \subseteq \overrightarrow{W_\gamma}$ and $\{\xi : \kappa(\xi) \leq \alpha\} \subseteq W_\gamma^{\uparrow}$, for $\alpha < \gamma$, are immediate from the definitions.

To show the reverse inclusions let $\xi \in \bigcap_{\gamma>\alpha} \overrightarrow{W_\gamma}$. Then for every $\varepsilon > 0$ there are infinitely many $n \in \mathbb{N}$ such that $K(\xi[0..n]) < (\alpha + \varepsilon) \cdot n$. Consequently, $\liminf_{w\to\xi} \frac{K(\xi[0..n])}{n} \leq \alpha + \varepsilon$. Since $\varepsilon > 0$ was chosen arbitrarily, the assertion is proved.

The proof of the other part is similar. ❑

Now, from Eq. (12) and the upper bound of Corollary 2.5 one obtains the following characterisation of $\underline{\kappa}(F)$ and $\kappa(F)$ similar to the relation between entropy and dimension in Lemma 3.3. For the case $W \in \Sigma_1$ this lemma was proved in [Hi05, Lemma 5.5] (see also [AH04]) in a different manner.

Lemma 4.1

$$\underline{\kappa}(F) = \inf\{H_V : F \subseteq \overrightarrow{V} \wedge V \in \Sigma_1 \cup \Pi_1\} = \inf\{H_V : F \subseteq \overrightarrow{V} \wedge V \in \Sigma_1\}$$
$$\kappa(F) = \inf\{H_W : F \subseteq W^{\uparrow} \wedge W \in \Sigma_1 \cup \Pi_1\} = \inf\{H_W : F \subseteq W^{\uparrow} \wedge W \in \Sigma_1\}$$

Proof. The inequalities "$\leq$" are immediate from Corollary 2.5.

To show the converse inequality for $\underline{\kappa}$ observe that Eqs. (12) and (8) imply $\underline{\kappa}(F) = \inf\{\gamma : \gamma > \underline{\kappa}(F) \wedge \gamma$ is left-computable$\}$ $\geq \inf\{H_{W_\gamma} : \gamma > \underline{\kappa}(F) \wedge \gamma$ is left-computable$\}$.

In view of Lemma 2.7 $W_\gamma \in \Sigma_1$ if only γ is left-computable. Thus the required inequality $\underline{\kappa}(F) \geq \inf\{H_V : F \subseteq \overrightarrow{V} \wedge V \in \Sigma_1\}$ follows.

The case of $\kappa(F)$ is proved in the same way. ❑

Here Lemma 4.1 proves a similar connection between lower and upper Kolmogorov complexity and the limit-operations $\overrightarrow{V}$ or $W^{\uparrow}$ as Lemma 3.3 did for Hausdorff and packing dimensions. In the sequel, these results will be used to obtain bounds on the Kolmogorov complexity of infinite strings via Hausdorff or packing dimension.

4.1 Lower bounds

It seems to be an obvious matter that similar to Corollary 2.4 large sets contain complex elements. This is established by the following lower bounds on $\underline{\kappa}$ and κ which were proved originally in [Ry86, Theorem 2] and [AH04], respectively.

Theorem 4.2 ([Ry86, AH04]) $\dim_H F \leq \underline{\kappa}(F)$ *and* $\dim_P F \leq \kappa(F)$

The proof is an easy consequence of Lemmas 3.3 and 4.1.

For the complexity functions derived from $H(\xi[0..n])$ or $K(\xi[0..n] \mid n)$ we obtain the following refined lower bounds.

Theorem 4.3 ([St93, CS06]) *Let* $F \subseteq X^\omega$ *and* $\mathbb{L}_\alpha(F) > 0$.

1. *Then for every* $f : \mathbb{N} \to \mathbb{N}$ *satisfying* $\sum_{n\in\mathbb{N}} r^{-f(n)} < \infty$ *there is a* $\xi \in F$ *such that* $K(\xi[0..n] \mid n) \geq_{\mathsf{ae}} \alpha \cdot n - f(n)$.

2. *Then there is a* $\xi \in F$ *such that* $H(\xi[0..n]) \geq_{\mathsf{ae}} \alpha \cdot n - c$ *for some constant* $c \in \mathbb{R}$.

For subsets which have not necessarily non-null α-dimensional measure but have certain structure properties we obtain also a tighter bound than the one given in the first inequality of Theorem 4.2.

The first one of these classes will referred to as balanced subsets of X^ω. In the subsequent Sections 5.1 and 5.2 we will introduce two more of those classes.

We call a function $g : \mathbb{N} \to \mathbb{N}$ of *sub-exponential growth* provided

$$\lim_{n\to\infty} g(n) \cdot (1+\varepsilon)^{-n} = 0 \text{ for all } \varepsilon > 0\,.$$

A subset $F \subseteq X^\omega$ is called *balanced* iff there is a function g of sub-exponential growth such that for all $w \in X^*$ the left derivative F/w satisfies

$$\mathbf{s}_{F/w}(n) \leq g(|w|) \cdot \frac{\mathbf{s}_{\mathbf{pref}(F)}(n+|w|)}{\mathbf{s}_{\mathbf{pref}(F)}(|w|)} = g(|w|) \cdot \frac{\sum_{|v|=|w|} \mathbf{s}_{\mathbf{pref}(F/v)}(|w|)}{\mathbf{s}_{\mathbf{pref}(F)}(|w|)}$$

for all $n \in \mathbb{N}$. Balanced sets have the property that the length-structure function of the nonempty derivatives $(F/v)_{|v|=m}$ do not exceed their average too much. In view of this property we obtain the following theorems (see [St89] and [vL87, St93]).

Theorem 4.4 ([St89]) *If* $F \subseteq X^\omega$ *is balanced and closed then* $\dim_H F = \underline{\dim}_B F$.

Theorem 4.5 *If* $F \subseteq X^\omega$ *is balanced and closed then there is a* $\xi \in F$ *such that*

$$K(\xi[0..n] \mid n) \geq_{\mathsf{ae}} \dim_H F \cdot n - o(n)\,.$$

4.2 Upper bounds

After the derivation of lower bounds on the Kolmogorov complexity of infinite words we turn to upper bounds. Here we need some computational constraints on the set F. To this end we introduce the low classes of the arithmetical hierarchy of ω-languages (see e.g. [Ro67, St97a]).

Definition 4.6 (Π_1-definable ω-languages) $F \subseteq X^\omega$ *is* Π_1-definable *if and only if there is a recursive language* $W_F \subseteq X^*$ $(W_F \in \Sigma_1 \cap \Pi_1)$ *such that*

$$\xi \in F \quad \longleftrightarrow \quad \forall w (w \sqsubset \xi \to w \in W_F)\,.$$

Definition 4.7 (Σ_2-definable ω-languages) $F \subseteq X^\omega$ *is* Σ_2-definable *if and only if there is a recursive set* $M_F \subseteq \mathbb{N} \times X^*$ *such that*

$$\xi \in F \quad \longleftrightarrow \quad \exists i (i \in \mathbb{N} \wedge \forall w (w \sqsubset \xi : (i, w) \in M_F)) \ .$$

The other classes are defined in a similar way. Observe that we can characterise several classes by recursive or recursively enumerable languages (see [Sc71, St86, St97a]).

Lemma 4.8 *For the classes* $\mathbf{\Sigma}_1, \mathbf{\Pi}_1, \mathbf{\Sigma}_2$ *and* $\mathbf{\Pi}_2$ *of the arithmetical hierarchy of* ω*-languages in* X^ω *the following identities hold true.*

$$\begin{aligned}
\mathbf{\Sigma}_1 &= \{W \cdot X^\omega : W \subseteq X^* \wedge W \in \Sigma_1 \cap \Pi_1\} \\
\mathbf{\Pi}_1 &= \{F : F \textit{ is closed in } (X^\omega, \rho) \wedge \mathbf{pref}(F) \in \Pi_1\} \\
\mathbf{\Sigma}_2 &= \{F : \exists W (W \subseteq X^* \wedge W \in \Sigma_1 \cap \Pi_1 \wedge F = W^\uparrow)\} \\
\mathbf{\Pi}_2 &= \{F : \exists W (W \subseteq X^* \wedge W \in \Sigma_1 \cap \Pi_1 \wedge F = \overrightarrow{W})\}
\end{aligned}$$

Other classes of interest are the following ones which are defined similar to the characterisation of $\mathbf{\Pi}_1$ in Lemma 4.8.

$$\begin{aligned}
\mathfrak{P} &= \{F : F \text{ is closed in } (X^\omega, \rho) \wedge \mathbf{pref}(F) \in \Pi_2\} \\
\mathfrak{S} &= \{F : F \text{ is closed in } (X^\omega, \rho) \wedge \mathbf{pref}(F) \in \Sigma_1\}
\end{aligned}$$

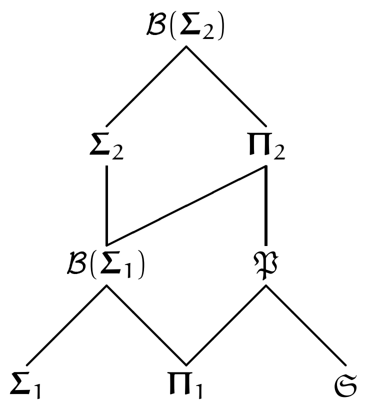

Figure 2: Inclusion relations between various classes of ω-languages ($\mathcal{B}(\mathbf{K})$ denotes the closure of $\mathbf{K}$ under Boolean operations)

Figure 2 presents the inclusion relation between these classes. All inclusions are proper, $\mathbf{\Sigma}_2$ and $\mathfrak{S}$ are incomparable, and $\mathbf{\Sigma}_1$ is not included in $\mathfrak{P}$. For instance, in Example 1.15 of [St93] ω-languages $E \in \mathfrak{S} \setminus \mathbf{\Sigma}_2$ and $F \in \mathbf{\Pi}_1 \setminus \mathfrak{S}$ are given, and $\mathbf{\Sigma}_1 \not\subseteq \mathfrak{P}$ follows from the fact that $\mathfrak{P}$ contains only closed sets.

First we obtain an exact estimate for $\underline{\kappa}(F)$ if F is Σ_2-definable (see Theorem 5 of [St98]).

Theorem 4.9 *If $F \subseteq X^\omega$ is Σ_2-definable then*

$$\dim_H F = \sup\{\underline{\kappa}(\xi) : \xi \in F\}.$$

Theorem 4.9 cannot be extended to higher classes of the arithmetical hierarchy. The proof of Lemma 6 in [St98] shows the following.

Lemma 4.10 *There are a countable subset $E \in \mathfrak{S}$ and an $F \in \mathbf{\Pi}_1$ such that $E \cap F = \{\zeta\}$, for some $\zeta \in X^\omega$, $\underline{\kappa}(\xi) = 1$ for all $\xi \in F$, and $\kappa(\xi) = 0$ for all $\xi \in E \setminus F$.*

Thus, for the ω-language E in Lemma 4.10 we have $\underline{\kappa}(\zeta) = \underline{\kappa}(E) = 1$ whereas $\dim_H E = \dim_P E = 0$ as E is countable. This shows also that the set E given in Lemma 4.10 is another witness for $\mathfrak{S} \setminus \mathbf{\Sigma}_2 \neq \emptyset$.

For sets in $\mathfrak{S}$ we obtain an upper bound via Corollary 2.5.

Lemma 4.11 *If $E \in \mathfrak{S}$ then $\underline{\kappa}(E) \leq \underline{\dim}_B E$.*

Similarly one obtains the following.

Lemma 4.12 *If $E \in \mathfrak{S} \cup \mathbf{\Pi}_1$ then $\kappa(E) \leq \overline{\dim}_B E$.*

Finally, Lemma 4.10 showed that $E \cap F = \{\zeta\}$ has $\underline{\kappa}(E \cap F) = 1$. Consequently, the bounds of Lemmas 4.11 and 4.12 do not extend to ω-languages of the form $E \cap F$ where $E \in \mathfrak{S}$ and $F \in \mathbf{\Pi}_1$.

5 Kolmogorov complexity for ω-power languages and regular ω-languages

As Lemma 4.10 showed the result of Theorem 4.9 cannot be extended to higher classes of the arithmetical hierarchy. In the preceding Theorem 4.5 we saw that structural properties might lead to tighter lower bounds. In this section classes of ω-languages which allow for more precise bounds are presented.

5.1 ω-power languages

The first class is connected to ω-languages exhibiting a certain kind of self-similarity, namely, ω-languages $F \subseteq X^\omega$ containing, for a certain set of words $W \subseteq X^*$ all of the shifts $w \cdot F$ where $w \in W$ in such a way that $F = \bigcup_{w \in W} w \cdot F$. Among these ω-languages the maximal ones have the form W^ω (see [St97b]). They satisfy the following properties.

Proposition 5.1 *Let* $W \subseteq X^*$. *Then* $W^\omega \subseteq \overrightarrow{W^*} \subseteq \mathcal{C}(W^\omega)$.

It is also obvious that W^ω is closed if W is finite, and W^ω is in the Borel class Π_2 if W is prefix-free. One cannot, however, bound the topological complexity of sets of the form W^ω as Finkel [Fi01] showed that for every $i \in \mathbb{N}$ the difference of the Borel classes $\Pi_{i+1} \setminus \Pi_i$ contains an ω-power languages W^ω where W is a context-free language.

Eq. (6.2) of [St93] and Corollary 3.9 of [Fa90] give the following formulae for Hausdorff and packing dimension.

Proposition 5.2
$$\begin{aligned} \dim_H W^\omega &= \dim_H \overrightarrow{W^*} = \mathsf{H}_{W^*} \\ \dim_P \mathcal{C}(W^\omega) &= \overline{\dim}_B W^\omega = \mathsf{H}_{\mathbf{pref}(W^*)} \end{aligned}$$

Moreover W^ω contains always an ω-word of highest upper Kolmogorov complexity (see [St81, St93]).

Lemma 5.3 *For* $W \subseteq X^*$ *let* $(m_i)_{i \in \mathbb{N}}$ *be a family of natural numbers and let* $(v_i)_{i \in \mathbb{N}}$ *be a family of words in* $\mathbf{pref}(W^*)$ *such that* $|v_i| < |v_{i+1}|$, $m_i/|v_i| < m_{i+1}/|v_{i+1}|$ *and* $K(v_i \mid |v_i|) \geq m_i$. *Then there is a* $\xi \in W^\omega$ *such that* $\kappa(\xi) \geq \sup\{m_i/|v_i| : i \in \mathbb{N}\}$.

As corollaries we obtain bounds on $\kappa(W^\omega)$.

Corollary 5.4 $\kappa(W^\omega) = \kappa(\mathcal{C}(W^\omega)) = \max\{\kappa(\xi) : \xi \in W^\omega\}$.

The next corollary follows also from Theorem 4.2 and Proposition 5.2.

Corollary 5.5 $\kappa(W^\omega) \geq \overline{\dim}_B W^\omega = \mathsf{H}_{\mathbf{pref}(W^*)}$

Since $W \in \Sigma_1$ implies $W^* \in \Sigma_1$ and $\mathcal{C}(W^\omega) \in \mathfrak{S}$, this corollary and Lemma 4.12 yield the following identity.

$$\kappa(W^\omega) = \overline{\dim}_B W^\omega \text{ if } W \in \Sigma_1 \tag{13}$$

Proposition 5.2 and Lemma 4.1 yield a similar estimate for $\underline{\kappa}(W^\omega)$ similar to Theorem 4.9 for Σ_2-definable sets.

Theorem 5.6 ([St93]) *If* $W \in \Sigma_1 \cup \Pi_1$ *then* $\dim_H W^\omega = \underline{\kappa}(W^\omega)$.

5.2 Regular ω-languages

The class of regular ω-languages is the one which is most extensively investigated, because it is the class of ω-languages definable by finite automata (cf. [St97a, Th90]). As we shall see below, this class behaves most regularly also in case of correspondences between complexity and dimension.

We refer to an ω-language $F \subseteq X^\omega$ as *regular* provided there are an $n \in \mathbb{N}$ and regular languages $W_i, V_i,\ i = 1, \ldots, n$ such that

$$F = \bigcup\nolimits_{i=1}^{n} W_i \cdot V_i^\omega .$$

It is known that all regular ω-languages are in $\mathcal{B}(\mathbf{\Sigma}_2)$.

First we mention properties of regular ω-languages with respect to dimensions which can be found in [St93, Corollary 4.4].

Proposition 5.7 *If* $F \subseteq X^\omega$ *is a regular* ω*-language closed in* (X^ω, ρ) *then* $\dim_H F = \overline{\dim}_B F$.

A particular case of this proposition follows already from Proposition 5.2 and the fact that $H_V = H_{\mathbf{pref}(V)}$ for regular languages $V \subseteq X^*$.

Corollary 5.8 *If* $W \subseteq X^*$ *is a regular language then* $\dim_H W^\omega = \overline{\dim}_B W^\omega$.

This yields the following as a corollary.

Corollary 5.9 *If* $F \subseteq X^\omega$ *is a regular* ω*-language then* $\dim_H F = \dim_P F$.

Proof. Observe that $\dim \bigcup_{i=1}^{n} W_i \cdot V_i^\omega = \max\{\dim V_i^\omega : 1 \leq i \leq n\}$ holds for $\dim \in \{\dim_H, \dim_P\}$. Now the assertion follows with Corollary 5.8.

❑

Regular ω-languages have non-null $\dim_H$-dimensional measure.

Theorem 5.10 ([St93, Theorem 4.7]) *Let* $F \subseteq X^\omega$ *be a nonempty regular* ω*-language. Then* $\mathbb{L}_{\dim_H F}(F) > 0$.

This enables us to apply Theorem 4.3 to obtain lower bounds on the complexity function for ω-words in regular ω-languages. Moreover, we can also transfer P. Martin-Löf's result [ML71] on complexity oscillations in X^ω to all nonempty regular ω-languages (see [St93, Theorem 4.12]).

Theorem 5.11 *Let $F \subseteq X^\omega$ be nonempty and regular.*

1. *If $f : \mathbb{N} \to \mathbb{N}$ satisfies $\sum_{n\in\mathbb{N}} r^{-f(i)} < \infty$ then there is a $\xi \in F$ such that $K(\xi[0..n] \mid n) \geq_{ae} \dim_H F \cdot n - f(n)$.*

2. *If $f : \mathbb{N} \to \mathbb{N}$ is a recursive function and satisfies $\sum_{n\in\mathbb{N}} r^{-f(i)} = \infty$ then $K(\xi[0..n] \mid n) \leq_{io} \dim_H F \cdot n - f(n)$ holds for all $\xi \in F$.*

A general linear upper bound on the complexity function for ω-words in regular ω-languages is the following one (cf. also [St81, St93]).

Theorem 5.12 1. *Let $W \subseteq X^*$ be a regular language. If $\dim_H W^\omega > 0$ then there is a constant $c_W \in \mathbb{R}$ such that for all $\xi \in W^\omega$ the bound $K(\xi[0..n]) \leq_{ae} \dim_H W^\omega \cdot n + c_W$ holds.*

2. *Let F be a regular ω-language such that $\dim_H F > 0$. Then for every $\xi \in F$ there is a constant $c_\xi \in \mathbb{R}$ such that $K(\xi[0..n]) \leq_{ae} \dim_H F \cdot n + c_\xi$.*

Proof. Since W is regular, $\mathbf{pref}(W^\omega)$ is also regular, thus $\mathbf{pref}(W^\omega) \in \Sigma_1 \cap \Pi_1$. According to Proposition 2.15 of [St93] there is a constant $c \in \mathbb{R}$ such that $s_{\mathbf{pref}(W^\omega)}(n) \leq c \cdot r^{\alpha \cdot n}$ for $\alpha = H_{\mathbf{pref}(W^\omega)}$ and all $n \in \mathbb{N}$. Consequently, $\sum_{i=0}^{n} s_{\mathbf{pref}(W^\omega)}(i) \leq c' \cdot r^{\alpha\cdot(n+1)}$ for a suitable constant $c' \in \mathbb{R}$. Now, our first assertion is a consequence of Corollaries 2.6 and 5.8.

The second one follows, because $\xi \in F = \bigcup_{i=1}^{n} W_i \cdot V_i^\omega$ implies that $\xi \in w_{\xi,i} \cdot V_i^\omega$ for suitable $i \in \{1, \ldots, n\}$ and $w_{\xi,i} \in W_i$. ❏

The theorem does not hold for $\dim_H F = 0$. In this case we have, in view of Proposition 4.12 of [St93] that a regular ω-language of Hausdorff dimension 0 is countable and consists entirely of recursive ω-words.

6 Subword complexity

It would be desirable to have an analogue to Lemma 4.1 for regular languages. For regular languages $V \subseteq X^*$, however, the identity $H_V = H_{\mathbf{pref}(V)}$ is true. In connection with the fact that $F \subseteq \overrightarrow{V}$ implies $\mathbf{pref}(F) \subseteq \mathbf{pref}(V)$ we obtain the identity $\inf\{H_V : F \subseteq \overrightarrow{V} \wedge V \text{ is regular}\} = \inf\{H_V : \mathbf{pref}(F) \subseteq \mathbf{pref}(V) \wedge V \text{ is regular}\}$.

Consequently, since every dense subset F of X^ω has $\mathbf{pref}(F) = X^*$, the infimum is 1 for arbitrary dense subsets of X^ω.

But if we restrict ourselves to single ω-words we obtain a variant of complexity, the so called *subword complexity*. It turns out that this subword complexity $\tau(\xi)$ of a string $\xi \in X^\omega$ is also closely related to the Hausdorff dimension of the regular ω-languages containing ξ. We start with some definitions. Let

$$\mathbf{infix}(\xi) := \{w : w \in X^* \wedge \exists v \exists \xi' (v \cdot w \cdot \xi' = \xi)\} \text{ and} \tag{14}$$

$$\mathbf{infix}_\infty(\xi) := \{w : w \in X^* \wedge \forall n \exists v \exists \xi' (|v| \geq n \wedge v \cdot w \cdot \xi' = \xi)\} \tag{15}$$

be the set of subwords of ξ and the set of subwords occurring infinitely often in ξ, respectively. We call $\tau(\xi) := H_{\mathbf{infix}(\xi)}$ the *subword complexity* of the string $\xi \in X^\omega$.

Then the following identity holds (cf. [St93, Section 5])

Lemma 6.1

$$\begin{aligned}\tau(\xi) = H_{\mathbf{infix}(\xi)} = H_{\mathbf{infix}_\infty(\xi)} &= \inf\{\dim_H F : F \textit{ is regular } \wedge \xi \in F\} \\ &= \inf\{\overline{\dim}_B F : F \textit{ is regular } \wedge \xi \in F\}\end{aligned}$$

Moreover, every nonempty regular ω-language F contains a recursive ω-word ξ_F of maximal subword complexity $\tau(\xi_F) = \dim_H F$.

Furthermore, we compare the functions $\underline{\kappa}$ and κ to this new complexity measure for infinite strings.

$$\underline{\kappa}(\xi) \leq \kappa(\xi) \leq \tau(\xi). \tag{16}$$

To this end we define the sets of ω-words (fibre sets in [CH94]) of specific complexity $\eta \in \{\underline{\kappa}, \kappa, \tau\}$

$$E_\alpha^{(\eta)} := \{\xi : \eta(\xi) \leq \alpha\} \quad \text{and} \quad F_\alpha^{(\eta)} := \{\xi : \eta(\xi) < \alpha\}. \tag{17}$$

The sets $E_\alpha^{(\underline{\kappa})}$ and $E_\alpha^{(\kappa)}$ were already considered in Eq. (12). The following inclusions are obvious.

$$F_\alpha^{(\eta)} \subseteq E_\alpha^{(\eta)} \text{ for } \eta \in \{\underline{\kappa}, \kappa, \tau\} \tag{18}$$

$$E_\alpha^{(\tau)} \subseteq E_\alpha^{(\kappa)} \subseteq E_\alpha^{(\underline{\kappa})} \text{ and } F_\alpha^{(\tau)} \subseteq F_\alpha^{(\kappa)} \subseteq F_\alpha^{(\underline{\kappa})} \tag{19}$$

It should be mentioned that Lemma 3.4 of [CH94] shows $F_\alpha^{(\eta)} \subset E_\alpha^{(\eta)}$ for $\eta \in \{\underline{\kappa}, \kappa\}$

The following relation between $\underline{\kappa}, \kappa$ and τ can be obtained from the results of [Ry84] or [CH94] and Lemma 6.1.

Theorem 6.2 *Each of the sets* $E_\alpha^{(\eta)}$, $F_\alpha^{(\eta)}$*,bb where* $\eta \in \{\underline{\kappa}, \kappa, \tau\}$*, has Hausdorff dimension* α*.*

We give a short proof.

Proof. In view of Eqs. (18) and (19) it suffices to show that $F_\alpha^{(\tau)} \subseteq E_\alpha^{(\underline{\kappa})}$.

Theorem 4.2 shows $\dim_H E_\alpha^{(\underline{\kappa})} = \dim_H \{\xi : \xi \in X^\omega \wedge \underline{\kappa}(\xi) \leq \alpha\} \leq \alpha$.

In order to show $\alpha \leq \dim_H F_\alpha^{(\tau)} = \dim_H \{\xi : \xi \in X^\omega \wedge \tau(\xi) < \alpha\}$ it suffices to construct a countable union of regular ω-languages $F_i \subseteq \{\xi : \xi \in X^\omega \wedge \tau(\xi) < \alpha\}$ such that $\sup\{\dim_H F_i : i \in \mathbb{N}\} = \alpha$.

Let $M_\alpha := \{(p,q) : p,q \in \mathbb{N} \wedge q \neq 0 \wedge \frac{p}{q} < \alpha\}$. For $(p,q) \in M_\alpha$ define $F_{(p,q)} := (X^p \cdot 0^{q-p})^\omega$. Each of the sets $F_{(p,q)}$ is regular, and one easily calculates $\dim_H F_{(p,q)} = \frac{p}{q}$. Then in view of Lemma 6.1 we have $\bigcup_{(p,q)\in M_\alpha} F_{(p,q)} \subseteq \{\xi : \xi \in X^\omega \wedge \tau(\xi) < \alpha\}$, and our assertion follows. ❑

Since $E_{\alpha'}^{(\eta)} \subset F_\alpha^{(\eta)}$, for $\alpha' < \alpha$ we obtain as a corollary to Theorem 6.2 and Lemma 3.4 of [CH94] that the families $(E_\alpha^{(\eta)})_{\alpha\in[0,1]}$ and $(F_\alpha^{(\eta)})_{\alpha\in[0,1]}$ form strictly increasing chains of ω-languages.

What concerns the packing dimension, Theorem 4.2 shows likewise $\dim_P E_\alpha^{(\kappa)} = \dim_P \{\xi : \xi \in X^\omega \wedge \kappa(\xi) \leq \alpha\} \leq \alpha$. Thus, in view of $\alpha = \dim_H F_\alpha^{(\tau)} \leq \dim_P F_\alpha^{(\tau)}$ and Eq (19), we obtain also the following.

Lemma 6.3 *Each of the sets* $E_\alpha^{(\eta)}$, $F_\alpha^{(\eta)}$*, where* $\eta \in \{\kappa, \tau\}$*, has packing dimension* α*.*

Lemma 6.3, however, does not hold for $\underline{\kappa}$. In Section 5 of [St05] it is mentioned that already $\dim_P \{\xi : \underline{\kappa}(\xi) = 0\} = 1$.

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