Randomness Relative to Cantor Expansions

Cristian S. Calude
Department of Computer Science
The University of Auckland,
Private Bag 92019
Auckland, New Zealand
Email: cristian@cs.auckland.ac.nz

Ludwig Staiger
Institut für Informatik
Martin-Luther-Universität Halle-Wittenberg
von-Seckemdorff-Platz 1
D - 06099 Halle, Germany
Email: staiger@informatik.uni-halle.de

Karl Svozil
Institut für Theoretische Physik,
Technische Universität Wien,
Wiedner Hauptstraße 8–10/136
A-1040 Wien, Austria
Email: svozil@tph.tuwien.ac.at

Abstract

Imagine a sequence in which the first letter comes from a binary alphabet, the second letter can be chosen on an alphabet with 10 elements, the third letter can be chosen on an alphabet with 3 elements and so on. Such sequences occur in various physical contexts, in which the coding of experimental outcome varies with scale. When can such a sequence be called *random*? In this paper we offer a solution to the above question using the approach to randomness proposed by Algorithmic Information Theory.

Contents

1 Motivation and examples

When it comes to comprehension and practical usefulness, the coding, or translation, from physics into a formal language may be of decisive importance: depending on the code chosen, what appears to be a disordered "mess" of data, may become structured and well understood (and vice versa). A typical example is the right choice of coordinate system utilizing the symmetry of a physical property. Thus far, little attention seems to have been given to the careful choice of number bases employed in coding physical entities; even to the extend that the coding sequence might use different bases for different positions.

Already Georg Cantor, the creator of naive set theory, considered such coding schemes. To obtain a better understanding, let us, as an example, examine the old British unit system, in which length can be measured in miles, furlongs, chains, yards, feet, hands, inches, lines [Drobot(1964)]. These scales relate in the following way: 1 mile = 8 furlongs = $8 \cdot 10$ chains = $8 \cdot 10 \cdot 22$ yards = $8 \cdot 10 \cdot 22 \cdot 3$ feet = $8 \cdot 10 \cdot 22 \cdot 3 \cdot 4$ hands = $8 \cdot 10 \cdot 22 \cdot 3 \cdot 4 \cdot 3$ inches = $8 \cdot 10 \cdot 22 \cdot 3 \cdot 4 \cdot 3 \cdot 12$ lines. Hence, the sequence of scales starts with $b_1 = 10, b_2 = 8, b_3 = 10, b_4 = 22, b_5 = 3, b_6 = 4, b_7 = 3, b_8 = 12$ and can be continued ad infinitum. For example, the number $0.963(11)232(10)00 \cdots 0 \cdots$ represents a length of 9 miles, 6 furlongs, 3 chains, 11 yards, 2 feet, 3 hands, 2 inches and 10 lines.

As a second example, consider a ball in gravitational fall impinging onto a board of nails with different numbers $b_n + 1$ of nails at different horizontal levels (here, n stands for the nth horizontal level and b_n is the basis corresponding to the position n). Let us assume that the layers are "sufficiently far apart" (and that there are periodic boundary conditions realizable by elastic mirrors). Then, depending on which one of the b_n openings the ball takes, one identifies the associated number (counted from 0 to $b_n - 1$) with the nth position $x_n \in \{0, \ldots, b_n - 1\}$ after the point. The resulting sequence leads to the real number whose Cantor expansion is $0.x_1 x_2 \cdots x_n \cdots$.

As a third example we consider a quantum correspondent of the board of nails harnessing irreducible complementarity and the randomness in the outcome of measurements on single particles. Take a quantised system with at least two complementary observables $\hat{A}, \hat{B}$, each one associated with N different outcomes a_i, b_j, $i, j \in \{0, \ldots, N-1\}$, respectively. Notice that, in principle, N could be a large (but finite) number. Suppose further that $\hat{A}, \hat{B}$ are "maximally" complementary in the sense that measurement of $\hat{A}$ totally randomises the outcome of $\hat{B}$ and *vice versa* (this should not be confused with optimal mutually unbiased measurements [Wootters and Fields(1989)]).

A real number $0.x_1x_2\cdots x_n\cdots$ in the Cantor expansion can be constructed from successive measurements of $\hat{A}$ and $\hat{B}$ as follows. Since all bases b_n used for the Cantor expansion are assumed to be bounded, choose N to be the least common multiple of all bases b_n. Then partition the N outcomes into even partitions, one per different base, containing as many elements as are required for associating different elements of the nth partition with numbers from the set $\{0,\ldots,b_n-1\}$. Then, by measuring

$$\hat{A},\hat{B},\hat{A},\hat{B},\hat{A},\hat{B},\ldots$$

successively, the nth position $x_n \in \{0,\ldots,b_n-1\}$ can be identified with the number associated with the element of the partition which contains the measurement outcome.

As an example, consider the Cantor expansion of a number in the bases 2, 6, and 9. As the least common multiple is 18, we choose two observables with 18 different outcomes; e.g., angular momentum components in two perpendicular directions of a particle of total angular momentum $\frac{9}{2}\hbar$ with outcomes in (units are in $\hbar$)

$$\left\{-\frac{9}{2},-4,-\frac{7}{2},\ldots,+\frac{7}{2},+4,+\frac{9}{2}\right\}.$$

Associate with the outcomes the set $\{0,1,2,\ldots,17\}$ and form the even partitions

$$\{\{0,1,2,3,4,5,6,7,8\},\{9,10,11,12,13,14,15,16,17\}\},$$
$$\{\{0,1,2\},\{3,4,5\},\{6,7,8\},\{9,10,11\},\{12,13,14\},\{15,16,17\}\},$$
$$\{\{0,1\},\{2,3\},\{4,5\},\{6,7\},\{8,9\},\{10,11\},\{12,13\},\{14,15\},\{16,17\}\},$$

(or any partition obtained by permuting the elements of $\{0,1,2,\ldots,17\}$) associated with the bases 2, 6, and 9, respectively.

Then, upon successive measurements of angular momentum components in the two perpendicular directions, the outcomes are translated into random digits in the bases 2, 6, and 9, accordingly.

As the above quantum example may appear "cooked up", since the coding is based on a uniform radix N expansion, one might consider successive measurements of the location and the velocity

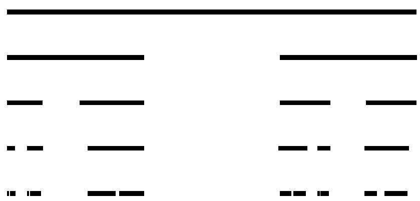

Figure 1: First construction steps of a generalised Cantor set.

of a single particle. In such a case, the value x_n is obtained by associating with it the click in a particular detector (or a range thereof) associated with spatial or momentum measurements. Any such arrangements are not very different in principle, since every measurement of a quantized system corresponds to registering a discrete event associated with a detector click [Summhammer(1989)].

This is a generalisation of self-similarity with intrinsic scale dependence. Geometric objects of this type might not scale in a self-similar manner but could be codable by a Cantor expansion.

Consider a generalized Cantor set obtained, for example, by cutting out in the nth construction step $1/(n+1)$th of each of the remaining segments (starting from the real interval $[0, 1]$ at $n = 1$) at a random position of $n + 1$ positions of equal length [Eberhard and Ross(1989)]. In Figure 1, the construction process is depicted.

Another example is the generalised Koch curve obtained by inserting in the nth construction step $n+1$ scaled down copies of the object obtained in the nth construction step at random positions. A different variation of the Koch curve is obtained if different objects (as compared to previous construction steps) are inserted. Any one of the above examples can be efficiently coded by Cantor expansions of random reals. For an efficient encoding, associate with every construction step a place in the expansion. Then, the basis chosen for this particular place in the expansion should be identified with the number of different segments in that construction step. For example, in the generalised Cantor set discussed above, there are $n + 1$ segments at the nth construction level; therefore, the basis chosen for the nth position should be $n + 1$. (This linear dependence is only an example, and much more general functions for the bases are possible.)

It is not too speculative to assume that this might reflect the physical property of different object formations at different (e.g., length or time) scales, which might be caused by non-similar interactions at different scales.

2 Varying Alphabets and the Cantor Expansion

In what follows we shall present methods to characterise and quantify Cantor encoded stochastic random sequences. Algorithmic Information Theory (see [Chaitin(1987), Chaitin(2001), Calude(2002)]) deals with random sequences over a finite (not necessarily binary) alphabet. A real number is random if its binary expansion is a binary random sequence; the choice of base is irrelevant (see [Calude(2002)] for various proofs).

Instead of working with a fixed alphabet we can imagine that the letters of a sequence are taken from a *fixed* sequence of alphabets. This construction was introduced by Cantor as a generalisation of the b–ary expansion of reals. More precisely, let

$$b_1, b_2, \ldots b_n, \ldots$$

be a fixed infinite sequence of positive integers greater than 1. Using a point we form the finite or infinite sequence

$$0.x_1 x_2 \ldots \tag{1}$$

such that $0 \leq x_n \leq b_n - 1$, for all $n \geq 1$. Consider the set of rationals

$$s_1 = \frac{x_1}{b_1}, s_2 = \frac{x_1}{b_1} + \frac{x_2}{b_1 b_2}, \ldots, s_n = s_{n-1} + \frac{x_n}{b_1 b_2 \cdots b_n}, \ldots \tag{2}$$

The above sum is bounded from above by 1,

$$0 \leq s_n \leq \sum_{i=1}^{n} \frac{b_i - 1}{b_1 b_2 \ldots b_i} = 1 - \frac{1}{b_1 b_2 \ldots b_n} < 1,$$

so there is a unique real number α that is the least upper bound of all partial sums (2). The sequence (1) is called the *Cantor expansion* of the real $\alpha \in [0, 1]$.

If $x_n = b_n - 1$, for all $n \geq 1$, then $s_n = 1 - 1/(b_1 b_2 \ldots b_n)$, so $\alpha = 1$. If $b_n = b$, for all $n \geq 1$, then the Cantor expansion becomes the classical b–ary expansion. If $x_n = 1$ and $b_n = n + 1$, for all $n \geq 1$, then $\alpha = e$.

The genuine strength of the Cantor expansion unfolds when various choices and interactions on different scales are considered.

The main result regarding Cantor expansions is the following theorem. Fix an infinite sequence of scales $b_1, b_2, \ldots$. Assume that we exclude Cantor expansions in which starting from some place after the point all the consecutive digits are $x_n = b_n - 1$. Then, every real number $\alpha \in [0,1]$ has a unique Cantor expansion (relative to $b_1, b_2, \ldots$) and its digits are determined by the following relations:

$$\rho_1 = \alpha, x_1 = \lfloor b_1 \rho_1 \rfloor, \rho_{n+1} = b_n \rho_n - x_n, x_{n+1} = \lfloor b_{n+1} \rho_{n+1} \rfloor.$$

Consequently, if we exclude Cantor expansions in which starting from some place after the point all the consecutive digits are $x_n = b_n - 1$. then given $\alpha \in [0,1]$ there is a unique sequence $\mathbf{x}^\alpha \in X^{(f)}$ whose Cantor expansion is exactly α. If $\mathbf{x} \in X^{(f)}$, then we denote by $\alpha^{\mathbf{x}}$ the real whose Cantor digits are given by the sequence $\mathbf{x}$, hence $\mathbf{x}^{\alpha^{\mathbf{x}}} = \mathbf{x}$ and $\alpha^{\mathbf{x}^\alpha} = \alpha$.

For more details regarding the Cantor expansion see [Hardy and Wright(1954), Drobot(1964)].

3 Notation and Basic Results

We consider $\mathbb{N}$ to be the set of non-negative integers. The cardinality of the set A is denoted by $\mathrm{card}(A)$. The base 2 logarithm is denoted by log.

If X is a set, then X^* denotes the free monoid (under concatenation) generated by X with e standing for the empty string. The length of a string $w \in X^*$ is denoted by $|w|$. We consider the space X^ω of infinite sequences (ω-words) over X. If $\mathbf{x} = x_1 x_2 \ldots x_n \ldots \in X^\omega$, then $\mathbf{x}(n) = x_1 x_2 \ldots x_n$ is the prefix of length n of $\mathbf{x}$. Strings and sequences will be denoted respectively by $x, u, v, w, \ldots$ and $\mathbf{x}, \mathbf{y}, \ldots$. For $w, v \in X^*$ and $\mathbf{x} \in X^\omega$ let $wv, w\mathbf{x}$ be the concatenation between w and $v, \mathbf{x}$, respectively.

By "$\sqsubseteq$" we denote the prefix relation between strings: $w \sqsubseteq v$ if there is a v' such that $wv' = v$. The relation "$\sqsubset$" is similarly defined for $w \in X^*$ and $\mathbf{x} \in X^\omega$: $w \sqsubset \mathbf{x}$ if there is a sequence $\mathbf{x}'$ such that $w\mathbf{x}' = \mathbf{x}$. The sets $\mathbf{pref}(\mathbf{x}) = \{w : w \in X^*, w \sqsubset \mathbf{x}\}$ and $\mathbf{pref}(B) = \bigcup_{\mathbf{x} \in B} \mathbf{pref}(\mathbf{x})$ are the languages of prefixes of $\mathbf{x} \in X^\omega$ and $B \subseteq X^\omega$, respectively. Finally, $wX^\omega = \{\mathbf{x} \in X^\omega : w \in \mathbf{pref}(\mathbf{x})\}$. The sets $(wX^\omega)_{w \in X^*}$ define the natural topology on X^ω.

Assume now that X is finite and has r elements. The unbiased discrete measure on X is the probabilistic measure $h(A) = \text{card}(A)/r$, for every subset of X. It induces the product measure μ defined on all Borel subsets of X^ω. This measure coincides with the Lebesgue measure on the unit interval, it is computable and $\mu(wX^\omega) = r^{-|w|}$, for every $w \in X^*$. For more details see [Martin-Löf(1966), Martin-Löf(1970), Calude(2002)].

In dealing with Cantor expansions we assume that the sequence of bases $b_1, b_2, \ldots b_n, \ldots$ is computable, i.e. given by a computable function $f : \mathbb{N} \to \mathbb{N} \setminus \{0, 1\}$. Let $X_i = \{0, \ldots, f(i) - 1\}$, for $i \geq 2$, and define the space

$$X^{(f)} = \prod_{i=1}^{\infty} X_i \subseteq \mathbb{N}^\omega .$$

The set

$$\mathbf{pref}(X^{(f)}) = \{w : w = w_1 w_2 \ldots w_n, w_i \in X_i, 1 \leq i \leq n\}$$

plays for $X^{(f)}$ the role played by X^* for X^ω.

Prefixes of a sequence $\mathbf{x} \in X^{(f)}$ are defined in a natural way and the set of all (admissible) prefixes will be denoted by $\mathbf{pref}(\mathbf{x})$. As we will report any coding to binary, the length of $w = w_1 w_2 \ldots w_n \in \mathbf{pref}(X^{(f)})$ is $\| w \| = \log(\prod_{i=1}^n f(i)); |w| = n$. In $X^{(f)}$ the topology is induced by the sets $[w]_f = \{\mathbf{x} \in X^f : w \in \mathbf{pref}(\mathbf{x})\}$ and the corresponding measure is defined by

$$\mu([w]_f) = \prod_{i=1}^{|w|} (f(i)^{-1}),$$

for every $w \in \mathbf{pref}(X^f)$. An open set is of the form $[A]_f = \{\mathbf{x} : \exists n(\mathbf{x}(n) \in A)\}$, for some set $A \subseteq \mathbf{pref}(X^{(f)})$. The open set $[A]_f$ is computably enumerable if A is computably enumerable.

The following two lemmas will be useful:

Let $0 \leq a < 2^m$ *and let* α, β *be two reals in the interval* $[a \cdot 2^{-m}, (a+1) \cdot 2^{-m}]$. *Then, the first* m *bits of* α *and* β *coincide, i.e., if* $\alpha = \sum_{i=1}^{\infty} x_i 2^{-i}$ *and* $\beta = \sum_{i=1}^{\infty} y_i 2^{-i}$, *then* $x_i = y_i$, *for all* $i = 1, 2, \ldots, m$.

Let $b_1, b_2, \ldots$ *be an infinite sequence of scales and* $a = \frac{j}{b_1 b_2 \ldots b_m} \in [0, 1]$. *Let* α, β *be two reals in the interval* $[a, a + 1/(b_1 b_2 \ldots b_m)]$. *Then, the first* m *digits of the Cantor expansions (relative to* $b_1, b_2, \ldots$*) of* α *and* β *coincide, that is, if* $\alpha = \sum_{i=1}^{\infty} x_i/(b_1 b_2 \ldots b_i)$ *and* $\beta = \sum_{i=1}^{\infty} y_i/(b_1 b_2 \ldots b_i)$, *then* $x_i = y_i$, *for all* $i = 1, 2, \ldots, m$.

4 Definitions of a Random Sequence Relative to the Cantor Expansion

In this section we propose five definitions for random sequences relative to their Cantor expansions and we prove that all definitions are mutually equivalent. We will fix a computable sequence of scales f.

We say that the sequence $\mathbf{x} \in X^f$ is *Cantor–random* if the real number $\alpha^{\mathbf{x}}$ is random (in the sense of Algorithmic Information Theory). e.g., the sequence corresponding to the binary expansion of α is random.

Next we define the notion of weakly Chaitin–Cantor random sequence. To this aim we introduce the Cantor self-delimiting Turing machine (shortly, a machine), which is a Turing machine C processing binary strings and producing elements of $\mathbf{pref}(X^{(f)})$ such that its program set (domain) $PROG_C = \{x \in \{0,1\}^* : C(x) \text{ halts}\}$ is a prefix-free set of strings. Sometimes we will write $C(x) < \infty$ when C halts on x and $C(x) = \infty$ in the opposite case.

The *program-size complexity* of the string $w \in \mathbf{pref}(X^{(f)})$ (relative to C) is defined by $H_C(w) = \min\{|v| : v \in \Sigma^*,\ C(y) = w\}$, where $\min \emptyset = \infty$. As in the classical situation the set of Cantor self-delimiting Turing machines is computably enumerable, so we can effectively construct a machine U (called *universal*) such that for every machine C, $H_U(x) \leq H_C(x) + O(1)$. In what follows we will fix a universal machine U and denote H_U simply by H.

The sequence $\mathbf{x} \in X^f$ is *weakly Chaitin-Cantor–random* if there exists a positive constant c such that for all $n \in \mathbb{N}$, $H(\mathbf{x}(n)) \geq \| x \| - c$.

The sequence $\mathbf{x} \in X^f$ is *strongly Chaitin-Cantor–random* if the following relation holds true: $\lim_{n\to\infty}(H(\mathbf{x}(n)) - \| x \|) = \infty$.

The sequence $\mathbf{x} \in X^f$ is *Martin-Löf-Cantor–random* if for every computably enumerable collection of computably enumerable open sets (O_n) in $X^{(f)}$ such that for every $n \in \mathbb{N}$, $\mu(O_n) \leq 2^{-n}$ we have $\mathbf{x} \notin \cap_{n=1}^{\infty} O_n$.

The sequence $\mathbf{x} \in X^f$ is *Solovay-Cantor–random* if for every computably enumerable collection of computably enumerable open sets (O_n) in $X^{(f)}$ such that $\sum_{n=1}^{\infty} \mu(O_n) < \infty$ the relation $\mathbf{x} \in O_n$ is true only for finitely many $n \in \mathbb{N}$.

Let us state a theorem. *Let* $\mathbf{x} \in X^{(f)}$. *Then, the following statements are equivalent:*

1. *The sequence* $\mathbf{x}$ *is weakly Chaitin-Cantor–random.*

2. *The sequence* $\mathbf{x}$ *is strongly Chaitin-Cantor–random.*

3. *The sequence* $\mathbf{x}$ *is Martin-Löf-Cantor–random.*

4. *The sequence* $\mathbf{x}$ *is Solovay-Cantor–random.*

These equivalences are direct translations of the classical proofs (see, for example, [Calude(2002)]).

Moreover, we have the following additional relations: *Let* $\mathbf{x} \in X^{(f)}$. *Then, the sequence* $\mathbf{x}$ *is weakly Chaitin-Cantor–random if* $\mathbf{x}$ *is Cantor–random. If the function* f *is bounded, then every weakly Chaitin-Cantor–random* $\mathbf{x}$ *is also Cantor–random sequence.*

Only the equivalence between the notions of Cantor–randomness and weakly Chaitin-Cantor–randomness will be proven.The argument is modification of the proof idea of Theorem 3 in [Staiger(2002)].

Assume first that $\mathbf{x} \in X^{(f)}$ is not Cantor–random and let $\alpha = \alpha^{\mathbf{x}}$. Let $\mathbf{y} = y_1 y_2 \ldots$ be the bits of the binary expansion of α. We shall show that $\mathbf{y}$ is not a binary random sequence.

Fix an integer $m \geq 1$ and consider the rational

$$\alpha(m) = \sum_{i=1}^{m} \frac{x_i}{b_1 b_2 \ldots b_m}.$$

We note that $w = x_1 x_2 \ldots x_m$ is in $\mathbf{pref}(X^{(f)})$ and $\| w \| = \log(b_1 b_2 \ldots b_m)$. Further on, $0 < \alpha(m) < \alpha$ and

$$\alpha - \alpha(m) \leq \sum_{t=m+1}^{\infty} \frac{x_t}{b_1 b_2 \ldots b_t} \leq \sum_{t=m+1}^{\infty} \frac{b_t - 1}{b_1 b_2 \ldots b_t} = \frac{1}{b_1 b_2 \ldots b_m}.$$

Next we define the following parameters:

$$M_m = \lfloor \log(b_1 b_2 \ldots b_m) \rfloor, \tag{3}$$

$$a_m = \lfloor \alpha(m) \cdot 2^{M_m} \rfloor. \tag{4}$$

and we note that

$$\alpha - \alpha(m) \leq \frac{1}{b_1 b_2 \ldots b_m} \leq 2^{-M_m}. \tag{5}$$

We are now in a position to prove the relation: for every integer $m \geq 1$,

$$[\alpha(m), \alpha] \subseteq \left[a_m \cdot 2^{-M_m}, (a_m + 2) \cdot 2^{-M_m}\right). \tag{6}$$

Indeed, in view of (5) and (4) we have $\alpha < (a_m + 2) \cdot 2^{-M_m}$ as:

$$\alpha \cdot 2^{-M_m} \leq \alpha(m) \cdot 2^{-M_m} + 1 < a_m + 2.$$

Again from (4), $a_m \leq \alpha(m) \cdot 2^{M_m}$.

Using (6), from $w = x_1 x_2 \ldots x_m$ plus two more bits we can determine $y_1 y_2 \ldots y_{M_m}$, that is, from the first m digits of the Cantor expansion of α and two additional bits we can compute the first M_m binary digits of α. In view of a result obtained earlier we obtain a computable function h which on an input consisting of a binary string v of length 2 and w produces as output $\mathbf{y}(M_m)$.

We are ready to use the assumption that **y** is random but **x** is not Cantor–random, that is, there is a universal self-delimiting Turing machine U^2 working on binary strings and there is a positive constant c such that for all $n \geq 1$,

$$H_{U^2}(\mathbf{y}(n)) \geq n - c, \tag{7}$$

and for every positive d there exists a positive integer l_d (depending upon d) such that

$$H(\mathbf{x}(l_d)) \leq \parallel \mathbf{x}(l_d) \parallel - d. \tag{8}$$

We construct a binary self-delimiting Turing machine C^2 such that for every $d > 0$, there exist two strings l_d and $v, s_{l_d} \in \{0,1\}^*$, such that $|v| = 2$, $|s_{l_d}| \leq \parallel \mathbf{x}(l_d) \parallel -d = \log(b_1 b_2 \ldots b_{l_d}) - d$ and $C^2(v, s_{l_d}) = \mathbf{y}(M_{l_d})$.

Consequently, in view of (7) and (8), for every d we have:

$$\begin{aligned}
M_{l_d} - c &\leq H_{U^2}(\mathbf{y}(M_{l_d})) \\
&\leq H_{C^2}(\mathbf{y}(M_{l_d})) + O(1) \\
&\leq |s_{l_d}| + 2 + O(1) \\
&\leq \log(b_1 b_2 \ldots b_{l_d}) + O(1) \\
&= M_{l_d} + O(1) - d,
\end{aligned}$$

a contradiction.

Recall that $\alpha = \sum_{i=1}^{\infty} x_i/(b_1 b_2 \ldots b_i) = \sum_{i=1}^{\infty} y_i 2^{-i}$. Now we prove that **x** is Cantor–random whenever **y** is random. Let $m \geq 1$ be an integer and let $\alpha_2(m) = \sum_{i=1}^{m} y_i 2^{-i}$. Given a large enough m we effectively compute the integer t_m to be the maximum integer $L \geq 1$ such that

$$2^{-m} \leq \frac{1}{b_1 b_2 \ldots b_L}. \tag{9}$$

We continue by proving that for all large enough $m \geq 1$:

$$[\alpha_2(m), \alpha] \subseteq \left[\alpha(t_m) - \frac{1}{b_1 b_2 \ldots b_{t_m}}, \alpha(t_m) + \frac{1}{b_1 b_2 \ldots b_{t_m}}\right]. \tag{10}$$

We note that $\alpha_2(m) < \alpha$ and

$$\begin{aligned} \alpha = \sum_{i=1}^{\infty} \frac{x_i}{b_1 b_2 \ldots b_i} &\leq \alpha(t_m) + \sum_{i=t_l}^{\infty} \frac{x_i}{b_1 b_2 \ldots b_i} \\ &\leq \alpha(t_m) + \sum_{i=t_l}^{\infty} \frac{b_i - 1}{b_1 b_2 \ldots b_i} \\ &\leq \alpha(t_m) + \frac{1}{b_1 b_2 \ldots b_{t_m}}. \end{aligned}$$

As $\alpha \leq \alpha(t_m) + 1/(b_1 b_2 \ldots b_{t_m})$ we only need to show that $\alpha(t_m) \leq \alpha_2(m) + 1/(b_1 b_2 \ldots b_{t_m})$. This is the case as otherwise, by (9), we would have:

$$\alpha(t_m) > \alpha_2(m) + \frac{1}{b_1 b_2 \ldots b_{t_m}} \geq \alpha_2(m) + 2^{-m} \geq \alpha,$$

a contradiction.

In case when f is bounded, assume by contradiction that $\mathbf{x}$ is Cantor–random but $\mathbf{y}$ is not random, that is there exists a positive constant c such that for all $n \geq 1$ we have:

$$H(\mathbf{x}(n)) \geq \log(b_1 b_2 \ldots b_n) - c, \tag{11}$$

and for every $d > 0$ there exists an integer $n_d > 0$ such that

$$H_{U^2}(\mathbf{y}(n_d)) < n_d - d. \tag{12}$$

In view of a result stated earlier and (10), there is a computable function F depending upon two binary strings such that $|v| = 2$, $F(\mathbf{y}(n_d), v) = \mathbf{x}(t_{n_d})$, so the partially computable function $F \circ U^2$ which maps binary strings in elements of $\mathbf{pref}(X^{(f)})$ is a Cantor self-delimiting Turing machine such that for every $d > 0$ there exists a binary string s_{n_d} of length less than $n_d - d$ and a binary string v of length 2 such that $F(U^2(s_{n_d}), v) = \mathbf{x}(t_{n_d})$.

As f is bounded, the difference $\mid t_{m+1} - t_m \mid$ is bounded. In view of (9), for large $m \geq 1$, $b_1 b_2 \ldots b_{t_m} > m - 1$, so we can write:

$$2^{-n_d} + \sum_{i=1}^{n_d} (1 - y_i) 2^{-i} = 1 - \sum_{i=1}^{n_d} y_i 2^{-i} \leq 1 - \frac{1}{b_1 b_2 \ldots b_{t_{n_d}}}, \tag{13}$$

$$\begin{aligned} n_d - c - 1 &\leq \log(b_1 b_2 \ldots b_{t_{n_d}}) - c \\ &\leq H_U(\mathbf{x}(t_{n_d})) \\ &\leq H_{F \circ U^2}(\mathbf{x}(t_{n_d})) + O(1) \\ &\leq |s_{n_d}| + 2 + O(1) \\ &\leq n_d - d + O(1), \end{aligned}$$

a contradiction.

Does the above result holds true for unbounded functions f? The answer is affirmative and has been independently obtained by L. Fortnow and A. Nies [Nies et al.(2004)]:

Let $\mathbf{x}$ *be a binary sequence. If there exists a computable infinite set* M *of positive integers and* $c > 0$ *such that for every* $m \in M$, $H_{U^2}(\mathbf{x}(m)) \geq m - c$, *then* $\mathbf{x}$ *is random.*

Here is a sketch of the proof. We construct the following self-delimiting Turing machine C: $C(u)$ tries to write the input u in the form $u = xy$ such that $U^2(x)$ is defined and $|U^2(x)| + |y|$ is the first element of R larger than $|U^2(x)|$. If all these conditions are satisfied, then $C(u) = U^2(x)y$.

First we note that C is self-delimiting. Secondly, if $H(\sigma) < |\sigma| - d$, and σ is a prefix of a string y such that the length of $w = \sigma y$ is the smallest element of R larger than $|\sigma|$, then we have $H_C(w) < |\sigma| + |y| - d$. Indeed, if $U^2(\sigma^*) = \sigma$, and $|\sigma^*| = H(\sigma)$, then $C(\sigma^* y) = \sigma y$ and $|\sigma^*| + |y| = H(\sigma) + |y| < |\sigma| + |y| - d$. Next let B be the constant necessary to U to simulate C, that is $H_{U^2}(u) \leq H_C(u) + B$, for all strings u.

Finally, assume that $\mathbf{x}$ is not weakly Chaitin–random, that is there exists a positive integer m such that $H_{U^2}(\mathbf{x}(m)) < m - (c + B)$. Consequently, taking t be the smallest element of R larger than m, we have: $H_{U^2}(\mathbf{x}(t)) \leq H_C(\mathbf{x}(t)) + B \leq t - c$, a contradiction.

It is interesting to note that in case of unbounded functions f we may have Cantor–random sequences $\mathbf{x} \in X^{(f)}$ which do not contain a certain letter, e.g. $0 \in X_i$. Let $f(i) = 2^{i+2}$. Then the measure of the set $F = \prod_{i=1}^{\infty} X_i'$, where $X_i' = X_i \setminus \{0\}$ satisfies $\mu(F) = \prod_{i=1}^{\infty}(1 - 2^{-i-1}) > 0$. Thus F contains a Cantor–random sequence $\mathbf{x}$. However, by construction, $\mathbf{x}$ does not contain the letter 0 which is in every X_i.

5 On the Meaning of Randomness in Cantor's Setting

So far, a great number of investigations have concentrated on the meaning and definition of randomness in the standard context, in which bases remain the same at all scales. That is, if one for instance "zooms into" a number by considering the next place in its expansion, it is always taken for granted that the same base is associated with different places.

From a physical viewpoint, if one looks into a physical property encoded into a real in, say, fixed decimal notation, then by taking the next digit amounts to specifying that physical property more precisely by a factor of ten. A fixed "zoom" factor may be the right choice if all physical properties such as forces and symmetries and boundary conditions remain the same at all scales. But this is hardly to be expected. Take, for instance, a "fractal" coastline. How is it generated? The origins of its geometry are the forces of the tidal and other forces on the land and coastal soil. That is, water moving back and forth, forming eddies, washing out little bays, and little bays within little bays, and little bays within little bays within little bays, ... and so on. There may be some structural components of this flow which results in scale dependence. Maybe the soil-water system forming the landscape will be "softer" at smaller scales, making bays relatively larger that their macroscopic counterparts. Indeed, eventually, at least at subatomic scales, the formation of currents and eddies responsible for the creation of ever smaller bays will break down.

In such cases, the base of the expansion might have to be modified in order to be able to maintain a proper relation between the coding of the geometric object formed by the physical system and the meaning of its number representation in terms of "zooming". All such processes are naturally stochastic, and therefore deserve a proper and precise formalisation in terms of random sequences in Cantor representations.

References

[Drobot(1964)] S. Drobot, *Real Numbers* (Prentice-Hall, Englewood Cliffs, New Jersey, 1964).

[Wootters and Fields(1989)] W. K. Wootters and B. D. Fields, Annals of Physics **191**, 363 (1989).

[Summhammer(1989)] J. Summhammer, in *The Concept of Probability*, edited by E. I. Bitsakis and C. A. Nicolaides (Kluwer Akademic Publishers, Amsterdam, 1989).

[Eberhard and Ross(1989)] P. H. Eberhard and R. R. Ross, Foundations of Physics Letters **2**, 127 (1989).

[Chaitin(1987)] G. J. Chaitin, *Algorithmic Information Theory* (Cambridge University Press, Cambridge, 1987).

[Chaitin(2001)] G. J. Chaitin, *Exploring Randomness* (Springer Verlag, London, 2001).

[Calude(2002)] C. Calude, *Information and Randomness—An Algorithmic Perspective* (Springer, Berlin, 2002), 2nd ed.

[Hardy and Wright(1954)] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers* (Cambridge University Press, London, 1954), third edition

[Martin-Löf(1966)] P. Martin-Löf, Information and Control **9**, 602 (1966).

[Martin-Löf(1970)] P. Martin-Löf, *Notes on Constructive Mathematics* (Almqvist & Wiksell, Stockholm, 1970).

[Staiger(2002)] L. Staiger, Theoretical Computer Science **284**, 455 (2002).

[Nies et al.(2004)] A. Nies, F. Stephan and S.A. Terwijn, CDMTCS Research Report 234, (2004).