POLYNOMIAL STABILITY OF OPERATOR SEMIGROUPS

ANDRÁS BÁTKAI, KLAUS-JOCHEN ENGEL, JAN PRÜSS, AND ROLAND SCHNAUBELT

ABSTRACT. We investigate polynomial decay of classical solutions of linear evolution equations. For bounded C_0–semigroups on a Banach space this property is closely related to polynomial growth estimates of the resolvent of the generator. For systems of commuting normal operators polynomial decay is characterized in terms of the location of the generator spectrum. The results are applied to systems of coupled wave-type equations.

1. INTRODUCTION

The asymptotic theory of operator semigroups provides powerful tools for the investigation of the (exponential) convergence to 0 of mild and classical solutions of the linear Cauchy problem

$$u'(t) + Au(t) = 0, \quad t \geq 0, \qquad u(0) = x, \tag{1.1}$$

where $-A$ generates the strongly continuous operator semigroup $(T(t))_{t\geq 0}$ on a Banach space X. In Section 2 we briefly review these results in order to provide the background for our paper.

However, weakly damped systems of linear wave equations can exhibit a type of behaviour not satisfactorily covered by semigroup theory so far: Classical solutions of (1.1) may converge to 0 polynomially, but not exponentially. Formulated in the framework of the evolution equation (1.1), certain systems lead to decay estimates of the form

$$\|T(t)x\| \leq Ct^{-\beta}\|A^\alpha x\|, \quad x \in D(A^\alpha),\ t > 0, \tag{1.2}$$

for some constants $\alpha, \beta > 0$. Such results were obtained in the recent papers [1], [2], [10], [15]; see also the references therein. These authors used energy type estimates which are more or less closely related to the specific problem posed on a Hilbert space. Observe that the estimate (1.2) with $\alpha = 0$ already implies exponential decay of the semigroup in operator norm. So we can exclude this case from our analysis.

An estimate like (1.2) typically holds if the spectrum of $-A$ is contained in open left half plane, but approaches the imaginary axis at $\pm i\infty$, see Section 5. One may further expect that the rate of approach of the spectrum is related to the constants α and β. We recall that this situation cannot occur if the semigroup is norm continuous at some $t_0 > 0$ or consists of positive operators on a Banach lattice, see e.g. Theorems II.4.18 and VI.1.10 in [7].

Date: April 13, 2004.

1991 *Mathematics Subject Classification.* Primary: 47D06; Secondary: 34G10, 35B40, 35L15, 35P15, 47A10.

Key words and phrases. Asymptotic behaviour, polynomial stability, C_0-semigroups, spectrum, Laplace transform, normal operators, coupled wave equation.

In Section 3 we first study general bounded C_0–semigroups on a Banach space X. Here we (almost) characterize the decay estimate (1.2) by an analogous growth property of the resolvent which in turn implies a geometric condition for the spectrum of $-A$ near $i\mathbb{R}$. This result relies in particular on a complex inversion formula of the Laplace transform applied to powers of the resolvent of $-A$. Unfortunately, we have to pay a price for the generality of our setting by loosing an arbitrarily small $\varepsilon > 0$ in the decay exponents. Further, for more specific situations one may hope for pure spectral criteria which are of course much easier to verify in applications. In Theorem 4.5 we in fact prove a sharp spectral criterion for systems of commuting normal operators on a Hilbert space X.7 This theorem follows from a corresponding characterization of polynomial decay for matrix multiplication semigroups on $L^p((\Omega,\mu),\mathbb{C}^n)$, because of the spectral theorem. The matrix multiplier result is proved via induction on the size n of the matrices, which requires a detailed spectral analysis. In the last section we apply Theorem 4.5 to coupled wave equations.

2. Stability concepts for operator semigroups

In this section we fix the notation and collect fundamental stability concepts and results for C_0–semigroups, in order to provide the background for our investigations. The proofs can be found in the monographs [3], [7], [11].

Let X be a Banach space. We write $G \in \mathcal{G}(X,M,w)$ if the linear operator G with domain $D(G)$ generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ satisfying $\|T(t)\| \leq Me^{wt}$ for $t \geq 0$. The exponential growth bound $\omega_0(G)$ is the infimum of such constants w. The resolvent operator of G is denoted by $R(\lambda,G) = (\lambda - G)^{-1}$ for λ contained in the resolvent set $\rho(G)$, $\sigma(G) = \mathbb{C} \setminus \rho(G)$ is the spectrum of G, and $s(G) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(G)\}$ is the spectral bound of G.

For $-A \in \mathcal{G}(X,M,w)$, we define the fractional powers $(d+A)^\alpha$ for $\alpha > 0$ and a fixed number $d > w$ by the formula

$$(d+A)^{-\alpha}x = \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{-\alpha}(\lambda + d + A)^{-1}x\,d\lambda$$

where Γ is any piecewise smooth path in the set $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > w - d,\ \lambda \notin [0,\infty)\}$ running from $\infty e^{-i\phi}$ to $\infty e^{i\phi}$ for some $0 < \phi < \pi/2$, cf. [7, Section II.5], [12, Section 2.7]. We further set $(d+A)^0 = I$. The operator $(d+A)^{-\alpha}$ is injective and bounded, hence it has a closed inverse denoted by $(d+A)^\alpha$. The domain $X_\alpha = D((d+A)^\alpha)$ is independent of the choice of $d > w$. We endow X_α with the graph norm of $(d+A)^\alpha$. If $w \in \rho(A)$ we can take $d = w$ in these definitions by deforming Γ appropriately.

For $\alpha \geq 0$ the fractional uniform exponential growth bound of the semigroup $T(\cdot)$ generated by $-A$ is defined by

$$\omega_\alpha(-A) = \inf\left\{a \in \mathbb{R} : \exists\, M \geq 1 \text{ such that } \left\|T(t)(d+A)^{-\alpha}\right\| \leq Me^{at},\ t \geq 0\right\}.$$

It is clear that

$$\omega_\beta(-A) \leq \omega_\alpha(-A) \leq \omega_0(-A) \qquad \text{for } \ 0 \leq \alpha \leq \beta,$$

where strict inequality is possible, see [16, Section 4]. Thus, if $\omega_\alpha(-A) < 0$, then all orbits $T(\cdot)x$ starting from $x \in X_\alpha$ converge to zero with an exponential speed. In particular, if

$\omega_1(-A) < 0$, then all orbits belonging to $C^1([0,+\infty), X)$ decay exponentially. We point out that this may happen even if there are unbounded orbits, cf. [11, Example 1.2.4].

One calls the semigroup uniformly exponentially stable if $\omega_0(-A) < 0$. The semigroup property implies that uniform exponential stability is equivalent to the fact that $\lim_{t\to\infty} \|T(t)\| = 0$. However, for our later investigations it is important to stress that the function $t \mapsto \|T(t)(d+A)^{-\alpha}\|$ does not satisfy the semigroup law anymore. Indeed, it Section 5 we treat examples where $\lim_{t\to\infty} \|T(t)(d+A)^{-\alpha}\| = 0$, but $\omega_\alpha(-A) = 0$.

For the study of these quantities one further introduces the *abscissa of growth order α* of the resolvent

$$s_\alpha(-A) = \inf\left\{\rho \geq s(-A) \,:\, \sup_{\operatorname{Re}\lambda > \rho} \frac{\|(\lambda+A)^{-1}\|}{1+|\operatorname{Im}\lambda|^\alpha} < \infty\right\}$$

for $\alpha \geq 0$. To simplify our notation, we introduce the symbol $\mathbb{C}_a$ to denote the open halfplane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > a\}$ for $a \in \mathbb{R}$, $\mathbb{C}_+ := \mathbb{C}_0$, and $\mathbb{C}_- = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$. A result of Latushkin and Shvidkoy, [9, Lemma 3.2], says that

$$s_\alpha(-A) = \inf\left\{\rho \geq s(-A) \,:\, \sup_{\lambda\in\mathbb{C}_\rho} \left\|(\lambda+A)^{-1}(d+A)^{-\alpha}\right\| < \infty\right\}.$$

The spectral bounds further satisfy the inequalities

$$s(-A) \leq s_\beta(-A) \leq s_\alpha(-A) \leq s_0(A) \qquad \text{for } 0 \leq \alpha \leq \beta,$$

where strict inequalities may occur even in Hilbert spaces X, see e.g. [11, Example 1.2.4]. Every semigroup on a Banach space X satisfies

$$s_\alpha(-A) \leq \omega_\alpha(-A).$$

Again, in general strict inequality is possible, see [16, Section 4]. Gearhart's theorem (see [11, Theorem 2.2.4]) implies that if X is a Hilbert space, then

$$s_0(-A) = \omega_0(-A).$$

Generalizing several previous results, Weis and Wrobel established in [16] the inequality

$$\omega_{\alpha+1}(-A) \leq s_\alpha(-A)$$

for $\alpha \geq 0$ and an arbitrary Banach space X. One can improve this inequality if one takes into account the geometry of X. We say that a Banach space X has *Fourier type* $p \in [1,2]$ if the Fourier transform extends from the Schwartz space $\mathcal{S}(\mathbb{R}, X)$ to a bounded operator from $L^p(\mathbb{R}, X)$ to $L^q(\mathbb{R}, X)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Clearly, every Banach space has Fourier type 1. It is known that Banach spaces having Fourier type 2 are isomorphic to Hilbert spaces and that the space $L^r(\mu)$ has Fourier type $\min\{r, s\}$ with $\frac{1}{r} + \frac{1}{s} = 1$, see for example the notes of [3, Section 1.8] for references on this subject. Weis and Wrobel proved that

$$\omega_{\alpha-1+\frac{2}{p}}(-A) \leq s_\alpha(-A)$$

if X has Fourier type $p \in [1,2]$. They also showed that these inqualities cannot be improved, in general.

So far we have considered general semigroups in a Banach space X. However, for several important classes of semigroups it is known that $s(-A) = \omega_0(-A)$; hence all the above quantities coincide. This happens for instance in the following cases.

a) The map $t \mapsto T(t)$ is continuous in the operator norm at some $t_0 > 0$. This holds in particular, if $T(\cdot)$ is analytic or $T(t_0)$ is a compact operator, see [7, Corollary IV.3.12].
b) The semigroup $T(\cdot)$ is essentially compact, i.e., $\|e^{-s(-A)t}T(t) - K\| < 1$ for some $t > 0$ and K compact linear operator, see [7, Theorem V.3.7].
c) The semigroup $T(\cdot)$ is a bounded group, see [7, Theorem IV.3.16].
d) The generator A is a normal operator on a Hilbert space.
e) The generator A is a multiplication operator on $L^p(\Omega, \mathbb{C}^n)$, see [6, Corollary IX.3.9] and Section 4 below.
f) The operators $T(t)$ are positive on $X = L^p(\Omega)$ or $X = C_0(\Omega)$, see [3, Section 5.3].

However, all the above collected results deal with exponential estimates of the orbits $T(\cdot)x$ for $x \in X_\alpha$. Therefore they cannot explain polynomial estimates like (1.2), which will be addressed in this paper. On the other hand, there has been considerable efforts to investigate strong stability of operator semigroups, i.e., $\lim_{t\to\infty} T(t)x = 0$ for all $x \in X$, see [3, Section 5.5]. In principle, these results can be applied to our situation, but they do not give decay estimates. Thus this line of research does not fit to our purposes, too.

Summing up, semigroup theory has not treated estimates of the type (1.2) so far. In our paper we want to close this gap at least partially using methods of spectral and Laplace transform theory.

3. Polynomial stability for general semigroups

We start our investigations with the following observation which allows to normalize estimate (1.2) if the semigroup is bounded.

Proposition 3.1. *(a) Assume that $-A \in \mathcal{G}(X, M, w)$. Let $\gamma \geq 1$. If $\|T(t)(d + A)^{-\alpha}\| \leq Ct^{-\beta}$ for $t > 0$ and some $\alpha, \beta > 0$, then $\|T(t)(d + A)^{-\alpha\gamma}\| \leq C'(\gamma)\, t^{-\beta\gamma}$ for $t > 0$.*

(b) Assume that $-A \in \mathcal{G}(X, M, 0)$ and that A is invertible. Then the following statements are equivalent with a constant $\alpha > 0$.

$$\|T(t)A^{-\alpha}\| \leq \frac{C}{t}, \quad t > 0, \tag{3.1}$$

$$\|T(t)A^{-\alpha\gamma}\| \leq \frac{C'(\gamma)}{t^\gamma}, \quad t > 0, \quad \text{for one/all } \gamma > 0. \tag{3.2}$$

Proof. (a) The assumption implies that

$$\|T(t)(d + A)^{-n\alpha}\| = \|[T(t/n)(d + A)^{-\alpha}]^n\| \leq (Cn^\beta)^n\, t^{-n\beta} \tag{3.3}$$

for $t > 0$ and $n \in \mathbb{N}$. Given $\gamma \geq 1$, we can write $\gamma = n + \tau$ for some $n \in \mathbb{N}$ and $\tau \in [0, 1)$. Combining estimate (3.3) with the moment inequality, see e.g. [7, Theorem II.5.34], we deduce

$$\begin{aligned}
\|T(t)(d + A)^{-\alpha\gamma}\| &= \|(d + A)^{(1-\tau)\alpha} T(t)(d + A)^{-(n+1)\alpha}\| \\
&\leq c\, \|T(t)(d + A)^{-n\alpha}\|^{1-\tau}\, \|T(t)(d + A)^{-(n+1)\alpha}\|^\tau \\
&\leq C'(\gamma)\, t^{-n\beta(1-\tau)}\, t^{-(n+1)\beta\tau} = C'(\gamma)\, t^{-\beta\gamma}.
\end{aligned}$$

(b.1) Assume that (3.2) holds for some $\gamma > 0$. We temporarily set $\delta = \alpha\gamma$ and obtain as above

$$\|T(t)A^{-n\delta}\| \leq C(n)\, t^{-n\gamma}$$

for $t > 0$ and $n \in \mathbb{N}$. The moment inequality now yields

$$\begin{aligned}\|T(t)A^{-n\delta\vartheta}\| = \|A^{n\delta(1-\vartheta)}T(t)A^{-n\delta}\| &\le c\left\|A^{n\delta}T(t)A^{-n\delta}\right\|^{1-\vartheta}\left\|T(t)A^{-n\delta}\right\|^{\vartheta}\\ &\le cM^{1-\vartheta}C(n)^{\vartheta}t^{-n\gamma\vartheta}\end{aligned}$$

for $\vartheta \in (0,1)$ and a constant c depending on the exponents. Choosing $\vartheta = \frac{1}{n\gamma}$ and $n > 1/\gamma$, we obtain inequality (3.1).

(b.2) Assume that (3.1) is satisfied and take some $\tilde{\gamma} > 0$. Then we deduce (3.2) for $\tilde{\gamma}$, if we replace in part (b.1) the number γ by 1 and set $\vartheta = \frac{\tilde{\gamma}}{n}$ for some $n > \tilde{\gamma}$. ☐

Definition 3.2. *Assume that $-A \in \mathcal{G}(X, M, w)$. We say that the semigroup $(T(t))_{t\ge 0}$ generated by $-A$ is **polynomially stable** if there exist constants $\alpha, \beta, C > 0$ such that*

$$\left\|T(t)(d+A)^{-\alpha}\right\| \le C\,t^{-\beta}, \qquad t > 0. \tag{3.4}$$

Note that the above definition is independent of d. If $(T(t))_{t\ge 0}$ is bounded and $0 \in \rho(A)$ we may normalize (3.4) to the estimate

$$\left\|T(t)A^{-\alpha}\right\| \le \frac{C}{t}, \qquad t > 0, \tag{3.5}$$

due to Proposition 3.1 (with a different α, in general). We next show that polynomial stability implies that the spectrum of A belongs to the open right halfplane and that its resolvent satisfies a certain bound.

Proposition 3.3. *Assume that $(-A) \in \mathcal{G}(X, M, w)$ and $\|T(t)(d+A)^{-\alpha}\| \le C/t$ for $t > 0$ and some $\alpha > 0$. Then $\sigma(-A) \subset \mathbb{C}_-$ and*

$$\|(\lambda + A)^{-1}A^{-\alpha-\varepsilon}\| \le M_\varepsilon \tag{3.6}$$

for $\operatorname{Re}\lambda \ge 0$ *and* $\varepsilon > 0$.

Proof. Proposition 3.1 yields

$$\|T(t)(d+A)^{-\alpha-\varepsilon}\| \le C\,t^{-1-\frac{\varepsilon}{\alpha}}$$

for $\varepsilon > 0$ and $t > 0$. Hence, the map $t \mapsto \|T(t)(d+A)^{-\alpha-\varepsilon}\|$ is integrable on $\mathbb{R}_+$. Recall that the Laplace transform of $T(\cdot)x$ is equal to $(\lambda + A)^{-1}x$ for $\operatorname{Re}\lambda > w$ and $x \in X$. By analytic continuation, the function $F_\varepsilon(\lambda) = (\lambda + A)^{-1}(d + A)^{-\alpha-\varepsilon}$ thus possesses a bounded holomorphic extension to $\overline{\mathbb{C}_0}$. Observe that $F_\varepsilon : \mathbb{C}_w \to \mathcal{L}(X, D(A))$ can also be extended holomorphically to $\overline{\mathbb{C}_0}$ since $AF_\varepsilon(\lambda) = (d + A)^{-\alpha-\varepsilon} - \lambda F_\varepsilon(\lambda)$ for $\operatorname{Re}\lambda > w$. As a result,

$$(\lambda + A)F_\varepsilon(\lambda)x = (d + A)^{-\alpha-\varepsilon}x \quad \text{and} \quad F_\varepsilon(\lambda)(\lambda + A)y = (d + A)^{-\alpha-\varepsilon}y$$

for $x \in X$, $y \in D(A)$, and $\operatorname{Re}\lambda \ge 0$. Therefore the part of $\lambda+A$ in the domain of $(d+A)^{\alpha+\varepsilon}$ (endowed with the graph norm) is invertible; hence $\lambda \in \rho(-A)$ by [7, Proposition IV.2.17]. Thus the operators $(\lambda+A)^{-1}(d+A)^{-\alpha-\varepsilon}$ are uniformly bounded for $\operatorname{Re}\lambda \ge 0$, which shows the assertion. ☐

It turns out that an estimate like (3.6) is already sufficient for the polynomial stability of a bounded semigroup, though with an arbitrarily small loss in the exponent of A^α. We first establish a result valid for general semigroups.

Proposition 3.4. *Assume that $-A \in \mathcal{G}(X, M, w)$ and that the function $\lambda \mapsto (\lambda+A)^{-1}(d+A)^{-\alpha}$, $\operatorname{Re}\lambda > w$, possesses a bounded holomorphic extension to $\operatorname{Re}\lambda \geq 0$, for some $\alpha > 0$. Then there is a constant $C(n,\delta)$ such that*

$$\|T(t)A^{-(n+1)\alpha-1-\delta}\| \leq C(n,\delta)\, t^{-n} \tag{3.7}$$

for $n \in \mathbb{N}$, $\delta \in (0,1]$, and all $t > 0$.

Proof. As in the proof of Proposition 3.3, we see that $(\lambda+A)^{-1}A^{-\alpha}$ is defined for $\operatorname{Re}\lambda \geq 0$ and bounded there by a constant C. We define

$$g(\lambda) = (\lambda + A)^{-n-1}A^{-(n+1)\alpha-1-\delta}$$

for $\operatorname{Re}\lambda \geq 0$. In order to estimate g, we first observe that

$$\|(\lambda + A)^{1-n}A^{-(n-1)\alpha}\| \leq C^{n-1}, \qquad \operatorname{Re}\lambda \geq 0,\ n \in \mathbb{N}. \tag{3.8}$$

Moreover,

$$\|\lambda(\lambda + A)^{-1}A^{-\alpha-1}\| = \|(I - A(\lambda + A)^{-1})A^{-\alpha-1}\| \leq C', \qquad \operatorname{Re}\lambda \geq 0. \tag{3.9}$$

Using the moment inequality, see e.g. [7, Theorem II.5.34], one deduces from (3.8) with $n = 2$ and (3.9) the estimate

$$\|(\lambda + A)^{-1}A^{-\alpha-\delta}\| \leq \frac{C''}{|\lambda|^{\delta}}, \qquad \operatorname{Re}\lambda \geq 0. \tag{3.10}$$

Combining (3.8), (3.9), and (3.10), we arrive at the inequality

$$\|g(\lambda)\| \leq \min\left\{\frac{C_n}{|\lambda|^{1+\delta}}, 1\right\} \tag{3.11}$$

for $\operatorname{Re}\lambda \geq 0$. We introduce the functions

$$f_a(t) := \frac{t^n}{n!}\, e^{-at}T(t)A^{-(n+1)\alpha-1-\delta}, \quad t \geq 0,\ a \geq 0, \qquad f(t) := f_0(t).$$

Due to e.g. formula (3.56) in [3], the Laplace transform of f_a is given by $\hat{f}_a(\lambda) = g(\lambda + a)$ for $\operatorname{Re}\lambda > w - a$. Thus the inversion formula proved in [3, Theorem 4.2.21] yields

$$f_a(t) = \lim_{R\to\infty} \frac{1}{R}\int_0^R \frac{1}{2\pi i}\int_{-ir}^{ir} e^{\lambda t} g(\lambda + a)\, d\lambda\, dr = \int_{-i\infty}^{+i\infty} e^{\lambda t} g(\lambda + a)\, d\lambda,$$

$$f(t) = \int_{-i\infty}^{+i\infty} e^{(a+\lambda)t} g(\lambda + a)\, d\lambda = \int_{a-i\infty}^{a+i\infty} e^{\lambda t} g(\lambda)\, d\lambda,$$

for $t \geq 0$. Due to (3.11), we can shift the path of integration to $a = 0$ and obtain

$$f(t) = \int_{-i\infty}^{+i\infty} e^{\lambda t} g(\lambda)\, d\lambda.$$

So (3.11) yields $\|f(t)\| \leq C(n,\delta)'$ which immediately implies (3.7). $\square$

Note that in the above proof we made essential use of the semigroup law. Example 4.2.9 in [11] shows that, for an arbitrary unbounded semigroup, one cannot hope to obtain the optimal decay estimate $\|T(t)A^{-\alpha}\| \leq C/t$ in (3.7) (with $n = 1$). Probably one could slightly improve Proposition 3.4 using ideas as in [11, §4.2]. But we do not pursue these matters since we are mostly interested in the case of bounded semigroups. Here we can now easily establish an almost optimal result.

Theorem 3.5. *We assume that $-A \in \mathcal{G}(X, M, 0)$ and $\sigma(A) \cap i\mathbb{R} = \emptyset$. Let $\alpha > 0$.*

(i) If $\|(i\rho + A)^{-1}A^{-\alpha}\| \le C$ for $\rho \in \mathbb{R}$, then for all $\varepsilon > 0$ there is a constant $C(\varepsilon) > 0$ such that $\|T(t)A^{-\alpha-\varepsilon}\| \le C(\varepsilon)\, t^{-1}$ for $t > 0$.

(ii) If $\|T(t)A^{-\alpha}\| \le C'\, t^{-1}$ for $t > 0$, then for all $\varepsilon > 0$ there is a constant $C(\varepsilon)' > 0$ such that $\|(\lambda + A)^{-1}A^{-\alpha-\varepsilon}\| \le C(\varepsilon)'$ for $\operatorname{Re}\lambda \ge 0$.

Proof. Notice that $\|(\lambda + A)^{-1}\| \le M/|\operatorname{Re}\lambda|$ for $\operatorname{Re}\lambda > 0$ since $\|T(t)\| \le M$. Thus the resolvent equation and the assumption in (i) imply that $(\lambda + A)^{-1}A^{-\alpha}$ is uniformly bounded for $\operatorname{Re}\lambda \ge 0$. So the first implication is a consequence of Propositions 3.4 and 3.1. The second implication follows immediately from Proposition 3.3. $\square$

We do not know whether one can omit the epsilons, in general. Observe that the conclusion in assertion (i) of the above theorem implies that $T(t)x \to 0$ as $t \to \infty$ for all $x \in X$.

For a better understanding of the condition in the previous theorem, we recall the following result by Latushkin and Shvidkoy, [9, Lemma 3.2].

Proposition 3.6. *Assume that $-A \in \mathcal{G}(X, M, w)$ and $\alpha > 0$. Let $S = \{\lambda \in \mathbb{C} : a < \operatorname{Re}\lambda < b\}$ be contained in $\rho(-A)$, for some $a < b$. Then the following assertions are equivalent.*

(i) $\|(\lambda + A)^{-1}(d + A)^{-\alpha}\| \le C$ for $\lambda \in S$ and a constant C.

(ii) $\|(\lambda + A)^{-1}\| \le C'\,(1 + |\lambda|^{\alpha})$ for $\lambda \in S$ and a constant C'.

In our situation, estimate (ii) allows to control the rate of approach of $\sigma(A)$ to the imaginary axis at $\pm i\infty$. Thus the following geometrical condition for $\sigma(A)$ is necessary for the polynomial stability of the semigroup.

Proposition 3.7. *We assume that $-A \in \mathcal{G}(X, M, 0)$, that $\sigma(A) \cap i\mathbb{R} = \emptyset$, and that*

$$\|(i\rho + A)^{-1}A^{-\alpha}\| \le C$$

for $\rho \in \mathbb{R}$ and constants $C, \alpha > 0$. Fix $\delta > 0$ such that $[0, \delta] \subset \rho(A)$. Then we have

$$|\operatorname{Im}\lambda| \ge C'\,(\operatorname{Re}\lambda)^{-1/\alpha} \qquad \text{for all } \lambda \in \sigma(A) \text{ with } \operatorname{Re}\lambda \le \delta.$$

Proof. As in the proof of Theorem 3.5 we see that $(\lambda + A)^{-1}A^{-\alpha}$ is uniformly bounded for $\operatorname{Re}\lambda \ge 0$. Proposition 3.6 and the continuity of the resolvent then yield

$$\frac{1}{\operatorname{Re}\lambda} \le \frac{1}{d(i\operatorname{Im}\lambda, \sigma(-A))} \le \|(i\operatorname{Im}\lambda + A)^{-1}\| \le C_1\,(1 + |\operatorname{Im}\lambda|^{\alpha}) \le C_2\,|\operatorname{Im}\lambda|^{\alpha}$$

for $\lambda \in \sigma(A)$ with $0 < \operatorname{Re}\lambda \le \delta$ and constants C_k not depending on λ. $\square$

It is clear that in general one cannot deduce asymptotic properties from pure spectral criteria. This can be seen if one multiplies the semigroup discussed in Paragraph 5.1 by $e^{-t/2}$. Then the spectral bound of the generator is $-1/2$, but there are exponentially growing orbits with initial values from the domain of the generator, see [11, Example 1.2.4].

4. Systems with commuting normal operators

We want to show that for systems with commuting normal operators one can get rid of the epsilon in Theorem 3.5 and that in this case the spectral condition in Proposition 3.7

is in fact sufficient for polynomial decay. As a preparation we deal with case of a single normal operator A.

Proposition 4.1. *Let H be a Hilbert space, let A be a normal operator on H with $\sigma(A) \subset \mathbb{C}_+$. Then the following are equivalent for $\alpha > 0$.*

(i) There exists $C > 0$ such that $\|T(t)A^{-\alpha}\| \leq C\,t^{-1}$ for $t \geq 0$.

(ii) There exists $C' > 0$ such that $\|(i\rho + A)^{-1}A^{-\alpha}\| \leq C'$ for $\rho \in \mathbb{R}$.

(iii) There exist $\delta, C'' > 0$ such that $|\operatorname{Im}\lambda| \geq C''\,(\operatorname{Re}\lambda)^{-1/\alpha}$ for $\lambda \in \sigma(A)$ with $\operatorname{Re}\lambda \leq \delta$.

Proof. Due to the spectral theorem for normal operators, see e.g. Theorems 13.21, 13.25, and 13.33 in [13], we have

$$\left\|tT(t)A^{-\alpha}\right\| = \sup_{\lambda\in\sigma(A)} t|\lambda|^{-\alpha}e^{-t\operatorname{Re}\lambda}, \qquad t \geq 0, \tag{4.1}$$

$$\left\|(i\rho + A)^{-1}A^{-\alpha}\right\| = \sup_{\lambda\in\sigma(A)} |\lambda|^{-\alpha}\,|i\rho+\lambda|^{-1}, \qquad \rho \in \mathbb{R}. \tag{4.2}$$

Assuming that (iii) holds, we thus obtain

$$\begin{aligned}\left\|tT(t)A^{-\alpha}\right\| &\leq \max\Big\{\sup_{\lambda\in\sigma(A),\operatorname{Re}\lambda\leq\delta} t|\operatorname{Im}\lambda|^{-\alpha}e^{-t\operatorname{Re}\lambda}\,,\ \sup_{\lambda\in\sigma(A),\operatorname{Re}\lambda>\delta} t|\lambda|^{-\alpha}e^{-t\operatorname{Re}\lambda}\Big\}\\ &\leq \max\Big\{\sup_{\lambda\in\sigma(A)} (C'')^{-\alpha}\,t\operatorname{Re}\lambda\,e^{-t\operatorname{Re}\lambda}, e\delta^{-1-\alpha}\Big\}\\ &\leq \max\{e(C'')^{-\alpha}, e\delta^{-1-\alpha}\}\end{aligned} \tag{4.3}$$

for $t \geq 0$, and analogously

$$\begin{aligned}\left\|(i\rho + A)^{-1}A^{-\alpha}\right\| &\leq \max\Big\{\sup_{\lambda\in\sigma(A),\operatorname{Re}\lambda\leq\delta} |\operatorname{Im}\lambda|^{-\alpha}\,|i\rho+\lambda|^{-1}\,,\ \sup_{\lambda\in\sigma(A),\operatorname{Re}\lambda>\delta} |\lambda|^{-\alpha}\,|i\rho+\lambda|^{-1}\Big\}\\ &\leq \max\Big\{\sup_{\lambda\in\sigma(A),\operatorname{Re}\lambda\leq\delta} (C'')^{-\alpha}\,\operatorname{Re}\lambda\,(\operatorname{Re}\lambda)^{-1}, \delta^{-1-\alpha}\Big\}\\ &= \max\{(C'')^{-\alpha}, \delta^{-1-\alpha}\}\end{aligned}$$

for $\rho \in \mathbb{R}$. So we have established the implications '(iii)$\Rightarrow$(i)' and '(iii)$\Rightarrow$(ii)'. Proposition 3.7 shows that (ii) implies (iii). Finally, assume that (i) is satisfied. Choosing $t = (\operatorname{Re}\lambda)^{-1}$, we deduce from (4.1) the estimate

$$|\lambda|^{-\alpha}\,(\operatorname{Re}\lambda)^{-1} = et|\lambda|^{-\alpha}e^{-t\operatorname{Re}\lambda} \leq eC \tag{4.4}$$

for $\lambda \in \sigma(A)$. There exists $\delta > 0$ such that $[0,\delta] \subset \rho(A)$, so that $|\lambda| \leq c\,|\operatorname{Im}\lambda|$ for $\lambda \in \sigma(A)$ with $\operatorname{Re}\lambda \leq \delta$. Hence, assertion (iii) follows from (4.4). ☐

With the same arguments one can also treat multiplication operators $Af = af$ for a function $a : \Omega \to \mathbb{C}$. These operators are endowed with their maximal domain $D(A) = \{f \in X : af \in X\}$ on a suitable function space X. Here we consider $X = C_0(\Omega)$ and $X = L^p(\Omega, \nu)$ for $1 \leq p < \infty$, a locally compact space Ω, and a regular Borel measure ν on Ω. Then $-A$ is densely defined and closed in X. It generates the bounded semigroup given by e^{-ta} if and only if the spectrum of $-A$ is contained in the closed left half plane. Moreover, $\sigma(A)$ coincides with the closed (resp., essential) range of a if $X = C_0$ (resp., $X = L^p$), see [6, §IX.1–3] or [7, §I.4]. Clearly, (4.1) and (4.2) still hold in the present situation, and so the proof of Proposition 4.1 also yields the next result.

Proposition 4.2. *Let Ω be a locally compact space and ν be a regular Borel measure on Ω. Assume that either*

(a) $X = L^p(\Omega,\nu)$ for $1 \le p < \infty$ and $a : \Omega \to \mathbb{C}$ is measurable with essential range in $\mathbb{C}_+$,

or that

(b) $X = C_0(\Omega)$ and $a : \Omega \to \mathbb{C}$ is continuous with with $\overline{a(\Omega)} \subset \mathbb{C}_+$.

In both cases let A be the multiplication operator corresponding to a on X. Then the following assertions are equivalent for $\alpha > 0$.

(i) There exists $C > 0$ such that $\|T(t)A^{-\alpha}\| \le C\,t^{-1}$ for $t \ge 0$.

(ii) There exists $C' > 0$ such that $\|(i\rho + A)^{-1}A^{-\alpha}\| \le C'$ for $\rho \in \mathbb{R}$.

(iii) There exist $\delta, C'' > 0$ such that $|\operatorname{Im}\lambda| \ge C''(\operatorname{Re}\lambda)^{-1/\alpha}$ for $\lambda \in \sigma(A)$ with $\operatorname{Re}\lambda \le \delta$.

Our main result in this section, Theorem 4.5, will be a consequence of Theorem 4.4 dealing with matrix multipliers. To prove the latter theorem, we need the next result due to Kreiss, [8]. In what follows, $B(x,r)$ is the open ball in $\mathbb{C}^n$ with center $x \in \mathbb{C}^n$ and radius $r > 0$, and $M_n(\mathbb{C})$ is the set of complex $n \times n$ matrices.

Theorem 4.3 (Kreiss)**.** *For a set $\mathcal{M} \subset M_n(\mathbb{C})$ the following assertions are equivalent.*

(i) There exists a constant $K_1 \ge 1$ such that

$$\left\|e^{tM}\right\| \le K_1$$

for all $t \ge 1$ and $M \in \mathcal{M}$.

(ii) There exist constants $K_2 \ge 1$ and $K_3 \ge 0$ satisfying the following property. For every $M \in \mathcal{M}$ there exists an invertible matrix $J_M \in M_n(\mathbb{C})$ with

$$\left\|J_M^{-1}\right\| + \|J_M\| \le K_2$$

such that

$$J_M^{-1} M J_M = \begin{pmatrix} \lambda_1^M & b_{12}^M & \dots & b_{1n}^M \\ 0 & \lambda_2^M & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_{n-1n}^M \\ 0 & \dots & 0 & \lambda_n^M \end{pmatrix},$$

where

$$\operatorname{Re}\lambda_n^M \le \operatorname{Re}\lambda_{n-1}^M \le \ldots \le \operatorname{Re}\lambda_1^M \le 0$$

and the upper diagonal entries b_{kl}^M satisfy the estimate

$$|b_{kl}^M| \le K_3\,|\operatorname{Re}\lambda_k^M| \quad \text{for all } 1 \le k < l \le n.$$

Here K_2 and K_3 only depend on n and K_1; and K_1 only depends on K_2, K_3, and n.

Let us now consider matrix multipliers $(Af)(\omega) = a(\omega)f(\omega)$ on $X = L^p((\Omega,\nu),\mathbb{C}^n)$ for $1 \le p < \infty$ and matrices $a(\omega) \in M_n(\mathbb{C})$ being measurable in ω. As in the scalar–valued case $n = 1$, we take $D(A) = \{f \in X : a(\cdot)f(\cdot) \in X\}$. If $\rho(A) \ne \emptyset$, then the spectrum of A is given by the essential union of $\sigma(a(\omega))$, i.e.,

$$\sigma(A) = \text{ess–}\bigcup_{\omega\in\Omega} \sigma(a(\omega)) := \{\lambda \in \mathbb{C} : \forall\,\varepsilon > 0 \;\; \nu\{\omega \in \Omega : \sigma(a(\omega)) \cap B(\lambda,\varepsilon)\} > 0\}.$$

In fact, one can find a function $\tilde{a}$ differing from a only on a set of measure 0 such that

$$\sigma(A) = \overline{\bigcup_{\omega\in\Omega} \sigma(\tilde{a}(\omega))}.$$

Moreover, $-A$ generates a bounded C_0–semigroup $(T(t))_{t\geq 0}$ on X if and only if $\operatorname{ess\,sup}_\omega |e^{-ta(\omega)}| \leq M$ for all $t \geq 0$, and then $T(t)f = e^{-ta(\cdot)}f(\cdot)$. These results can be found in [6, §IX.1-3.].

Theorem 4.4. *Let $X = L^p((\Omega,\nu),\mathbb{C}^n)$ for $1 \leq p < \infty$, $n \in \mathbb{N}$, Ω be a locally compact space, ν be a regular Borel measure on Ω, and $a(\omega) \in M_n(\mathbb{C})$ be measurable in $\omega \in \Omega$. Assume that $\operatorname{ess\,sup}_\omega |e^{-ta(\omega)}| \leq M$ for all $t \geq 0$ and that $\sigma(A) \cap i\mathbb{R} = \emptyset$ for the associated multiplication operator. Then the following statements are equivalent for $\alpha > 0$.*

(i) There exists $C > 0$ such that $\|T(t)A^{-\alpha}\| \leq C\, t^{-1}$ for $t \geq 0$.

(ii) There exists $C' > 0$ such that $\|(i\rho + A)^{-1}A^{-\alpha}\| \leq C'$ for $\rho \in \mathbb{R}$.

(iii) There exist $\delta, C'' > 0$ such that $|\operatorname{Im}\lambda| \geq C''\,(\operatorname{Re}\lambda)^{-1/\alpha}$ for $\lambda \in \sigma(A)$ with $\operatorname{Re}\lambda \leq \delta$.

Proof. (a) The implication '(ii)⇒(iii)' was proved in Proposition 3.7.

(b) We suppose that (iii) holds. Redefining $a(\omega)$ on a set of measure 0 we may assume that $e^{-ta(\omega)}$ and $a(\omega)^{-1}$ are uniformly bounded and that $\sigma(a(\omega))$ belongs to $\mathbb{C}_+$ and satisfies (iii) for all $\omega \in \Omega$. Observe that we only have to show that

(A) $|te^{-ta}a^{-\alpha}| \leq C$ for $t \geq 0$ and all matrices $a \in M_n(\mathbb{C})$ satisfying $|e^{-ta}| \leq M$, $|a^{-1}| \leq M'$, $\sigma(-a) \subset \mathbb{C}_-$ and $|\operatorname{Im}\lambda| \geq C''\,(\operatorname{Re}\lambda)^{-1/\alpha}$ for $\lambda \in \sigma(a)$ with $\operatorname{Re}\lambda \leq \delta$, where the constant C only depends on the strictly positive constants n, M, M', C'', δ, α.

Assertion (A) is proved by induction over the dimension n. The case $n = 1$ was settled in Proposition 4.2. Let $n \in \mathbb{N}$ be given and suppose that (A) has been verified for all dimensions $m \in \{1, \cdots, n\}$. Let $a \in M_{n+1}(\mathbb{C})$ satisfy the assumptions in (A). In view of Theorem 4.3, we can assume that

$$-a = \begin{pmatrix} \lambda_1 & b_{12} & \dots & b_{1n+1} \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_{nn+1} \\ 0 & \dots & 0 & \lambda_{n+1} \end{pmatrix}, \tag{4.5}$$

$$\operatorname{Re}\lambda_{n+1} \leq \operatorname{Re}\lambda_n \leq \ldots \leq \operatorname{Re}\lambda_1 < 0, \qquad |b_{kl}| \leq K\,|\operatorname{Re}\lambda_k|,$$

for $1 \leq k < l \leq n+1$, and a constant $K \geq 1$ only depending on n and M. We assume that the constant $\delta > 0$ in (A) is less than $2(C'')^{\frac{\alpha}{1+\alpha}}$ and that $2\,|\operatorname{Im}\lambda_1| \geq |\operatorname{Re}\lambda_1|$ if $|\operatorname{Re}\lambda_1| \leq \delta/2$, replacing the given δ by a smaller one if necessary.

(b.1) If $s(-a) < -\delta/2$, then $\|e^{-ta}\| \leq ce^{-\delta t/4}$ with a constant only depending on n, δ, K, and M. Hence, (A) holds in this case.

(b.2) We now consider the case $s(-a) \geq -\delta/2$. We set

$$r_0 = \frac{|\operatorname{Re}\lambda_1|}{4n+2}.$$

Then there is a radius $r \in \{r_0, 2r_0, \cdots, (2n+1)r_0\}$ such that $d(\lambda_k, \Gamma) \geq r_0$ for $\Gamma := \partial B(\lambda_1, r)$ and $k \in \{1, \cdots, n+1\}$. Notice that $r \leq |\operatorname{Re}\lambda_1|/2$. The representation (4.5) of

$-a$ and an induction argument show that the components r_{kl} of $(\mu+a)^{-1}$, $\mu \in \Gamma$, can be estimated by

$$|r_{kl}| \le \frac{c_n}{|\mu-\lambda_k|} \max_{k+1\le j\le l} \frac{|\operatorname{Re}\lambda_l|}{|\mu-\lambda_l|}\,\frac{|\operatorname{Re}\lambda_{l-1}|}{|\mu-\lambda_{l-1}|}\cdots\frac{|\operatorname{Re}\lambda_j|}{|\mu-\lambda_j|}$$

for $1 \le k < l \le n+1$ and a constant only depending on the data in (A), whereas $r_{kk} = (\mu-\lambda_k)^{-1}$ and $r_{kl} = 0$ for $k > l$. Since $|\mu-\lambda_k| \ge r_0 = (4n+2)^{-1}\,|\operatorname{Re}\lambda_1|$, we have

$$\ell(\Gamma)\sup_{\mu\in\Gamma}|(\mu+a)^{-1}| \le c_n' \sup_{\mu\in\Gamma}\max_{j=1,\cdots,n+1}\left\{1, \frac{|\operatorname{Re}\lambda_j|^n}{|\mu-\lambda_j|^n}\right\},$$

where $\ell(\Gamma)$ is the length of the curve. Let $\mu\in\Gamma$ and $j\in\{1,\cdots,n\}$. If $|\operatorname{Re}\lambda_j| \ge 2|\operatorname{Re}\lambda_1|$, then

$$\frac{|\operatorname{Re}\lambda_j|}{|\mu-\lambda_j|} \le \frac{|\operatorname{Re}\lambda_j|}{|\operatorname{Re}\lambda_j| - |\operatorname{Re}\mu|} \le \frac{|\operatorname{Re}\lambda_j|}{|\operatorname{Re}\lambda_j| - \frac{3}{2}|\operatorname{Re}\lambda_1|} \le 4\,.$$

If $|\operatorname{Re}\lambda_j| \le 2|\operatorname{Re}\lambda_1|$, then

$$\frac{|\operatorname{Re}\lambda_j|}{|\mu-\lambda_j|} \le \frac{2\,|\operatorname{Re}\lambda_1|}{r_0} = 8n+4.$$

Putting these observations together, we arrive at

$$\left|\frac{1}{2\pi i}\int_\Gamma f(\mu)\,(\mu+a)^{-1}\,d\mu\right| \le C(n)\sup_{\mu\in\Gamma}|f(\mu)| \tag{4.6}$$

for every bounded measurable function f defined on a neighbourhood of Γ. Again the constant only depends on the data in (A).

If we take f being equal to 1 on a neighbourhood of $B(\lambda_1, r)$, then estimate (4.6) implies that the spectral projection P for $-a$ corresponding to $B(\lambda_1, r)$ is bounded uniformly with respect to the data in (A). Let $m = \dim P\mathbb{C}^{n+1}$ and $l = n+1-m$. Since P commutes with a, the matrices aP and $a(I-P)$ satisfy our assumptions with uniform constants. If $P \ne I$, then $m, l \in \{1,\cdots,n\}$, and we can apply our induction hypothesis to deduce that (A) holds in this case.

(b.3) It remains to consider the case $P = I$. Then all eigenvalues λ_k of $-a$ belong to the set $B(\lambda_1, r)$. So (4.6) yields

$$|te^{-ta}a^{-\alpha}| = \left|\frac{1}{2\pi i}\int_\Gamma te^{t\mu}(-\mu)^{-\alpha}\,(\mu+a)^{-1}\,d\mu\right| \le C(n)\sup_{\mu\in\Gamma} te^{t\operatorname{Re}\mu}\,|\mu|^{-\alpha}\,. \tag{4.7}$$

Using $r \le \frac{1}{2}|\operatorname{Re}\lambda_1| \le |\operatorname{Im}\lambda_1|$, the spectral assumption in (A), and $|\operatorname{Re}\lambda_1| \le \delta/2 \le (C'')^{\frac{\alpha}{1+\alpha}}$, we further estimate

$$\begin{aligned}
|\operatorname{Im}\mu|\,|\operatorname{Re}\mu|^{1/\alpha} &\ge (|\operatorname{Im}\lambda_1| - \tfrac{1}{2}|\operatorname{Re}\lambda_1|)\,2^{-1/\alpha}|\operatorname{Re}\lambda_1|^{1/\alpha} \\
&\ge (C''\,|\operatorname{Re}\lambda_1|^{-1/\alpha} - \tfrac{1}{2}|\operatorname{Re}\lambda_1|)\,2^{-1/\alpha}|\operatorname{Re}\lambda_1|^{1/\alpha} \\
&= 2^{-1/\alpha}C'' - 2^{-1-1/\alpha}\,|\operatorname{Re}\lambda_1|^{(\alpha+1)/\alpha} \\
&\ge 2^{-1-1/\alpha}C''
\end{aligned}$$

for $\mu\in\Gamma$. In view of (4.7) and the estimates in (4.3), this inquality shows that (A) holds also in this case. So we conclude that (A) is verified for the dimension $n+1$, and thus in fact for all dimensions.

(c) We want to prove '(i) $\Rightarrow$ (ii)'. As in (b), it suffices to show this implication for all matrices $a \in M_n(\mathbb{C})$ satisfying the assumptions with uniform constants and having uniformly bounded inverses, provided we obtain (ii) with a uniform constant C'. Moreover, we can again suppose that a is given as in (4.5). Then

$$\begin{aligned} te^{-ta}a^{-\alpha}e_1 &= \frac{1}{2\pi i}\int_\Gamma te^{t\mu}(-\mu)^{-\alpha}(\mu+a)^{-1}e_1\,d\mu \\ &= \frac{1}{2\pi i}\int_\Gamma te^{t\mu}(-\mu)^{-\alpha}((\mu-\lambda_1)^{-1},0,\cdots,0)^T\,d\mu, \end{aligned}$$

for $e_1 = (1,0,\cdots,0)^T$ and a suitable path Γ around $\sigma(-a)$. So (i) implies that

$$\left|te^{t\lambda_1}(-\lambda_1)^{-\alpha}\right| = \left|\frac{1}{2\pi i}\int_\Gamma te^{t\mu}(-\mu)^{-\alpha}(\mu-\lambda_1)^{-1}\,d\mu\right| \le C$$

for all $t \ge 0$. Taking $t = -(\operatorname{Re}\lambda_1)^{-1}$, we see that $|\operatorname{Re}\lambda_1|^{-1} \le eC|\lambda_1|^{\alpha}$. Using that a is uniformly bounded, we thus derive (iii) for λ_1 and a sufficiently small $\delta > 0$ with a constant $\tilde{C}''$ only depending on the data. Now we can argue as in part (b) and verify that (ii) holds. $\square$

We can now easily deduce the main result of this section from the above theorem. Let H be a Hilbert space and A_{kl} $(k,l = 1,\cdots,n)$ be normal operators on H whose spectral resolutions mutually commute. In Section X.3 of [14] it shown that there are measurable function $f_{kl} : \mathbb{R} \to \mathbb{C}$ and a selfadjoint bounded operator B on H such that $A_{kl} = f_{kl}(B)$. Due to the spectral theorem in the version of [5, Corollary X.5.3] (and the proof given there), there is a locally compact space Ω, a regular Borel measure ν on Ω, a multiplication operator M on $L^2(\Omega,\nu)$, and a unitary operator $U : H \to L^2(\Omega,\nu)$ such that $UBU^{-1} = M$. Thus $UA_{kl}U^{-1}$ is also a multiplication operator on $L^2(\Omega,\nu)$ corresponding to a measurable function $a_{kl} : \Omega \to \mathbb{C}$. We next consider the matrix operator $\mathcal{A} = [A_{kl}]$ initially defined on

$$D_0 = \left(\bigcap_{k=1}^{n} D(A_{k1})\right) \times \cdots \times \left(\bigcap_{k=1}^{n} D(A_{kn})\right) \subset H^n.$$

Using the transformation $\mathcal{U} = \operatorname{diag}(U,\cdots,U) : H^n \to L^2(\Omega,\mathbb{C}^n)$, one sees that $\mathcal{A}$ possesses a closure (denoted by the same symbol) such that $\mathcal{U}\mathcal{A}\mathcal{U}^{-1}$ is equal to the matrix multiplicator $Af = a(\cdot)f$ on $L^2(\Omega,\mathbb{C}^n)$, where $a(\omega) = [a_{kl}(\omega)]$, see [6, Proposition IX.6.2]. Therefore, $-\mathcal{A}$ generates a bounded C_0–semigroup $(\mathcal{T}(t))_{t\ge0}$ if and only if $\|e^{-ta}\|_\infty \le c$ for $t \ge 0$ and then $\mathcal{T}(t) = \mathcal{U}e^{-ta}\mathcal{U}^{-1}$. Moreover,

$$\sigma(\mathcal{A}) = \text{ess-}\bigcup_{\omega\in\Omega}\sigma(a(\omega)).$$

Now, Theorem 4.4 immediately implies the following result.

Theorem 4.5. *Let H be a Hilbert space and A_{kl} $(k,l = 1,\cdots,n)$ be normal operators on H whose spectral resolutions mutually commute. Define $\mathcal{A}$ in $\mathcal{H} = H^n$ as above. Assume that $-\mathcal{A}$ generates a bounded C_0–semigroup $(\mathcal{T}(t))_{t\ge0}$ and that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$. Then the following statements are equivalent for $\alpha > 0$.*

(i) There exists $C > 0$ such that $\|\mathcal{T}(t)\mathcal{A}^{-\alpha}\| \le Ct^{-1}$ for $t \ge 0$.
(ii) There exists $C' > 0$ such that $\|(i\rho + \mathcal{A})^{-1}\mathcal{A}^{-\alpha}\| \le C'$ for $\rho \in \mathbb{R}$.

(iii) There exist $\delta, C'' > 0$ *such that* $|\operatorname{Im}\lambda| \geq C'' (\operatorname{Re}\lambda)^{-1/\alpha}$ *for* $\lambda \in \sigma(\mathcal{A})$ *with* $\operatorname{Re}\lambda \leq \delta$.

5. Applications

5.1. **The semigroup of Zabczyk.** The semigroup presented here is a slight modification of the famous example due to J. Zabczyk, cf. [11, Example 1.2.4], [7, Counterexample IV.3.4] or [3, Example 5.1.10]. Consider the Hilbert space $X = \bigoplus_{n\in\mathbb{N}} \mathbb{C}^n$. For the $n \times n$ matrix

$$A_n := \begin{pmatrix} 0 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 \\ 0 & \dots & \dots & 0 \end{pmatrix},$$

we define the operator $-A := \bigoplus_{n\in\mathbb{N}}(A_n + in - 1)$ with maximal domain $D(A)$. As calculated in the references given above, this operator generates the bounded strongly continuous semigroup

$$T(t) = \bigoplus_{n\in\mathbb{N}} \left(e^{-t} e^{int} e^{A_n t}\right), \tag{5.1}$$

and $s(-A) = -1$. To estimate the resolvent of A, we use the inequality

$$|R(\lambda, A_n + in - 1)| \leq \frac{1}{|\lambda + 1 - in| - 1}.$$

Take $\lambda = i\rho \in i\mathbb{R}$ and denote by $N \in \mathbb{N}$ the natural number such that $|\rho - N|$ is minimal. Then we obtain

$$\begin{aligned} \|(i\rho + A)^{-1}\|^2 &\leq |R(i\rho, A_N + iN - 1)|^2 + \sum_{n\neq N} \frac{1}{(|i\rho + 1 - in| - 1)^2} \\ &\leq |R(i\rho, A_N + iN - 1)|^2 + C \end{aligned}$$

for a suitable constant C independent of ρ. Since

$$R(i\rho, A_N + iN - 1) = \begin{pmatrix} \frac{1}{\lambda} & \frac{1}{\lambda^2} & \frac{1}{\lambda^3} & \cdots & \frac{1}{\lambda^N} \\ 0 & \frac{1}{\lambda} & \frac{1}{\lambda^2} & \cdots & \frac{1}{\lambda^{N-1}} \\ 0 & 0 & \frac{1}{\lambda} & \cdots & \frac{1}{\lambda^{N-2}} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \frac{1}{\lambda} \end{pmatrix}$$

with $\lambda = i(\rho - N) + 1$, the operator norm of this matrix becomes maximal if $\rho = N$. Thus we arrive at

$$\|(i\rho + A)^{-1}\|^2 \leq \|(iN + A)^{-1}\|^2 = N^2 + C \leq C' |\rho|^2$$

Theorem 3.5 thus yields that the semigroup (5.1) satisfies

$$\left\|T(t)A^{-1-\varepsilon}\right\| \leq \frac{C_\varepsilon}{t}$$

for all $\varepsilon > 0$.

5.2. **Weakly coupled wave equations.** If one couples a conservative with a damped wave equation, it is a priori not clear whether and in which way the resulting system is damped again. As a model problem for such phenomena we study the equations

$$
\begin{aligned}
&\partial_{tt}u(t,x) - \Delta u(t,x) + b\partial_t u(t,x) + \gamma u(t,x) - \kappa v(t,x) = 0, \qquad t \geq 0,\ x \in \Omega,\\
&\partial_{tt}v(t,x) - \Delta v(t,x) + \gamma v(t,x) - \kappa u(t,x) = 0, \qquad t \geq 0,\ x \in \Omega,\\
&u(t,x) = 0, \quad v(t,x) = 0, \qquad t \geq 0,\ x \in \partial\Omega,\\
&u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x),\ v(0,x) = v_0(x),\ \partial_t v(0,x) = v_1(x), \quad x \in \Omega,
\end{aligned}
\tag{5.2}
$$

on a bounded domain $\Omega \subset \mathbb{R}^N$ with boundary $\partial\Omega$ of class C^2 and for constants $b, \kappa > 0$ and $\gamma \geq \kappa$, see [2, Example 6.1]. We reformulate this partial differential equation as the second order evolution equation

$$
\begin{aligned}
&\ddot{u}(t) + Au(t) + b\dot{u}(t) - \kappa v(t) = 0, \qquad t \geq 0,\\
&\ddot{v}(t) + Av(t) - \kappa u(t) = 0, \qquad t \geq 0,\\
&u(0) = u_0,\ \dot{u}(0) = u_1, \quad v(0) = v_0,\ \dot{v}(0) = v_1,
\end{aligned}
\tag{5.3}
$$

on $H = L^2(\Omega)$ for the operator $A = -\Delta + \gamma$, with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. (Observe that our notation differs from that of the previous sections.) In what follows we will only use that $A = A^* \geq \omega > \kappa > 0$ and $b > 0$. As in [2], we rewrite (5.3) as a first order system on $\mathcal{X} = D(A^{1/2}) \times H \times D(A^{1/2}) \times H$ (endowed with the canonical scalar product) employing the operator matrix

$$
\mathcal{A} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ A & b & -\kappa & 0 \\ 0 & 0 & 0 & -1 \\ -\kappa & 0 & A & 0 \end{pmatrix}
$$

with domain $D(\mathcal{A}) = D(A) \times D(A^{1/2}) \times D(A) \times D(A^{1/2})$. (For (5.2) we have $D(A^{1/2}) = H_0^1(\Omega)$, of course.) One can check that $D(\mathcal{A}^n) = D(A^n) \times D(A^{n/2}) \times D(A^n) \times D(A^{n/2})$ (see [2, Lemma 3.1]). Using the bounded perturbation theorem and a suitable equivalent 'energy norm' on $\mathcal{X}$, it is shown in [2, §3] that $-\mathcal{A}$ generates a bounded C_0–semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $\mathcal{X}$. Moreover, the solution $w \in C^1(\mathbb{R}_+, X) \cap C(\mathbb{R}_+, D(\mathcal{A}))$ of

$$
\dot{w}(t) + \mathcal{A}w(t) = 0, \quad t \geq 0, \qquad w(0) = w_0,
\tag{5.4}
$$

with $w_0 = (u_0, u_1, v_0, v_1) \in D(\mathcal{A})$ gives the solutions $u, v \in C^2(\mathbb{R}_+, H) \cap C(\mathbb{R}_+, D(A))$ of (5.3) via $w(t) = (u(t), \dot{u}(t), v(t), \dot{v}(t))$. Then u and v also solve the equations (5.2) in $L^2(\Omega)$ w.r.t. the space variables, where the boundary conditions are understood in the sense of traces.

We can further transform the first order problem (5.4) to the setting of Theorem 4.5 by means of the isomorphism $J : \mathcal{X} \to H^4$, $Jx = (A^{1/2}x_1, x_2, A^{1/2}x_3, x_4)$. We will show below that the spectrum of $\mathcal{A}$ is contained in the open right half plane and that $|\operatorname{Im}\lambda| \geq c\,|\operatorname{Re}\lambda|^{-1/2}$ for $\lambda \in \sigma(-\mathcal{A})$ with real part close to 0. Thus Theorem 4.5 and Proposition 3.1 imply that $\|\mathcal{T}(t)\mathcal{A}^{-1}\| \leq ct^{-1/2}$. Proposition 3.1 yields better decay for more regular initial data. These facts were already shown in [2, Example 6.1] by other methods. However, our spectral analysis shows that the decay exponent is optimal, and it explains why precisely this type of damping occurs.

To compute the spectrum of $\sigma(-\mathcal{A})$ we have to solve the resolvent equation $(\lambda+\mathcal{A})x = y$ with $x \in D(\mathcal{A})$, for a given $y \in \mathcal{X}$. This equation means that

$$\begin{aligned}\lambda x_1 - x_2 &= y_1\\ Ax_1 + (\lambda+b)x_2 - \kappa x_3 &= y_2\\ \lambda x_3 - x_4 &= y_3\\ -\kappa x_1 + Ax_3 + \lambda x_4 &= y_4\,.\end{aligned}$$

The above system is equivalent to

$$\begin{aligned}x_2 &= \lambda x_1 - y_1\\ x_4 &= \lambda x_3 - y_3\\ (\lambda^2+b\lambda+A)x_1 &= \kappa x_3 + (\lambda+b)y_1 + y_2\\ p(\lambda,A)x_3 &= \kappa(\lambda+b)y_1 + \kappa y_2 + (A+\lambda(\lambda+b))\,(\lambda y_3 + y_4)\,,\end{aligned} \tag{5.5}$$

where we temporarily assume that $y_3, y_4 \in D(A)$ and use the polynomial

$$p(\lambda,a) = (a+(\lambda+b)\lambda)\,(a+\lambda^2) - \kappa^2 = \lambda^4 + b\lambda^3 + 2a\lambda^2 + ab\lambda + a^2 - \kappa^2.$$

Because of $A = A^* \geq \omega$, the operator $\lambda^2 + b\lambda + A$ is invertible whenever $\operatorname{Re}\lambda \geq -\delta$ for some $\delta > 0$ depending on b and ω. Since $|p(\lambda,a)| \to \infty$ as $a \to \infty$, the operator $p(\lambda,A)$ is invertible if and only if $p(\lambda,a) \neq 0$ for all $a \in \sigma(A)$. In this case also $a^2\,|p(\lambda,a)|^{-1}$ is bounded for $a \geq 0$, so that $p(\lambda,A)^{-1} : H \to D(A^2)$ is continuous. As a consequence, the equations (5.5) give the unique solution $x \in D(\mathcal{A})$ of $(\lambda+\mathcal{A})x = y$, for each $y \in \mathcal{X}$, provided that $p(\lambda,a) \neq 0$ and $\lambda^2+b\lambda+a \neq 0$ for all $a \in \sigma(A)$. On the other hand, assume that $\lambda \in \rho(-\mathcal{A})$. For a given $\eta \in H$ we define $y = (0, \eta/\kappa, 0, 0) \in \mathcal{X}$ and $x = (\lambda+\mathcal{A})^{-1}y$. Then (5.5) implies that $p(\lambda,A)x_3 = \eta$ and that $x_3 \in D(A)$ is the unique solution of this equation. Summing up,

$$\sigma(-\mathcal{A}) \cap \mathbb{C}_{-\delta} = \{\lambda \in \mathbb{C}_{-\delta} : p(\lambda,a) = 0 \ \text{ for some } a \in \sigma(A)\}.$$

So let us solve the 'characteristic equation' $p(\lambda,a) = 0$. Since $-\mathcal{A}$ generates a bounded semigroup, we only have to consider the case $\operatorname{Re}\lambda \leq 0$. First let $\lambda = i\tau \in i\mathbb{R}$ for some $a \geq \omega > \kappa > 0$. If $p(i\tau,a) = 0$, then

$$0 = \operatorname{Im} p(i\tau,a) = -b\tau^3 + ab\tau.$$

Hence, $\tau = 0$ or $\tau^2 = a$. But in both cases

$$\operatorname{Re} p(i\tau,a) = \tau^4 - 2a\tau^2 + a^2 - \kappa^2 \neq 0.$$

Therefore the spectrum of $-\mathcal{A}$ is contained in the open left half plane. If there are $\lambda \in \sigma(-\mathcal{A})$ with $\operatorname{Re}\lambda \to 0$, then we have $|\operatorname{Im}\lambda| \to \infty$ and thus $a \to \infty$. So we can restrict ourselves to $a \geq a_0$ for a large $a_0 > 0$, and $\operatorname{Re}\lambda \in (-\delta, 0)$. We set $\varepsilon = a^{-1/2}$ and $z = \varepsilon\lambda$. Then $p(\lambda,a) = 0$ is equivalent to the equation

$$\kappa^2\varepsilon^4 = (1 + z^2 + b\varepsilon z)(1 + z^2). \tag{5.6}$$

Note that for $\varepsilon = 0$ (5.6) has the two double solutions i and $-i$. Due to the Theorems A.4.1 and A.5.4 in [4], we know that for sufficiently small $\varepsilon > 0$ there are four distinct solutions $z = z(\varepsilon)$ of (5.6) given by a series in $\sqrt{\varepsilon}$. Inserting these series into (5.6) and comparing coefficients, we first deduce that the series is in fact a power series. Moreover, we obtain the expansions

$$z_{1,2}(\varepsilon) = \pm i - \tfrac{\kappa^2\varepsilon^3}{2b} + \mathcal{O}(\varepsilon^4) \quad \text{and} \quad z_{3,4}(\varepsilon) = \pm i - \tfrac{b\varepsilon}{2} + \mathcal{O}(\varepsilon^2)$$

of the solutions of (5.6). This leads to the solutions

$$\lambda_{1,2} = \lambda_{1,2}(a) = \pm i\sqrt{a} - \tfrac{\kappa^2}{2b}\tfrac{1}{a} + \mathcal{O}(a^{-3/2}), \quad \lambda_{3,4} = \lambda_{3,4}(a) = \pm i\sqrt{a} - \tfrac{b}{2} + \mathcal{O}(a^{-1/2})$$

of $p(\lambda, a) = 0$, for each sufficiently large $a \in \sigma(A)$. As a result, close to imaginary axis there are only the spectral values

$$\pm i\sqrt{a} - \tfrac{\kappa^2}{2b}\tfrac{1}{a} + \mathcal{O}(a^{-3/2}), \quad \text{for large } a \in \sigma(A);$$

i.e., $|\operatorname{Im}\lambda| \geq c|\operatorname{Re}\lambda|^{-1/2}$ for $\lambda \in \sigma(-\mathcal{A})$ with small $|\operatorname{Re}\lambda|$. As observed above, this fact shows that $\|\mathcal{T}(t)\mathcal{A}^{-1}\| \leq ct^{-1/2}$ and that this decay exponent is optimal. Thus we have shown that

$$\|\nabla u(t)\|_2^2 + \|\dot{u}(t)\|_2^2 + \|\nabla v(t)\|_2^2 + \|\dot{v}(t)\|_2^2 \leq c\,t^{-1}\left(\|u_0\|_{H^2}^2 + \|u_1\|_{H^1}^2 + \|v_0\|_{H^2}^2 + \|v_1\|_{H^1}^2\right)$$

for all classical solutions of (5.2).

Further observe the first term on the right hand side of (5.6) represents the damped equation and the second one the undamped equation, whereas the left hand side gives the coupling. The spectral values for the full system thus result from a perturbation of those of the two separated systems. The interesting part $\lambda_{1,2}$ of the spectrum is shifted to the left from $i\mathbb{R}$ to the hyperbolas $\pm i\sqrt{a} - \frac{\kappa^2}{2b}\frac{1}{a}$ (asymptotically for $a \to \infty$).

5.3. **A weakly coupled wave and plate equation.** In the same setting as above we study the Petrowsky-wave system

$$\begin{cases} \partial_{tt}u(t,x) + \Delta^2 u(t,x) + b\partial_t u(t,x) - \kappa v(t,x) = 0, & (t,x) \in \mathbb{R}_+ \times \Omega, \\ \partial_{tt}v(t,x) - \Delta v(t,x) - \kappa u(t,x) = 0, & (t,x) \in \mathbb{R}_+ \times \Omega, \end{cases} \tag{5.7}$$

for $b, \kappa > 0$ together with the initial conditions of (5.2) and the boundary conditions

$$v(\cdot,t) = u(\cdot,t) = \Delta u(\cdot,t) = 0 \quad \text{on } \partial\Omega \text{ for all } t > 0,$$

see [2, Example 6.4]. Since the arguments used below are similar to those in the previous example, we only sketch them. Again we work with the Hilbert space $H = L^2(\Omega)$ and the Dirichlet–Laplacian $A = -\Delta_D$ on H. We assume that $\kappa < \omega^{3/2}$, where ω is the lower bound of A (i.e, the Poincaré constant on Ω).

The weakly coupled wave and plate equation can also be written as an abstract Cauchy problem in the product space $\mathcal{X} = D(A) \times H \times D(A^{\frac{1}{2}}) \times H$ using the operator

$$\mathcal{A} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ A^2 & b & -\kappa & 0 \\ 0 & 0 & 0 & -1 \\ -\kappa & 0 & A & 0 \end{pmatrix}$$

with domain $D(\mathcal{A}) = D(A^2) \times D(A) \times D(A) \times D(A^{\frac{1}{2}})$. This operator generates a bounded semigroup on $\mathcal{X}$ and is unitary equivalent to a matrix consisting of commuting normal operators in the space H^4. Hence we can apply Theorem 4.5.

Using analogous arguments as in the previous example, we obtain

$$\sigma(-\mathcal{A}) \cap \mathbb{C}_{-\delta} = \{\lambda \in \mathbb{C}_{-\delta} \,:\, p(\lambda, a) = 0 \text{ for some } a \in \sigma(A)\}\,,$$

for small $\delta > 0$, where

$$p(\lambda, a) = (a^2 + b\lambda + \lambda^2)(a + \lambda^2) - \kappa^2.$$

Setting $\varepsilon = a^{-1}$ and $z = \lambda\varepsilon$, the equation $p(\lambda, a) = 0$ is equivalent to

$$(1 + z^2 + b\varepsilon z)(\varepsilon + z^2) = \kappa^2\varepsilon^4. \tag{5.8}$$

For $\varepsilon = 0$, there are two single roots $+i$ and $-i$ and the double root 0. For small $\varepsilon > 0$, we thus obtain two roots $z_{1,2}(\varepsilon)$ of (5.8) given by series in $\sqrt{\varepsilon}$ and two roots $z_{3,4}(\varepsilon)$ given by a power series, see [4, Sections A.4, A.5]. Inserting the series into (5.8) and comparing coefficients, we deduce

$$z_{1,2}(\varepsilon) = \pm i\sqrt{\varepsilon}\Big(1 - \frac{\kappa^2}{2}\,\varepsilon^3 - \frac{\kappa^2}{2}\,\varepsilon^4\Big) - \frac{b\kappa^2}{2}\varepsilon^5 + \mathcal{O}(\varepsilon^{\frac{11}{2}}), \qquad z_{3,4}(\varepsilon) = \pm i - \frac{b}{2}\,\varepsilon + \mathcal{O}(\varepsilon^2);$$

which yields

$$\lambda_{1,2}(a) = \pm i\sqrt{a}\Big(1 - \frac{\kappa^2}{2}\,a^{-3} - \frac{\kappa^2}{2}\,a^{-4}\Big) + \frac{b\kappa^2}{2}\,a^{-4} + \mathcal{O}(a^{-\frac{9}{2}}), \qquad \lambda_{3,4}(a) = \pm ia - \frac{b}{2} + \mathcal{O}(a^{-1})$$

for large $a \geq 0$. This means that

$$|\operatorname{Im}\lambda| \geq C\,|\operatorname{Re}\lambda|^{-1/8}$$

for $\lambda \in \sigma(-\mathcal{A})$ with small $\operatorname{Re}\lambda$, so that Theorem 4.5 implies $\|\mathcal{T}(t)\mathcal{A}^{-1}\| \leq ct^{-1/8}$; i.e.

$$\|D^2u(t)\|_2^2 + \|\dot{u}(t)\|_2^2 + \|\nabla v(t)\|_2^2 + \|\dot{v}(t)\|_2^2 \leq c\,t^{-1/4}\,(\|u_0\|_{H^4}^2 + \|u_1\|_{H^2}^2 + \|v_0\|_{H^2}^2 + \|v_1\|_{H^1}^2)$$

for all classical solutions of (5.7).

We note that in [2] only polynomial decay for initial data in $D(\mathcal{A}^4)$ was established (besides data in the test function space), without describing $D(\mathcal{A}^4)$ and its norm explicitely.

Acknowledgments

A. B. was supported by the OTKA grant Nr. F034840, by the FKFP grant Nr. 0049/2001 and by the Research Training Network HPRN-CT-2000-00116 of the European Union.

References

[1] F. Alabau-Boussouira, *Indirect boundary stabilization of weakly coupled hyperbolic systems*, SIAM J. Control Optim. **41** (2002), 511–541.

[2] F. Alabau, P. Cannarsa, V. Komornik, *Indirect internal stabilization of weakly coupled evolution equations*, J. Evol. Equ. **2** (2002), 127–150.

[3] W. Arendt, C.J.K. Batty, M. Hieber, F. Neubrander, *Vector Valued Laplace Transforms and Cauchy Problems,* Birkhäuser Verlag, Monographs in Mathematics **96**, 2001.

[4] H. Baumgärtel, *Analytic Perturbation Theory for Matrices and Operators,* Oper. Theory Adv. Appl. **15**, Birkhäuser Verlag, Basel, 1985.

[5] N. Dunford, J.T. Schwartz, *Linear Operators. Part II: Spectral Theory,* Interscience, New York, 1963.

[6] K.-J. Engel, *Operator Matrices and Systems of Evolution Equations,* book manuscript, available at: http://www.fa.uni-tuebingen.de/people/klen/omsee.ps

[7] K.-J. Engel, R. Nagel, *One-Parameter Semigroups for Linear Evolution,* Springer-Verlag, Graduate Texts in Mathematics **194**, 1999.

[8] H.O. Kreiss, *Über Matrizen die beschränkte Halbgruppen erzeugen*, Math. Scand. **7** (1959), 71–81.

[9] Y. Latushkin, R. Shvidkoy, *Hyperbolicity of semigroups and Fourier multipliers*, in: A.A. Borichev, N.K. Nikolski (eds.), 'Systems, approximation, singular integral operators, and related topics (Bordeaux, 2000),' Oper. Theory Adv. Appl. **129**, Birkhäuser Verlag, Basel, 2001, pp. 341–363.

[10] G. Lebeau, E. Zuazua, *Decay rates for the three-dimensional linear system of thermoelasticity*, Arch. Rational Mech. Anal. **148** (1999), 179–231.

[11] J.M.A.M. v. Neerven *The Asymptotic Behaviour of Semigroups of Linear Operators*, Birkhäuser verlag, Oper. Theory Adv. Appl. **88**, Birkhäuser Verlag, Basel, 1996.

[12] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer–Verlag, 1983.

[13] W. Rudin, *Functional Analysis*, Tata McGraw–Hill, 1974.

[14] B. Sz.–Nagy, *Spektraldarstellung linearer Transformationen des Hilbertschen Raumes*, Springer–Verlag, Corrected Reprint, 1967.

[15] L.R. Tcheugoué Tébou, *Well-posedness and energy decay estimates for the damped wave equation with L^r localizing coefficient*, Comm. Partial Differential Equations **23** (1998), 1839–1855.

[16] L. Weis, V. Wrobel, *Asymptotic behavior of C_0-semigroups in Banach spaces*, Proc. Amer. Math. Soc. **124** (1996), 3663–3671.

A. Bátkai, ELTE TTK, Department of Applied Analysis, Pf. 120, H-1518 Budapest, Hungary

E-mail address: batka@cs.elte.hu

K.-J. Engel, Università di L'Aquila, Dipartmento di Matematica, Sezione Ingeneria, Località Monteluco, I-67040 Roio Poggio, Italy

E-mail address: engel@ing.univaq.it

J. Prüss, R. Schnaubelt, FB Mathematik und Informatik, Martin–Luther–Universität, 06099 Halle, Germany

E-mail address: pruess@mathematik.uni-halle.de

E-mail address: schnaubelt@mathematik.uni-halle.de