THE DOMAIN OF ELLIPTIC OPERATORS ON $L^p(\mathbb{R}^d)$ WITH UNBOUNDED DRIFT COEFFICIENTS

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Abstract. We show that the elliptic operator $Au = \operatorname{div}(a\nabla u) + b \cdot \nabla u$ has the domain $D(A) = \{u \in W^{2,p}(\mathbb{R}) : b \cdot \nabla u \in L^p(\mathbb{R})\}$ on $L^p(\mathbb{R}^d)$, $1 < p < \infty$, and that A generates a C_0–semigroup on $L^p(\mathbb{R}^d)$. Here the diffusion coefficients $a(x)$ are uniformly elliptic and the drift coefficients $b(x)$ can grow as $|x| \log |x|$. Our approach relies on a Dore–Venni type theorem on sums of non commuting operators in $L^p(\mathbb{R}^d)$. The description of the domain implies global regularity of the density ρ of the invariant measure μ of the corresponding transition probabilities (if μ exists). We prove that $\rho \in W^{2,q}(\mathbb{R}^d)$ for all $1 < q < \infty$.

1. Introduction

Regularity properties of elliptic operators with unbounded coefficients have been intensively studied in recent years, mainly motivated by applications to stochastic processes and stochastic differential equations. In the present paper we investigate the operator

$$Au = \operatorname{div}(a\nabla u) + b \cdot \nabla u =: A_0 u + Bu$$

on $\mathbb{R}^d$. The diffusion part A_0 is supposed to be uniformly elliptic and the drift coefficients b may be unbounded. We want to show that A on its minimal domain

$$D(A) = D(A_0) \cap D(B) = \{u \in W^{2,p}(\mathbb{R}^d) : b \cdot \nabla u \in L^p(\mathbb{R}^d)\} \tag{1.1}$$

generates a C_0–semigroup $T(\cdot)$ in $L^p(\mathbb{R}^d)$, $1 < p < \infty$; see Theorem 2.4. This fact has been established in [15] for the prototypical example, the Ornstein–Uhlenbeck operator

$$A_{OU} u(x) = \operatorname{tr}(a_0 D^2 u) + b_0 x \cdot \nabla u(x) \tag{1.2}$$

(where $a_0 = a_0^T > 0$ and b_0 are real matrices). We point out that the Ornstein–Uhlenbeck semigroup is not analytic on $L^p(\mathbb{R}^d)$ if $b_0 \neq 0$, see [18] and also [11]. Thus the parabolic equation

$$\partial_t u(t,x) = A_{OU} u(t,x), \quad t > 0, \ \ x \in \mathbb{R}^d,$$

does not satisfy the standard parabolic regularity properties if one measures them in the 'global norm' of $L^p(\mathbb{R}^d)$, although the domain of A_{OU} is 'optimal.'

In fact, the Ornstein–Uhlenbeck semigroup is analytic on suitably weighted L^p spaces, where the domain of the A_{OU} becomes a weighted Sobolev space due to [15]. This result can be extended to larger classes of drift terms b as established in [7] and [16], see also the references therein. Nevertheless it is important to compute the domain $D(A)$ of A in the unweighted space $L^p(\mathbb{R}^d)$. In Section 3 we prove that a suitable embedding of $D(A)$ implies global regularity of the density ρ of the invariant measure μ for the transition

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probabilities corresponding to $T(\cdot)$ (assuming that μ exists). More precisely, we prove in Theorem 3.1 that $\rho \in W^{2,q}(\mathbb{R}^d)$ for all $1 < q < \infty$, so that ρ and $\nabla\rho$ are continuous and vanish at infinity.

It seems that global regularity of invariant measures has not been studied as thoroughly as local regularity. We are only aware of a series of papers by Bogachev, Krylov, Röckner, see e.g. [2], [4], and also [12]. The methods, assumptions and results of these papers are quite different from ours. For recent results on local regularity we refer to [3], [4], and the references therein.

In the recent paper [13], it was shown that the domain of A is indeed given by (1.1) assuming globally Lipschitz continuity of b and an additional condition on the oscillation of a at infinity. In our hypothesis (H), we weaken the assumptions on b allowing for growth as $|x|\log|x|$, cf. (2.3). Observe that if $|b(x)|$ grows as $|x|^{1+\varepsilon}$, $\varepsilon > 0$, then it can happen that the semigroup does not even exist on $L^p(\mathbb{R}^d)$, see (2.4).

In [13] the authors reduced the problem to the case of bounded coefficients by a change of variables. We proceed in a completely different way and apply a non–commutative Dore–Venni type theorem from [17] to the operator sum $A = A_0 + B$. As preliminary steps we study the flow semigroup generated by B and the commutator of A_0 and B. Thereby we considerably improve the approach of [15].

Notation. C_b^k (resp., C_c^k), $k \in \mathbb{N}_0$, is the space of k–times continuously differentiable functions which are bounded together with their derivatives (resp., which have compact support). Moreover, $x \cdot y = (x|y)$ denotes the standard scalar product in $\mathbb{R}^d$, M^T the transpose of the matrix M, Db the Jacobian of a vector field b, and $\nabla f = (\partial_1 f, \cdots, \partial_d f)^T$ the gradient of a function f. The adjoint of an operator C is designated by C^*. We write c_d for a generic constant only depending on the space dimension d. We refer to [9] and [19] for unexplained concepts from the theory of operator semigroups.

2. The generation result

We study the differential operator $Au = \operatorname{div}(a\nabla u) + b\cdot\nabla u$ supposing that the coefficients $a = (a_{ij})$ and $b = (b_i)$ satisfy

(H) $a \in C_b^1(\mathbb{R}^d, \mathbb{R}^{d^2})$, $a_{ij}(x) = a_{ji}(x)$, $\alpha_1 I \le a(x) \le \alpha_2 I$, $b \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, and

$$|(Db(x)a(x)\xi|\xi)| \le \beta_1\,, \qquad \xi \in \mathbb{R}^d,\ |\xi| = 1, \tag{2.1}$$

$$\Big|\sum_{k=1}^d b_k(x)\,\partial_k a_{ij}(x)\Big| \le \beta_2, \qquad i,j = 1,\cdots,d, \tag{2.2}$$

for $x \in \mathbb{R}^d$ and constants $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$.

Observe that (2.1) for $a_{ij}(x) = \delta_{ij}$ is equivalent to the two–sided disspativity estimate

$$|(b(x) - b(y)|x - y)| \le \beta_1\,|x-y|^2\,, \qquad x, y \in \mathbb{R}^d.$$

Our hypotheses are more general than those of [13], where (H) was assumed with (2.1) replaced by the global Lipschitz continuity of $b : \mathbb{R}^d \to \mathbb{R}^d$. Assumption (H) allows for vector fields b growing a bit more than linearly, as seen by the example

$$a(x,y) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \qquad b(x,y) = \ln r \begin{pmatrix} -y/\beta \\ x/\alpha \end{pmatrix}, \quad r \ge 1, \tag{2.3}$$

on $\mathbb{R}^2$. Here $r = \sqrt{x^2+y^2}$, b is extended in a smooth way to $\mathbb{R}^2$, and $\alpha > \beta > 0$. It should be observed that the above functions a and b do not satisfy the inequality $|(a(x)Db(x)\xi|\xi)| \le c\,|\xi|^2$ for $x, \xi \in \mathbb{R}^d$ with $|\xi| = 1$. Moreover, a and $b_1(x,y) = \ln r\,(-y,x)^T$ violate (2.1).

If the drift grows as $|x|^{1+\varepsilon}$ for some $\varepsilon > 0$, then it can happen that A does not generate a C_0–semigroup on $L^p(\mathbb{R}^d)$, even if $d = 1$. As an example we consider the operator

$$Au = u'' + bu', \qquad b(x) = -\operatorname{sign}(x)\,|x|^{1+\varepsilon}, \quad x \in \mathbb{R}. \tag{2.4}$$

We suppose that the domain of A is contained in $C_b(\mathbb{R})$ (which is true if $D(A) = \{u \in W^{2,p}(\mathbb{R}) : bu' \in L^p(\mathbb{R})\}$ as in Theorem 2.4). Assume for a contradiction that $\lambda - A$ is invertible in $L^p(\mathbb{R})$ for some $\lambda > 0$ and $p \in (1,\infty)$. Take $0 \le f \in C_c(\mathbb{R})$ with $f \ne 0$. Then there exists a function $u \in D(A)$ such that $(\lambda - A)u = f$. Consider $V(x) = x^2$ and $W(x) = \varepsilon\,(2\lambda)^{-1} + |x|^{-\varepsilon}$ for $|x| \ge 1$. Observe that $(\lambda - A)V(x) \ge 0$ and $(\lambda - A)W(x) \le 0$ for sufficiently large $|x|$. Since $u, f \in C_b(\mathbb{R})$, we can apply (the proof of) Theorem 3.20 of [14] which says that $u(x) \ge \delta W(x)$ for large $|x|$ and some $\delta > 0$. This estimate contradicts $u \in L^p(\mathbb{R})$. Hence $\lambda - A$ is not invertible for $\lambda > 0$ and A is not a generator on $L^p(\mathbb{R})$. Observe that in this case the ordinary differential equation (2.9) below has global solutions in forward time.

We collect some simple consequences of (H). Estimate (2.1) immediately implies that

$$|Db(x)a(x) + a(x)Db(x)^T| \le 2\,\beta_1\,, \qquad x \in \mathbb{R}^d. \tag{2.5}$$

Since $|a^{-1}(x)| \le \alpha_1^{-1}$, the inquality (2.5) further yields

$$|a(x)^{-1}Db(x) + Db(x)^T a(x)^{-1}| \le 2\,\beta_1\,\alpha_1^{-2}\,, \qquad x \in \mathbb{R}^d. \tag{2.6}$$

Similarly we can deduce that

$$\begin{aligned} |\operatorname{div} b(x)| &= |\operatorname{tr} Db(x)| = \tfrac{1}{2}\,|\operatorname{tr}(Db(x) + a(x)Db(x)^T a(x)^{-1})| \\ &\le c_d\,|Db(x) + a(x)Db(x)^T a(x)^{-1}| \le c_d\,\beta_1\,\alpha_1^{-1} \end{aligned} \tag{2.7}$$

for $x \in \mathbb{R}^d$. The components of the matrix $a(x)^{-1}$ are denoted by $r_{ij}(x)$. Then we have $\partial_k r_{ij}(x) = -[a(x)^{-1}\,(\partial_k a(x))\,a(x)^{-1}]_{ij}$ and thus obtain

$$\Big|\sum_{k=1}^d b_k(x)\,\partial_k r_{ij}(x)\Big| = \Big|\sum_{n,m=1}^d r_{im}(x) r_{nj}(x) \sum_{k=1}^d b_k(x)\,\partial_k a_{mn}(x)\Big| \le c_d\,\beta_2\,\alpha_1^{-2} \tag{2.8}$$

for $x \in \mathbb{R}^d$ and $i, j = 1, \cdots, d$.

We next construct the translation semigroup induced by the vector field b. Let $\varphi(t,x)$ be the maximal local flow solving the ordinary differential equation

$$u'(t) = b(u(t)), \quad t \in (t_0(x), t_1(x)), \qquad u(0) = x \in \mathbb{R}^d. \tag{2.9}$$

Then $v(t) = \partial_k \varphi(t,x)$ satisfies the variational equation

$$v'(t) = Db(\varphi(t,x))v(t), \quad t \in (t_0(x), t_1(x)), \qquad v(0) = e_k, \tag{2.10}$$

where e_k is the k–th unit vector in $\mathbb{R}^d$.

Lemma 2.1. *If (H) holds, then $\varphi(t,x)$ exists for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$. Moreover, $|\partial_k\,\varphi(t,x)| \le Me^{\gamma\,|t|}$ for $t \in \mathbb{R}$, $x \in \mathbb{R}^d$, and some constants $M > 0$ and $\gamma \in \mathbb{R}$.*

Proof. Set $\Phi(t) = (a(\varphi(t,x))^{-1}b(\varphi(t,x))|b(\varphi(t,x)))$ for a fixed $x \in \mathbb{R}^d$ and $t \in [0, t_1(x))$. Then (2.9) implies

$$\begin{aligned}\Phi'(t) = &\sum_{i,j,k}(\partial_k r_{ij}(\varphi(t,x)))\, b_k(\varphi(t,x))\, b_i(\varphi(t,x))\, b_j(\varphi(t,x)) \\ &+ ([a(\varphi(t,x))^{-1}Db(\varphi(t,x)) + Db(\varphi(t,x))^T a(\varphi(t,x))^{-1}]b(\varphi(t,x))|b(\varphi(t,x))).\end{aligned}$$

Estimates (2.8) and (2.6) and assumption (H) thus yield

$$\Phi'(t) \le c\,|b(\varphi(t,x))|^2 \le c\,\alpha_2\,\Phi(t), \quad \text{hence} \quad \Phi(t) \le \Phi(0)\,e^{c\alpha_2 t}$$

for a constant $c > 0$ depending on d, α_1, β_1, β_2. Employing (H) once more, we obtain

$$|b(\varphi(t,x))|^2 \le \alpha_2 \Phi(t) \le \frac{\alpha_2}{\alpha_1}\,|b(x)|^2\, e^{c\alpha_2 t}$$

for $x \in \mathbb{R}^d$ and $t \in [0, t_1(x))$. This estimate further implies that

$$\tfrac{d}{dt}\,|\varphi(t,x)|^2 = 2\,(b(\varphi(t,x))|\varphi(t,x)) \le \sqrt{\alpha_2/\alpha_1}\,|b(x)|e^{\frac{1}{2}tc\alpha_2}\,(1 + |\varphi(t,x)|^2).$$

Integrating this inequality and using Gronwall's lemma, we see that $|\varphi(t,x)|$ remains bounded as $t \to t_1(x)$ if $t_1(x) < \infty$; hence $t_1(x) = \infty$. One verifies that $t_0(x) = -\infty$ by reversing time.

To prove the second assertion, we let $\Psi(t) = (a(\varphi(t,x))^{-1}\partial_k\varphi(t,x)|\partial_k\varphi(t,x))$ for $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$. As above we deduce $\Psi'(t) \le \gamma'\,\Psi(t)$ and thus

$$|\partial_k\varphi(t,x))|^2 \le \alpha_2\Psi(t) \le \alpha_2\Psi(0)\,e^{\gamma'|t|} = \alpha_2 r_{kk}(x)\,e^{\gamma'|t|} \le \frac{\alpha_2}{\alpha_1}\,e^{\gamma'|t|}$$

for $t \in \mathbb{R}$, $x \in \mathbb{R}^d$, and some constant $\gamma' > 0$. □

Let $p \in (1, \infty)$ and $p' = p/(p-1)$. In view of the above lemma we can define

$$(S(t)f)(x) = f(\varphi(t,x)), \qquad f \in L^p(\mathbb{R}^d),\ x \in \mathbb{R}^d,\ t \in \mathbb{R}.$$

Lemma 2.2. *Assume that (H) holds. Then $S(\cdot)$ defined above is a positive C_0–group on $L^p(\mathbb{R}^d)$ and $W^{1,p}(\mathbb{R}^d)$ satisfying $\|S(t)\|_{L^p} \le e^{w_p|t|}$ and $\|S(t)\|_{W^{1,p}} \le Me^{c|t|}$ for $t \in \mathbb{R}$, $1 < p < \infty$, and some constants $M \ge 1$, $c \in \mathbb{R}$, $w_p = c_d\beta_1\alpha_1^{-1}p^{-1}$. Its generator B on $L^p(\mathbb{R}^d)$ is given by $Bf = g$ on $D(B) = \{f \in L^p(\mathbb{R}^d) : \exists\, g \in L^p(\mathbb{R}^d)$ s.t. $\langle g, \phi\rangle = -\langle f, \operatorname{div}(\phi b)\rangle\ \forall \phi \in C_c^1(\mathbb{R}^d)\}$, where the brackets denote the duality of L^p–$L^{p'}$. In particular, if $f \in W^{1,p}(\mathbb{R}^d)$ and $b \cdot \nabla f \in L^p(\mathbb{R}^d)$, then $f \in D(B)$ and $Bf = b \cdot \nabla f$. Moreover, $C_c^1(\mathbb{R}^d)$ is a core for B.*

Proof. (1) Observe that $x = \varphi(-t, \varphi(t,x))$ for $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$ and that $D_x\varphi(t,x)$ is a fundamental solution of (2.10). Thus a change of variables, the Abel–Liouville formula, and (2.7) yield

$$\begin{aligned}\int_{\mathbb{R}^d} |S(t)f(x)|^p\,dx &= \int_{\mathbb{R}^d} |f(y)|^p\,|\det(D_y\varphi(-t,y))|\,dy \\ &= \int_{\mathbb{R}^d} |f(y)|^p\,\exp\Big(\int_0^{-t}(\operatorname{div} b)(\varphi(s,y))\,ds\Big)\,dy \\ &\le e^{c_d\beta_1\alpha_1^{-1}|t|}\,\|f\|_p^p\end{aligned}$$

for $f \in L^p(\mathbb{R}^d)$ and $t \in \mathbb{R}$. It is then easy to check that $S(t)$ is a positive C_0–group on $L^p(\mathbb{R}^d)$ which satisfies the required estimate. Moreover,

$$\partial_k(S(t)f)(x) = \partial_k f(\varphi(t,x)) = \sum_{j=1}^{d} S(t)(\partial_j f)(x)\, \partial_k \varphi_j(t,x)$$

for $f \in C_c^1(\mathbb{R}^d)$. Combined with Lemma 2.1, this identity implies that $S(t)$ is also a C_0–group on $W^{1,p}(\mathbb{R}^d)$ which is exponentially bounded uniformly in p.

(2) Let B be the generator of $S(t)$ on $L^p(\mathbb{R}^d)$. We set $\tilde{B}f := g$ for $f \in D(\tilde{B}) := \{f \in L^p(\mathbb{R}^d) : \exists\, g \in L^p(\mathbb{R}^d) \text{ s.t. } \langle g, \phi\rangle = -\langle f, \operatorname{div}(\phi b)\rangle\ \forall \phi \in C_c^1(\mathbb{R}^d)\}$. In order to prove $B = \tilde{B}$, we first observe that $C_c^1(\mathbb{R}^d)$ is contained in $D(B)$ and $Bf = b\cdot\nabla f$ for $f \in C_c^1(\mathbb{R}^d)$. Because $C_c^1(\mathbb{R}^d)$ is invariant under $S(t)$, the space $C_c^1(\mathbb{R}^d)$ is even a core for B. We can thus approximate a given $f \in D(B)$ with respect to the graph norm of B by $f_n \in C_c^1(\mathbb{R}^d)$. Using this approximation, one easily derives that $B \subset \tilde{B}$.

(3) So it remains to check that $\tilde{B} - w$ is injective for some $w > w_p$. If f belongs to the kernel of $\tilde{B} - w$, then

$$0 = \langle(\tilde{B} - w)f, \phi\rangle = -\langle f, w\phi + (\operatorname{div} b)\phi + b \cdot \nabla\phi\rangle$$

for $\phi \in C_c^1(\mathbb{R}^d)$. Since $-b$ satisfies the same assumptions as b, parts (1) and (2) show that $B'g = -b \cdot \nabla g$ defined on $C_c^1(\mathbb{R}^d)$ has a closure in $L^{p'}(\mathbb{R}^d)$ which is a generator. The operator $-b \cdot \nabla - \operatorname{div} b$ is a bounded perturbation of B' due to (2.7), so that $C_c^1(\mathbb{R}^d)$ is also a core for the closure of $-b \cdot \nabla - \operatorname{div} b$ on $L^{p'}(\mathbb{R}^d)$. Taking a sufficiently large w, we thus deduce $f = 0$. $\square$

We further define

$$A_0 f = \operatorname{div}(a\nabla f), \qquad A_1 = I - A_0, \qquad B_w = wI - B, \tag{2.11}$$

where $D(A_0) = W^{2,p}(\mathbb{R}^d)$ and w is a fixed number larger than the growth bounds of $S(\cdot)$ on $L^p(\mathbb{R}^d)$ and $W^{1,p}(\mathbb{R}^d)$ for all $1 < p < \infty$. In the following lemma we do not suppose any properties of the coefficients a and b besides being sufficiently smooth.

Lemma 2.3. *Let f be a test function f, $a \in C^2(\mathbb{R}^d, \mathbb{R}^{d^2})$, and $b \in C^2(\mathbb{R}^d, \mathbb{R}^d)$. Then*

$$[A_1, B_w]f = (A_0 B - BA_0)f = \operatorname{div}\{[aDb^T + Db\, a - (\operatorname{div} b)a - (b\cdot\nabla)a]\,\nabla f\} + (\operatorname{div} b)A_0 f.$$

Proof. The assertion is a consequence of the following calculations.

$$\begin{aligned}
[A_1, B_w]f &= \sum_{i,j,k} \Big(\partial_i[a_{ij}\partial_j(b_k\partial_k f)] - b_k\partial_{ik}(a_{ij}\partial_j f)\Big) \\
&= \sum_{i,j,k} \partial_i[(a_{ij}(\partial_j b_k) + (\partial_j b_i)a_{jk})\,\partial_k f)] - \sum_{i,j,k} \partial_i((\partial_j b_i)a_{jk}\partial_k f) \\
&\qquad + \sum_{i,j,k} \partial_i(a_{ij}b_k\partial_{jk} f) - \sum_{i,j,k} b_k\partial_{ik}(a_{ij}\partial_j f) \\
&= \operatorname{div}[(aDb^T + Db\, a)\nabla f] - \sum_{i,j,k}(\partial_{ij}b_i)a_{jk}\partial_k f - \sum_{i,j,k}(\partial_j b_i)\partial_i(a_{jk}\partial_k f) \\
&\qquad + \sum_{i,j,k} \partial_i(a_{ij}b_k\partial_{jk} f) - \sum_{i,j,k} b_i\partial_{ij}(a_{jk}\partial_k f)
\end{aligned}$$

$$
\begin{aligned}
&= \operatorname{div}((aDb^T + Db\, a)\nabla f) - \sum_{j,k} (\partial_j (\operatorname{div} b))\, a_{jk} \partial_k f - \sum_{i,j,k} \partial_j (b_i \partial_i (a_{jk} \partial_k f)) \\
&\quad + \sum_{i,j,k} \partial_j (a_{jk} b_i \partial_{ik} f) \\
&= \operatorname{div}((aDb^T + Db\, a)\nabla f) - \sum_{j,k} \partial_j ((\operatorname{div} b) a_{jk} \partial_k f) - \sum_{i,j,k} \partial_j (b_i (\partial_i a_{jk}) \partial_k f) \\
&\quad + (\operatorname{div} b) A_0 f. \qquad \square
\end{aligned}
$$

We now come to our first main result where we show that A with minimal domain generates a C_0–semigroup $T(\cdot)$ on $L^p(\mathbb{R}^d)$. We recall that in general $T(\cdot)$ is *not* analytic. In fact, the map $t \mapsto T(t)$ is even nowhere continuous in operator norm in the special case of the Ornstein–Uhlenbeck operator A_{OU} given by (1.2), see [11], [18].

Theorem 2.4. *Assume that (H) holds and let $1 < p < \infty$. Then the operator $Af = \operatorname{div}(a\nabla f) + b \cdot \nabla f$ with domain $D(A) = \{f \in W^{2,p}(\mathbb{R}^d) : b \cdot \nabla f \in L^p(\mathbb{R}^d)\}$ generates a positive C_0–semigroup $T(\cdot)$ on $L^p(\mathbb{R}^d)$ such that $\|T(t)\|_{L^p} \le e^{w_p t}$, where w_p is given by Lemma 2.1. Moreover, the semigroups obtained on $L^p(\mathbb{R}^d)$ and $L^q(\mathbb{R}^d)$ coincide on $L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ for $1 < p, q < \infty$.*

Proof. We use the notation introduced in (2.11). We want to employ Corollary 2 of [17] in order to show that $A - \kappa$ with the asserted domain $D(A)$ is invertible in $L^p(\mathbb{R}^d)$, for some $\kappa > 0$. Observe that $-A_1$ and $-B_w$ generate positive, contractive C_0–semigroups on $L^p(\mathbb{R}^d)$ and that the semigroup generated by $-A_1$ is analytic for every $1 < p < \infty$. Moreover, A_1 is self adjoint on $L^2(\mathbb{R}^d)$. Using the transference principle (see [6, §4] and [5, Thm.5.8]) and Riesz–Thorin interpolation, we see that there are angles $\varphi_A \in (0, \pi/2)$ and $\varphi_B > \pi/2$ such that $\varphi_A + \varphi_B < \pi$ and

$$
\|(\lambda + A_1)^{-1}\| \le \frac{c}{1 + |\lambda|}, \quad \|(\mu + B_w)^{-1}\| \le \frac{c}{|\mu|}, \qquad \|A_1^{is}\| \le c e^{\varphi_A |s|}, \quad \|B_w^{is}\| \le c e^{\varphi_B |s|}
$$

for $s \in \mathbb{R}$, $|\arg \lambda| < \pi - \varphi_A$, $|\arg \mu| < \pi - \varphi_B$, and a constant $c > 0$. (We refer to [1] for an introduction to imaginary powers of sectorial operators.) In order to apply [17, Cor.2], it thus remains to establish the commutator estimate

$$
\|A_1(\lambda + A_1)^{-1} \left(A_1^{-1}(\mu + B_w)^{-1} - (\mu + B_w)^{-1} A_1^{-1}\right)\| \le \frac{\tilde{c}}{|\lambda|^{1/2}\, |\mu|^2} \tag{2.12}
$$

for $|\arg \lambda| < \pi - \varphi_A$, $|\arg \mu| < \pi - \varphi_B$, and a constant $\tilde{c}$. We denote the operator in (2.12) by $C(\lambda, \mu)$. Formally, (2.12) follows in a straightforward way from Lemma 2.3 and our assumptions. However, due to lack of regularity it takes some effort to relate $C(\lambda, \mu)$ with the commutator of A_1 and B_w.

To that purpose, we first approximate the coefficients a_{ij} and b_i locally in $C^1(\mathbb{R}^d)$ by $a_{ij}^{(k)}, b_i^{(k)} \in C^2(\mathbb{R}^d)$, $k \in \mathbb{N}$. The corresponding differential operators are designated by $A_0^{(k)}$, $B^{(k)}$, $A_1^{(k)}$, and $B_w^{(k)}$. Standard cut–off and mollifying procedures allow us to define bounded operators $T_n : W^{2,p}(\mathbb{R}^d) \to W^{2,p}(\mathbb{R}^d)$ which map $W^{2,p}(\mathbb{R}^d)$ into test functions and converge strongly to the identity in $W^{2,p}(\mathbb{R}^d)$ as $n \to \infty$. We now introduce the functions

$$
u_{kn} = A_1^{-1}[A_1^{(k)} B_w^{(k)} - B_w^{(k)} A_1^{(k)}] T_n A_1^{-1} f
$$

for $f \in D(B)$. Lemma 2.3 shows that

$$u_{kn} = A_1^{-1/2} A_1^{-1/2} \operatorname{div}[\{a^{(k)}(Db^{(k)})^T + Db^{(k)} a^{(k)} - (\operatorname{div} b^{(k)})a^{(k)} - \textstyle\sum_i b_i^{(k)} \partial_i a^{(k)}\} \nabla T_n A_1^{-1} f]$$
$$+ A_1^{-1}(\operatorname{div} b^{(k)}) A_0^{(k)} T_n A_1^{-1} f.$$

Since $\partial_k (A_1^{-1/2})^* = \partial_k (A_1^*)^{-1/2}$ is bounded on $L^{p'}(\mathbb{R}^d)$, we can extend $A_1^{-1/2} \operatorname{div}$ (defined on $W^{1,p}(\mathbb{R}^d)^d$, say) to a bounded operator $U : L^p(\mathbb{R}^d)^d \to L^p(\mathbb{R}^d)$. Thus the limit of u_{kn} as $k \to \infty$ exists in $L^p(\mathbb{R}^d)$ and is equal to

$$u_n := A_1^{-1/2} U[\{aDb^T + Db\, a - (\operatorname{div} b)a - \textstyle\sum_i b_i \partial_i a\} \nabla T_n A_1^{-1} f] + A_1^{-1}(\operatorname{div} b) A_0 T_n A_1^{-1} f.$$

Because of (2.5), (2.7), and (H), the functions in the brackets $\{\cdots\}$ and $\operatorname{div} b$ are bounded. Therefore we obtain

$$u := \lim_{n\to\infty} u_n = A_1^{-1/2} U[\{aDb^T + Db\, a - (\operatorname{div} b)a + \textstyle\sum_i b_i \partial_i a\} \nabla A_1^{-1} f] + A_1^{-1}(\operatorname{div} b) A_0 A_1^{-1} f$$

in $L^p(\mathbb{R}^d)$, and the crucial estimate

$$\|A_1^{1/2} u\|_p \le c' \, \|f\|_p \tag{2.13}$$

holds for a constant c' (only depending on p and the constants in (H)). We further define

$$C_{kn}(\lambda, \mu) = A_1(\lambda + A_1)^{-1}(\mu + B_w)^{-1} A_1^{-1}[A_1^{(k)} B_w^{(k)} - B_w^{(k)} A_1^{(k)}] T_n A_1^{-1} (\mu + B_w)^{-1}.$$

For a given function $g \in L^p(\mathbb{R}^d)$ we now set $f = (\mu + B_w)^{-1} g \in D(B)$. By the above results, $C_{kn}(\lambda, \mu) g$ converges in $L^p(\mathbb{R}^d)$ to

$$\tilde{C}(\lambda, \mu) g := A_1^{1/2}(\lambda + A_1)^{-1} \, A_1^{1/2}(\mu + B_w)^{-1} A_1^{-1/2} A_1^{1/2} u$$

as first $k \to \infty$ and then $n \to \infty$. Using $D(A_1^{1/2}) = W^{1,p}(\mathbb{R}^d)$, Lemma 2.2, and estimate (2.13), we arrive at

$$\|\tilde{C}(\lambda, \mu) g\| \le \frac{c''}{|\lambda|^{1/2} \, |\mu|} \, \|f\|_p \le \frac{c\, c''}{|\lambda|^{1/2} \, |\mu|^2} \, \|g\|_p \tag{2.14}$$

for $|\arg \lambda| < \pi - \varphi_A$, $|\arg \mu| < \pi - \varphi_B$, and a constant c''.

On the other hand, for a test function ϕ we compute

$$\langle u_{kn}, A_1^* \phi \rangle = \langle A_1 u_{kn}, \phi \rangle = \langle [B^{(k)} A_1^{(k)} - A_1^{(k)} B^{(k)}] T_n A_1^{-1} f, \phi \rangle$$
$$= -\langle A_1^{(k)} T_n A_1^{-1} f, \operatorname{div}(\phi b^{(k)}) \rangle - \langle b^{(k)} \cdot \nabla T_n A_1^{-1} f, \operatorname{div}((a^{(k)})^T \nabla \phi) \rangle,$$

where we use the L^p–$L^{p'}$ duality. Letting $k \to \infty$, we deduce

$$\langle u_n, A_1^* \phi \rangle = -\langle A_1 T_n A_1^{-1} f, \operatorname{div}(\phi b) \rangle - \langle b \cdot \nabla T_n A_1^{-1} f, A_1^* \phi \rangle,$$

Now we can take the limit as $n \to \infty$, and obtain

$$\langle u, A_1^* \phi \rangle = -\langle f, \operatorname{div}(\phi b) \rangle - \langle b \cdot \nabla A_1^{-1} f, A_1^* \phi \rangle = \langle Bf, \phi \rangle - \langle b \cdot \nabla A_1^{-1} f, A_1^* \phi \rangle,$$

because of $f \in D(B)$. For $\psi = A_1^* \phi$ this identity yields

$$\langle u, \psi \rangle = \langle A_1^{-1} Bf, \psi \rangle - \langle b \cdot \nabla A_1^{-1} f, \psi \rangle.$$

Since test functions are a core for A_1^*, the set of the functions ψ is dense in $L^{p'}(\mathbb{R}^d)$. Consequently, $b \cdot \nabla A_1^{-1} f \in L^p_{loc}(\mathbb{R}^d)$ belongs to $L^p(\mathbb{R}^d)$. Lemma 2.2 now shows that $A_1^{-1} f \in D(B)$, and thus

$$u = A_1^{-1} Bf - BA_1^{-1} f$$

for $f \in D(B)$. This equality implies

$$\tilde{C}(\lambda,\mu)g = A_1(\lambda + A_1)^{-1}(\mu + B_w)^{-1}[B_w A_1^{-1} - A_1^{-1}B_w](\mu + B_w)^{-1}g = C(\lambda,\mu)g,$$

i.e., (2.14) is in fact the required estimate (2.12).

Thus Corollary 2 of [17] shows that $\nu + A_1 + B_w = \nu + 1 + w - A$ with domain $D(A_0) \cap D(B)$ is invertible on $L^p(\mathbb{R}^d)$ for some $\nu > 0$. So $A = A_0 + B$ has the required domain due to Lemma 2.2. Since $-A_1$ and $-B_w$ generate contractive C_0–semigroups, the remaining assertions follow, e.g., from Trotter's product formula, [9, Cor.III.5.8]. ☐

3. Invariant measures

In this section we assume that (H) holds. We want to construct a semigroup on $C_b(\mathbb{R}^d)$ corresponding to A. Let $0 \leq f \in C_b(\mathbb{R}^d)$ and $r > 0$. We consider the parabolic problem

$$\begin{aligned} \partial_t u(t,x) &= Au(t,x), \qquad |x| < r,\ t > 0,\\ u(t,x) &= 0, \qquad |x| = r,\ t > 0,\\ u(t,x) &= f(x), \qquad |x| \leq r. \end{aligned}$$

Due to the maximum principle, the classical solutions u_r converge monotonically to a positive function u on $\mathbb{R}_+ \times \mathbb{R}^d$ as $r \to \infty$, and we have $\|u\|_\infty \leq \|f\|_\infty$. One can now check that there is a semigroup of positive contractions $T_\infty(t)$ on $C_b(\mathbb{R}^d)$ such that $u(t,x) = (T_\infty(t)f)(x)$ is a classical solution of

$$\partial_t u(t,x) = Au(t,x), \quad t > 0,\ x \in \mathbb{R}^d,$$

for $f \in C_b(\mathbb{R}^d)$. The semigroup is given by

$$T_\infty(t)f(x) = \int_{\mathbb{R}^d} p(t,x,y)f(y)\,dy \tag{3.1}$$

for a positive function p being continuous in $(t,x) \in (0,\infty)\times\mathbb{R}^d$ for a.e. $y \in \mathbb{R}^d$. Moreover, the mapping $\mathbb{R}_+ \ni t \mapsto T_\infty(t)f(x)$ is continuous uniformly for x in compact subsets of $\mathbb{R}^d$, where $f \in C_b(\mathbb{R}^d)$. Finally, we can extend $T_\infty(t)$ to a contraction on $L^\infty(\mathbb{R}^d)$, which has the strong Feller property (i.e., it maps $L^\infty(\mathbb{R}^d)$ into $C_b(\mathbb{R}^d)$), and $T_\infty(t)$ is irreducible (i.e., $T_\infty(t)f(x) > 0$ for $0 \leq f \in C_b(\mathbb{R}^d)$ with $f \neq 0$). The representation (3.1) also shows that $T_\infty(t)$ preserves bounded pointwise convergence. These facts have been shown in [14, §4] for operators in non–divergence form with coefficients being locally Hölder continuous. The arguments given there can easily be adapted to our setting replacing Schauder estimates by estimates in Sobolev spaces. A Borel probability measure μ on $\mathbb{R}^d$ is called *invariant* for $T_\infty(\cdot)$ if

$$\int_{\mathbb{R}^d} T_\infty(t)f\,d\mu = \int_{\mathbb{R}^d} f\,d\mu \tag{3.2}$$

for all bounded Borel functions f. An invariant measure for $T_\infty(\cdot)$ exists if e.g.

$$\limsup_{|x|\to\infty}\,(\operatorname{tr} a(x) + b(x)\cdot x) < 0,$$

see e.g. [10], [14, Cor.6.6]. We want to establish gobal regularity properties of the density of an invariant measure μ, assuming that μ exists. Let us first check that indeed $d\mu = \rho dx$ for some $0 < \rho \in L^1(\mathbb{R}^d)$.

Since $T_\infty(\cdot)$ is irreducible, the support of an invariant measure is equal to $\mathbb{R}^d$. We know that $0 \le T_\infty(t)\mathbb{1} \le \mathbb{1}$. If there were a point $x_0 \in \mathbb{R}^d$ where $T_\infty(t)\mathbb{1}(x_0) < 1$, then there would exist a number $\delta \in [0,1)$ and an non-empty, open set $O \subseteq \mathbb{R}^d$ such that $T_\infty(t)\mathbb{1} \le \delta$ on O. But this fact leads to the contradiction

$$1 = \int_{\mathbb{R}^d} \mathbb{1}\, d\mu = \int_{\mathbb{R}^d} T_\infty(t)\mathbb{1}\, d\mu \le \mu(\mathbb{R}^d \setminus O) + \delta\mu(O) < 1,$$

because of the invariance of μ. As a result, $T_\infty(t)\mathbb{1} = \mathbb{1}$ for $t \ge 0$. We now define

$$P(t,x,\Gamma) = \int_\Gamma p(t,x,y)\, dy = T_\infty(t)\mathbb{1}_\Gamma(x)$$

for a Borel set $\Gamma \subset \mathbb{R}^d$, $t \ge 0$, $x \in \mathbb{R}^d$. It is easy to check that $P(t,x,\cdot)$ is a Markovian transition function with corresponding transition semigroup $T_\infty(\cdot)$ on $C_b(\mathbb{R}^d)$ in the sense of [8]. Moreover, Proposition 2.1.1 in [8] implies that $T_\infty(\cdot)$ is stochastically continuous. Then Hasminskii's and Doob's Theorem, see Proposition 4.1.1 and Theorem 4.2.1 in [8], show that μ has a strictly positive density ρ and that μ is the unique invariant measure.

We next want to prove that $T_\infty(t)f$ coincides with $T(t)f$ for $f \in L^p(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$. We define $A_\infty u = Au$ for $u \in D(A_\infty) = \{u \in \bigcap_{p>1} W^{2,p}_{loc}(\mathbb{R}^d) : u, Au \in C_b(\mathbb{R}^d)\}$. Then the Laplace transform of $T_\infty(\cdot)$ (defined pointwise) is the resolvent $R(\lambda, A_\infty)$ for $\operatorname{Re}\lambda > 0$ by Propositions 5.1 and 5.7 of [14]. (Here one needs that $T_\infty(t)\mathbb{1} = \mathbb{1}$; the arguments in [14] work again in our setting.) Let $f \in C_c(\mathbb{R}^d)$. Temporarily, we denote the generator of $T(\cdot) = T_p(\cdot)$ on $L^p(\mathbb{R}^d)$ by A_p, $1 < p < \infty$. Due to Theorem 2.4 the function $u = R(w, A_p)f$ does not depend on p and belongs $W^{2,p}(\mathbb{R}^d)$ for all $1 < p < \infty$, so that $u \in C_b(\mathbb{R}^d)$. Since further $Au = A_p u = wu - f \in C_b(\mathbb{R}^d)$, we conclude that $u \in D(A_\infty)$ and $A_\infty u = Au = A_p u$. Hence, the resolvents for A_∞ and A_p coincide on $C_c(\mathbb{R}^d)$, so that $T_p(t)f = T_\infty(t)f$, $t \ge 0$, by the uniqueness of the Laplace transform. This equality thus holds for all $f \in L^p(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$. So we drop the subscripts p and ∞.

We next show that $\rho \in W^{2,p}(\mathbb{R}^d)$ for all $p < \infty$. Hence, ρ is continuously differentiable and $\rho(x)$, $\nabla\rho(x)$ tend to 0 as $|x| \to \infty$ by Sobolev's embedding theorem. In Theorem 1.1 of [2] it was proved that $\rho \in W^{1,2}(\mathbb{R}^d)$ supposing global Lipschitz continuity of a_{kl} and $b \in L^2(\mu)$ (where A is written in non–divergence form).

Theorem 3.1. *Assume that (H) holds and that $T(t)$ possesses an invariant measure μ. Then μ is unique and has a density ρ which belongs to all $W^{2,q}(\mathbb{R}^d)$, $1 < q < \infty$. In particular, $\rho \in C_0^1(\mathbb{R}^d)$.*

Proof. By the above remarks, we only have to show that the density $0 \le \rho \in L^1(\mathbb{R}^d)$ in fact belongs to $W^{2,q}(\mathbb{R}^d)$ for each $1 < q < \infty$. Take $f \in L^p(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$, $p > d/2$, and $w > 0$ as in the previous section. Using (3.2), we calculate

$$\begin{aligned}\int_{\mathbb{R}^d} R(w,A)f(x)\,\rho(x)\,dx &= \int_{\mathbb{R}^d}\int_0^\infty e^{-wt}T(t)f(x)\,\rho(x)\,dt\,dx \\ &= \int_0^\infty e^{-wt}\int_{\mathbb{R}^d} T(t)f(x)\,\rho(x)\,dx\,dt \\ &= \frac{1}{w}\int_{\mathbb{R}^d} f(x)\rho(x)\,dx.\end{aligned}$$

Thus Theorem 2.4 and Sobolev's embedding theorem yield

$$\Big|\int f\rho\,dx\Big| \le \|wR(w,A)f\|_\infty\,\|\rho\|_1 \le c\,\|f\|_p\,.$$

This means that $\rho\in L^{p'}(\mathbb{R}^d)$, where $p'=p/(p-1)$. So we obtain

$$\int T(t)f\,\rho\,dx = \int f\rho\,dx$$

for all $f\in L^p(\mathbb{R}^d)$. Since the weak generator of a C_0–semigroup is equal to its generator, see e.g. [19, Thm.2.1.3], the above equality implies $\rho\in D(A^*)$. Let $f\in D(A)$ and $g\in\mathcal{D}:=\{v\in W^{2,p'}(\mathbb{R}^d): b\cdot\nabla v\in L^{p'}(\mathbb{R}^d)\}$. By Lemma 2.2 there are $g_n\in C_c^1(\mathbb{R}^d)$ such that $g_n\to g$ and $b\cdot\nabla g_n\to b\cdot\nabla g$ in $L^{p'}(\mathbb{R}^d)$. So we obtain

$$\begin{aligned}\langle Af,g\rangle &= \langle A_0f,g\rangle + \lim_{n\to\infty}\langle b\cdot\nabla f,g_n\rangle\\ &= \langle f,A_0^*g\rangle - \lim_{n\to\infty}\langle f, b\cdot\nabla g_n + \operatorname{div}(b)g_n\rangle\\ &= \langle f, A_0^*g - b\cdot\nabla g - \operatorname{div}(b)g\rangle.\end{aligned}$$

Consequently, the adjoint A^* extends the operator $A':=A_0^*-b\cdot\nabla-\operatorname{div}(b)$ defined on $\mathcal{D}$. But A' is a bounded perturbation of a generator thanks to (2.7) and Theorem 2.4. This shows that $A'=A^*$ and $D(A^*)=\mathcal{D}\subset W^{2,p'}(\mathbb{R}^d)$. Hence $\rho\in L^q(\mathbb{R}^d)$ with $q=dp'/(d-2p')$ if $d>2p'$, and thus $\rho\in L^r(\mathbb{R}^d)$ for $1\le r\le q$. If $d\le 2p'$, we have $\rho\in L^r(\mathbb{R}^d)$ for all $1\le r\le q<\infty$. In both cases we obtain as above $\rho\in W^{2,r}(\mathbb{R}^d)$ for $1<r\le q$. In the first case we can iterate the above procedure, replacing p' by q. In finitely many steps we arrive at the assertion. □

Remark 3.2. The above argument can be used whenever one knows suitable properties of $D(A)$ for the generator A of the transition semigroup on $L^p(\mathbb{R}^d)$. For instance, we obtain $\rho\in L^{p'}(\mathbb{R}^d)$ if $D(A)\hookrightarrow L^\infty(\mathbb{R}^d)$.

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