Maximal Regularity for Evolution Equations in L_p-Spaces

Lectures given by
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1 Maximal L_p-Regularity

Let X_0 be a Banach space with norm $|\cdot|_0$, and let A be a linear, closed, densely defined operator in X_0. As usual, domain kernel and range of A are denoted by $\mathcal{D}(A)$, $\mathcal{N}(A)$, $\mathcal{R}(A)$, respectively, and $\sigma(A)$ and $\rho(A)$ mean spectrum and resolvent set of A. We set $X_1 = \mathcal{D}(A)$ equipped with the graph norm of A, and denote the norm in X_1 by $|\cdot|_1$. In the sequel, we always assume that A is unbounded, to avoid trivialities.

Recall that A is called **pseudo-sectorial** if $(-\infty, 0) \subset \rho(A)$, and

$$\sup_{t>0} |t(t+A)^{-1}|_{\mathcal{B}(X_0)} < \infty.$$

The spectral angle ϕ_A is defined by

$$\phi_A := \inf\{\phi > 0 : \ \sup\{|z(z+A)^{-1}|_{\mathcal{B}(X_0)} : \ |\arg(z)| < \pi - \phi\} < \infty\}.$$

If in addition $\mathcal{D}(A)$ and $\mathcal{R}(A)$ are dense in X_0, then A is called **sectorial.**

Let $J = [0, \infty)$ or $[0, T]$ for some $T > 0$ and let $f : J \to X_0$ be a function. We consider the inhomogeneous initial value problem

$$\dot{u}(t) + Au(t) = f(t) \quad t \in J, \ u(0) = u_0, \tag{1.1}$$

in $L_p(J; X_0)$ for $p \in (1, \infty)$.

The definition of maximal L_p-regularity for (1.1) is as follows.

Definition 1.1. *A is said to belong to the class $\mathcal{MR}_p(J; X_0)$ - and we say that there is* **maximal L^p-regularity** *for (1.1) - if for each $f \in L_p(J; X_0)$ there exists a unique $u \in H^1_p(J; X_0) \cap L_p(J; X_1)$ satisfying (1.1) in the $L_p(J; X)$ sense, with $u_0 = 0$.*

The closed graph theorem implies then that there exists a constant $C > 0$ such that

$$||u||_{L_p(J;X_0)} + ||\dot{u}||_{L_p(J;X_0)} + ||Au||_{L_p(J;X_0)} \leq C||f||_{L_p(J;X_0)}. \tag{1.2}$$

We denote the solution operator $f \mapsto u$ by R. It is well known that there is maximal L_p regularity for (1.1) only if A generates an analytic semigroup. If $J = [0, \infty)$, then the semigroup is even of negative exponential type. We state this as

Proposition 1.2. *Let $A \in \mathcal{MR}_p(J; X_0)$ for some $p \in (1, \infty)$.*

Then the following assertions are valid.

(i) *If $J = [0, T]$ then there is $\omega \geq 0$ and $M \geq 1$ such that $\{z \in \mathbb{C} : \mathrm{Re} z \leq -\omega\} \subset \rho(A)$ and the estimate*

$$|z(z+A)^{-1}|_{\mathcal{B}(X_0)} \leq M, \quad \mathrm{Re} z \geq \omega,$$

is valid. In particular, $\omega + A$ is sectorial with spectral angle $< \pi/2$.

(ii) *If $J = \mathbb{R}_+$ then $\mathbb{C}_- := \{z \in \mathbb{C} : \mathrm{Re} z < 0\} \subset \rho(A)$ and there is a constant $M \geq 1$ such that*

$$|(z+A)^{-1}|_{\mathcal{B}(X_0)} \leq \frac{M}{1+|z|}, \quad \mathrm{Re} z > 0,$$

is valid. In particular, A is sectorial with spectral angle $< \pi/2$ and $0 \in \rho(A)$.

Proof. Consider first the case $J = [0,T]$. Choose $\mu \in \mathbb{C}_+$, $x \in X_0$, and define $f_\mu \in L_p([0,\infty); X_0)$ by $f_\mu(t) = e^{-\mu t}x$. Let $u_\mu(t;x) = R(f_\mu)(t)$ where R denotes the solution operator for (1.1). Set

$$U_\mu x := 2\mathrm{Re}\mu \int_0^T e^{-\bar{\mu}t} u_\mu(t;x)dt = \frac{2\mathrm{Re}\mu}{\mu}[\int_0^T e^{-\bar{\mu}t}\dot{u}_\mu(t;x)dt - e^{-\mu T}u_\mu(T;x)].$$

The maximal regularity property for (1.1) implies that there exists a constant $C > 0$ such that

$$|U_\mu|_{\mathcal{B}(X_0)} \leq \frac{C}{1+|\mu|}, \quad \mathrm{Re}\mu > \omega,$$

where ω is sufficiently large. In fact, we have with Hölder's inequality

$$|U_\mu x|_0 \leq 2(p')^{-1/p'}(\mathrm{Re}\mu)^{1-1/p'}||u_\mu||_{L_p(J;X_0)} \leq (\mathrm{Re}\mu)^{1/p}C||f_\mu||_{L_p(J;X_0)} \leq C|x|_0,$$

as well as

$$\begin{aligned} |U_\mu x|_0 &\leq 2\mathrm{Re}\mu|\mu|^{-1}[(p'\mathrm{Re}\mu)^{-1/p'} + e^{-\mathrm{Re}\mu T}T^{1/p'}]||\dot{u}_\mu||_{L_p(J;X_0)} \\ &\leq |\mu|^{-1}(\mathrm{Re}\mu)^{1/p}C||f_\mu||_{L_p(J;X_0)} \leq |\mu|^{-1}C|x|_0, \end{aligned}$$

by the maximal regularity estimate (1.2).

Next we multiply (1.1) with $f = f_\mu$ by $e^{-\bar{\mu}t}$ and integrate over J. This yields by closedness of A and an integration by parts

$$\begin{aligned} (1 - e^{-2\mathrm{Re}\mu T})x &= 2\mathrm{Re}\mu \int_0^T e^{-\bar{\mu}t} f_\mu(t)dt = 2\mathrm{Re}\mu \int_0^T e^{-\bar{\mu}t}[\dot{u}_\mu(t;x) + Au_\mu(t;x)]dt \\ &= (\bar{\mu} + A)U_\mu x + 2(\mathrm{Re}\mu)e^{-\bar{\mu}T}u_\mu(T;x), \end{aligned}$$

which after rearrangement becomes

$$(\bar{\mu} + A)U_\mu x = x - V_\mu x, \quad V_\mu x := e^{-2\mathrm{Re}\mu T}x + 2(\mathrm{Re}\mu)e^{-\bar{\mu}T}u_\mu(T;x).$$

Estimating as before we obtain

$$|V_\mu x|_0 \leq [e^{-2\mathrm{Re}\mu T} + Ce^{-\mathrm{Re}\mu T}(T\mathrm{Re}\mu)^{1/p'}]|x|_0,$$

from which we see that there is $\omega > 0$ such that $|V_\mu|_{\mathcal{B}(X_0)} \leq 1/2$, for each $\mathrm{Re}\mu \geq \omega$. This then shows that $\bar{\mu} + A$ is surjective for all such μ and - in case it is also injective - we get

$$(\bar{\mu} + A)^{-1} = U_\mu(1 - V_\mu)^{-1}, \quad \mathrm{Re}\mu \geq \omega.$$

The estimate on U_μ then shows that $\omega + A$ is sectorial with spectral angle $< \pi/2$.

To show injectivity, observe that by uniqueness of solutions we have $u_\mu(t;Ax) = Au_\mu(t;x)$. Now suppose that $x \in \mathcal{N}(\mu + A)$; then

$$Au_\mu(t;x) = u_\mu(t;Ax) = u_\mu(t;-\mu x) = -\mu u_\mu(t,x),$$

hence the equation for $u_\mu(t;x)$ becomes

$$\dot{u}_\mu(t;x) - \mu u_\mu(t;x) = e^{-\mu t}x, \quad t > 0, \; u_\mu(0,x) = 0.$$

Thus

$$u_\mu(t;x) = [\int_0^t e^{\mu(t-s)}e^{-\mu s}ds]x = (2\mu)^{-1}(e^{\mu t} - e^{-\mu t})x = \mu^{-1}\sinh(\mu t)x,$$

and assuming $|x|_0 = 1$, the maximal regularity estimate (1.2) implies the inequality

$$\int_0^T |\sinh(\mu t)|^p dt = ||\mu u_\mu||^p_{L_p(J;X_0)} \leq C^p ||f_\mu||^p_{L_p(J;X_0)} \leq C^p/p\mathrm{Re}\mu,$$

which yields $\mathrm{Re}\mu \leq \omega$, for some sufficiently large $\omega > 0$.

For the case $J = \mathbb{R}_+$ the proof is simpler; one deduces in the same way the relation $(\bar{\mu} + A)^{-1} = U_\mu$ with $\omega = 0$. □

There is variant of maximal L_p-regularity if one requires for the solution of (1.1) only $u \in C(\mathbb{R}_+; X_0)$ and $\dot{u}, Au \in L_p(\mathbb{R}_+; X_0)$. We call the class of operators with this weaker property ${}_0\mathcal{MR}_p(\mathbb{R}_+; X_0)$. The proof of Proposition 1.2 shows that then in (ii) the condition $0 \in \rho(A)$ is dropped. More precisely we have

Corollary 1.3. *Suppose $A \in {}_0\mathcal{MR}_p(\mathbb{R}_+; X_0)$.*

Then A is pseudo-sectorial in X_0 with spectral angle $< \pi/2$.
Moreover, $A \in \mathcal{MR}_p(\mathbb{R}_+; X_0)$ if and only if $A \in {}_0\mathcal{MR}_p(\mathbb{R}_+; X_0)$ and $0 \in \rho(A)$.

Proposition 1.2 shows that for a finite interval $J = [0, T]$ its length $T > 0$ plays no role for maximal L_p-regularity, and up to a shift of A, without loss of generality, we may consider $J = [0, \infty)$ and may assume that $-A$ is the generator of an analytic semigroup of negative exponential type.

Assuming the latter, it is well-known that there exists a solution $u \in H^1_p(\mathbb{R}_+; X_0) \cap L_p(\mathbb{R}_+; \mathcal{D}(A))$ satisfying (1.1) with $f = 0$ in the $L_p(\mathbb{R}_+; X_0)$ sense if and only if $u_0 \in X_p := (X_0, X_1)_{1-\frac{1}{p},p}$. In fact, this follows easily from the well-known basic characterization of X_p in terms of A, i.e.

$$X_p = \mathcal{D}_A(1 - 1/p, p) := \{x \in X_0 : \, Ae^{-At}x \in L_p(\mathbb{R}_+; X_0)\}.$$

In the sequel, we denote by $|\cdot|_p$ a norm on X_p.

Unfortunately, the converse of Proposition 1.2 is false, in general, as a very recent result due to Kalton and Lancien [14] shows. Actually, it is a formidable task to prove that a given operator A belongs to $\mathcal{M}_p(\mathbb{R}_+; X_0)$. We want to explain the difficulty in more detail. Obviously, the variation of parameters formula

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}f(s)ds, \quad t \geq 0,$$

implies that there is maximal L^p regularity for (1.1) if and only if the operator $\mathcal{R}_{max}$ defined by

$$\mathcal{R}_{max}f := A\int_0^t e^{-A(t-s)}f(s)ds$$

acts as a bounded operator on $L^p(J; X_0)$. It is nontrivial to show this since the kernel of this convolution operator on the half-line is Ae^{-At} which has a nonintegrable singularity near $t = 0$, behaving like $1/t$, as follows from the well-known, best possible estimate

$$|Ae^{-At}|_{\mathcal{B}(X_0)} \leq \frac{Me^{-\eta t}}{t}, \quad t > 0,$$

valid for exponentially stable analytic semigroups. Therefore $\mathcal{R}_{max}$ is a *singular* integral operator on $L_p(\mathbb{R}_+; X_0)$ with operator-valued kernel.

However, a positive result shows that maximal L_p-regularity is independent of p. We state this as

Theorem 1.4. *Suppose $A \in {}_0\mathcal{MR}_{p_0}(\mathbb{R}_+; X_0)$ for some $p_0 \in (1, \infty)$.*
Then $A \in {}_0\mathcal{MR}_p(\mathbb{R}_+; X_0)$ for all $p \in (1, \infty)$

This result is due to Sobolevskii [20] and was rediscovered several times, e.g. by Cannarsa and Vespri [3]. Its proof is based on the famous theorem of Benedek, Cálderon and Panzone [2]. For an uptodate proof see Hieber [12].

A very useful property of the classes ${}_0\mathcal{MR}_p(\mathbb{R}_+; X_0)$ is their invariance under perturbations. This follows from the variation of parameters formula and is given here without proof.

Proposition 1.5. *Suppose $A \in \mathcal{MR}_p(\mathbb{R}_+; X_0)$ and let B be linear operator in X_0 with $\mathcal{D}(B) \supset \mathcal{D}(A)$ and assume there are constants $\alpha, \beta \geq 0$ such that*

$$|Bx|_0 \leq \alpha |x|_0 + \beta |Ax|_0, \quad x \in \mathcal{D}(A).$$

Then there are $\beta_0 > 0$ and $\omega \geq 0$ such that $\beta < \beta_0$ implies $\omega + A + B \in \mathcal{MR}_p(\mathbb{R}_+; X_0)$.

Next we write the operator $\mathcal{R}_{max}$ as a convolution on $\mathbb{R}$ with operator-valued kernel $K(t)$ given by

$$K(t) := \begin{cases} AT(t) & t > 0 \\ 0 & t \leq 0 \, . \end{cases}$$

This representation calls for the use of Fourier transforms. In the special case where $X_0 = H$ is a Hilbert space, the Fourier transform $\mathcal{F}$ acts as an isometric isomorphism on $L_2(\mathbb{R}, H)$, due to Parseval's theorem. By taking Fourier transforms and employing the convolution theorem for $\mathcal{F}$, we obtain

$$\mathcal{F}(\mathcal{R}_{max} f)(s) = A(is - A)^{-1}(\mathcal{F}f)(s), \qquad f \in L_2(\mathbb{R}, H).$$

The analyticity of e^{-At} implies that $\sup_{s \in \mathbb{R}} ||A(is - A)^{-1}|| < \infty$ and it follows from Parseval's identity that

$$\mathcal{R}_{max} \in \mathcal{B}(L_2(\mathbb{R}, H)).$$

This result is folklore and it is encouraging. Since the operator-valued version of Mikhlin's Fourier multiplier results holds also for operator-valued multipliers, we may deduce the following positive result on maximal L_p-regularity which goes back to de Simon [5]. It also can be obtained by Theorem 1.4 with $p_0 = 2$.

Theorem 1.6. *Let $1 < p < \infty$, and suppose that $X_0 = H$ is a Hilbert space.*
Then $A \in {}_0\mathcal{MR}_p(\mathbb{R}_+; X_0)$ if and only if A is pseudo-sectorial with spectral angle $\phi_A < \pi/2$.

Theorem 1.6 solves the problem of maximal L_p-regularity in case $X_0 = H$ is a Hilbert space. Since the proof relies on the Hilbert-space-valued Mikhlin theorem it breaks down for more general Banach spaces X_0. In fact, Pisier has shown that whenever the Mikhlin theorem holds for $L_p(\mathbb{R}; X_0)$ with any $\mathcal{B}(X_0)$-valued multiplier $M \in C^1(\mathbb{R} \setminus \{0\}; \mathcal{B}(X_0))$ satisfying the Mikhlin condition

$$\sup\{|M(\rho)|_{\mathcal{B}(X_0)} + |\rho M'(\rho)|_{\mathcal{B}(X_0)} : \rho \in \mathbb{R} \setminus \{0\}\} < \infty,$$

then X_0 must be isomorphic to a Hilbert space (unpublished). Therefore, for a long time it was believed that this way the problem of characterizing maximal L_p-regularity could not be solved.

In 1973 Brezis asked the question whether for $X_0 = L_q(\Omega, \mu)$, $1 < q < \infty$, *any* negative generator of a bounded analytic semigroup has the property of maximal L_p-regularity. The paper of Kalton and Lancien [14] answers this in the negative for $q \neq 2$. In fact, they proved that if X_0 is such that, for some fixed $p \in (1, \infty)$, *every* sectorial operator A in X_0 with spectral angla $< \pi/2$ belongs to ${}_0\mathcal{MR}_p(\mathbb{R}_+; X_0)$, then X_0 must be isomorphic to a Hilbert space. Thus a special multiplier of the form $M(\rho) = A(i\rho + A)^{-1}$ is enough in the result of Pisier mentioned before. However, so far no specific example of an $-A$ in some $X_0 = L_q(G)$ is known which generates a bounded analytic C_0-semigroup but $A \notin \mathcal{M}_p(X_0)$. Of particular interest would be of such an example where A is a partial differential operator.

In 1975 Da Prato and Grisvard were able to prove maximal L_p-regularity via the sum method for real interpolation spaces between X_0 and X_1. By means of this method Dore and Venni [10] succeeded to prove $A \in \mathcal{M}_p(X_0)$ provided X_0 is a Banach space of class $\mathcal{HT}$ and A admits bounded imaginary powers A^{is} with power angle $\theta_A < \pi/2$.

Prüss and Sohr [19] removed some extra assumptions imposed in [10]. The Dore-Venni approach is intimately related to the functional calculus of A, it uses in particular the inverse Mellin transform.

Another interesting result was obtained in Lamberton [15]. He proved maximal L_p-regularity for $X_0 = L_q(G)$, $1 < p, q < \infty$, provided $-A$ generates a bounded analytic C_0-semigroup which acts on all spaces $L_r(G)$ as a contraction, $1 \leq r \leq \infty$. Observe that for pde's this approach is necessarily restricted to second order differential opertors A, in particular elliptic systems are not covered.

The special case $X_0 = L_q(G)$ was also considered in Hieber and Prüss [13]. In this paper maximal L_p-regularity is shown in case the semigroup e^{-At} generated by $-A$ admits heat kernel estimates. The proof relies on Calderon-Zygmund theory, Interpolation and on the theorem of Benedek, Calderón and Panzone. In Prüss [18] this approach is generalized to fractional evolution equations of parabolic type, using Poisson estimates for the kernel of the resolvent $(\lambda + A)^{-1}$ of A rather than heat kernel estimates.

Another recent result which is striking was given by Weis [21]. He obtained a characterization of the class $\mathcal{M}_p(X_0)$ in case X_0 is a Banach space of class $\mathcal{HT}$ in terms of $\mathcal{R}$-boundedness. We state this as

Theorem 7.1. *Let X_0 be a Banach space of class $\mathcal{HT}$, and let A be pseudo-sectorial with spectral angle $\phi_A < \pi/2$.*

Then $A \in \mathcal{M}_p(X)$ if and only if the set $\{i\rho(i\rho + A)^{-1} : \rho \in \mathbb{R}\}$ is $\mathcal{R}$-bounded.

The proof of this result depends heavily on a recent operator-valued Fourier-multiplier theorem valid in spaces of class $\mathcal{HT}$. We are not going to explain or prove this result here, we refer to our recent exposition of the theory in Denk, Hieber and Prüss [7]. The aims of these lectures are not in the theory but rather in applications of maximal L_p-regularity.

2 Normally Elliptic Boundary Value Operators

The most famous example of an operator in L_p-space which enjoys the property of maximal L_p-regularity is the Laplacian with Dirichlet or Neumann boundary conditions. This was first proved by Ladyzhenskaya, Solonnikov and Ural'tseva [16], even in a more general setting of single second order differential operators. Here we present without proof a fairly general result even for the vector-valued case which is very recent; see Denk, Hieber and Prüss [7].

Let $G \subset \mathbb{R}^n$ be a bounded domain of class C^{2m} with boundary ∂G, where $m \in \mathbb{N}$, and let E be a Banach space of class $\mathcal{HT}$. We consider the following operator

$$\begin{aligned} Au(x) &= \mathcal{A}(x, D)u, \ x \in G, \qquad \text{with domain} \\ \mathcal{D}(A) &= \{u \in H_p^{2m}(G; E) : \ \mathcal{B}_j(x, D)u(x) = 0 \text{ on } \partial G\}. \end{aligned}$$

Here we use multiindex notation with $D = -i\partial_x$, $j = 1, \ldots, m$, and $m_j \leq 2m - 1$. The differential operators are given by

$$\begin{aligned} \mathcal{A}(x, D) &= \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha, \quad t \in J, \ x \in G, \\ \mathcal{B}_j(x, D) &= \sum_{|\beta| \leq m_j} b_\beta^j(x) D^\beta, \quad t \in J, \ x \in \partial G. \end{aligned}$$

The coefficents $a_\alpha(x)$ and $b_\beta^j(x)$ for fixed x are bounded linear operators in E.

We are interested in maximal L_p-regularity for A in the space $X = L_p(G; E)$ for $1 < p < \infty$. In the following we denote by $\mathcal{D}_\#$ the principal part of the differential operator $\mathcal{D}$. Let $\Sigma_\phi = \{z \in \mathbb{C} : \ z \neq 0, |\arg z| < \phi\}$, and $\nu(x)$ be the outer normal of ∂G at $x \in \partial G$. We impose the following assumptions on the coefficients.

(E) *For all $x \in \bar{G}$, $\xi \in \mathbb{R}^n$, $|\xi| = 1$, $\sigma(\mathcal{A}_\#(x, \xi)) \subset \mathbb{C}_+ = \Sigma_{\pi/2}$,*
i.e. $\mathcal{A}(x, D)$ is **normally elliptic**.

(LS) *For all $x \in \partial G$, $\xi \in \mathbb{R}^n$, with $\xi \cdot \nu(x) = 0$, $\lambda \in \bar{\mathbb{C}}_+$, $\lambda \neq 0$, and $h \in E^m$, the ode-system*

$$\begin{aligned} &\lambda v(y) + \mathcal{A}_\#(x, \xi + i\nu(x)\partial_y)v(y) = 0, \quad y > 0 \\ &\mathcal{B}_{j\#}(x, \xi + i\nu(x)\partial_y)v(0) = h_j, \quad j = 1, \ldots, m \end{aligned}$$

*admits a unique solution $v \in C_0(\mathbb{R}_+; E)$. (***Lopatinskii-Shapiro condition***)*

(SD) *There are $r_k \geq p$ with $\frac{n}{2mr_k} < 1 - \frac{k}{2m}$ such that*

$$\begin{aligned} &a_\alpha \in L_{r_k}(G; \mathcal{B}(E)), \quad \textit{for all } |\alpha| = k < 2m, \\ &a_\alpha \in C(\bar{G}; \mathcal{B}(E)), \quad \textit{for all } |\alpha| = 2m. \end{aligned}$$

(SB) *There are* $r_{jk} \geq p$ *with* $\frac{n-1}{2mr_{jk}} < 1 - \frac{k}{2m} - \frac{1}{2mp}$, *such that for all* $|\beta| = k \leq m_j$

$$b^j_\beta \in W^{2m-k-1/p}_{r_{jk}}(\partial G; \mathcal{B}(E))).$$

We call a boundary value operator A as defined before *normally elliptic*, if it satisfies conditions **(E)**, **(LS)**, **(SD)**, **(SB)**. The following result is proved in Denk, Hieber and Prüss [7].

Theorem. *Suppose conditions* **(E)**, **(LS)**, **(SD)**, **(SB)** *are satisfied,* $1 < p < \infty$, *and let* A *be defined as above, i.e.* A *is normally elliptic.*

Then there is $\omega \in \mathbb{R}$ *such that the operator* $A + \omega$ *in* $X_0 = L_p(G; E)$ *belongs to* $\mathcal{MR}_p(\mathbb{R}_+; X_0)$.

It should be observed that assuming the regularity assumptions **(SD)** on the coefficients $a_\alpha(x)$, it can be shown that the normally ellipticity **(E)** is necessary for maximal L_p-regularity of A. This was observed first for finite-dimensional E by Amann (1989). If one also considers maximal regularity with inhomogeneous boundary data and assumes regularity condition **(SB)** for the boundary coefficients, then the Lopatinskii-Shapiro condition **(LS)** is necessary as well. These facts are proved in Denk, Hieber and Prüss [8]. Another recent result concerns the fact that A admits an $\mathcal{H}^\infty$-calculus, if in additon the top-order coefficients $a_\alpha(x)$, $\alpha = 2m$, are Hölder-continuous on $\bar{G}$; see Denk, Dore, Hieber, Prüss and Venni [6].

3 Quasilinear Evolution Equations

Let X_0 be a Banach space, $X_1 \hookrightarrow X_0$ densely, $J = [0, a]$, $J_0 = [0, a_0]$, and let $1 < p < \infty$. In this section we consider the quasilinear problem

$$\dot{u}(t) + A(t, u(t))u(t) = F(t, u(t)), \quad t \in J, u(0) = u_0. \tag{3.1}$$

Here $u_0 \in X_p := (X_0, X_1)_{1-1/p,p}$, $A : J_0 \times X_p \to \mathcal{B}(X_1, X_0)$ is continuous, and $F : J_0 \times X_p \to X_0$ satisfies assumptions of Caratheodory type, i.e. $F(\cdot, u)$ is measurable for each $u \in X_p$, $F(t, \cdot)$ is continuous for a.a. $t \in J_0$. Moreover, we assume the following conditions on Lipschitz continuity of A and F.

(A) For each $R > 0$ there is a constant $L(R) > 0$ such that

$$|A(t, u)v - A(t, \bar{u})v|_0 \leq L(R)|u - \bar{u}|_p |v|_1, \quad t \in J,\ u, \bar{u} \in X_p, |u|_p, |\bar{u}|_p \leq R,\ v \in X_1.$$

(F) $f(\cdot) := F(\cdot, 0) \in L_p(J; X)$; for each $R > 0$ there is a function $\phi_R \in L_p(J)$ such that

$$|F(t, u) - F(t, \bar{u})|_0 \leq \phi_R(t)|u - \bar{u}|_p, \quad a.a. t \in J,\ u, \bar{u} \in X_p, |u|_p, |\bar{u}|_p \leq R.$$

Let $A_0 := A(0, u_0)$ and suppose that the problem

$$\dot{w}(t) + A_0 w(t) = g(t), \quad t \in J_0,\ w(0) = 0, \tag{3.2}$$

has the property of L_p-maximal regularity. Then for each function $g \in Y_0 := L_p(J;X_0)$, $a \le a_0$ there is a unique solution w of (3.2) of class

$$w \in Y_1 := {}_0H^1_p(J;X_0) \cap L_p(J;X_1).$$

By the closed graph theorem, then there is a constant $C_0 > 0$ such that

$$||w||_1 := ||w||_{L_p(J;X_1)} + ||\dot{w}||_{L_p(J;X_0)} \le C_0 ||g||_0 := ||g||_{L_p(J;X_0)}.$$

By the embedding theorem we have $Y_1 \hookrightarrow C(J;X_p)$ and there is a constant $C_1 > 0$ such that

$$||w||_{p,\infty} := \sup_{t\in J} |w(t)|_p \le C_1 ||w||_1, \quad w \in Y.$$

Observe that neither C_0 nor C_1 depend on a, since functions in Y_1 have zero trace at $t = 0$.

We want to solve (3.1) locally by means of maximal regularity via the contraction mapping theorem. For this purpose we first introduce a reference solution u^* which is defined as the solution of the linear problem

$$\dot{w}(t) + A_0 w(t) = f(t), \quad t \in J, \ w(0) = u_0, \tag{3.3}$$

where $f(t) = F(t,0)$ belongs to $L_p(J;X_0)$ by condition **(F)**. By the maximal regularity property we have

$$u^* \in H^1_p(J_0;X_0) \cap L_p(J_0;X_1).$$

Then we introduce a ball by means of

$$\mathbb{B}_r := \{v \in Y_0 : \ v - u^* \in Y_1, ||v - u^*||_1 \le r\}.$$

Given $v \in \mathbb{B}_r$ solve the linear problem

$$\dot{u}(t) + A_0 u(t) = F(t,v) + (A(0,u_0) - A(t,v(t))v(t), \quad t \in J, \ u(0) = u_0.$$

Define the map T on $\mathbb{B}_r$ by $Tv := u$; we want to show that $T(\mathbb{B}_r) \subset \mathbb{B}_r$ and that T is a contraction in the norm of Y_1, provided $a > 0$ is sufficiently small, and $r \in (0,1]$ is chosen properly.

For this purpose we estimate as follows. Maximal regularity yields

$$||Tv - u^*||_1 = ||u - u^*||_1 \le C_0[||F(\cdot,u) - f(\cdot)||_0 + ||A(0,u_0)v - A(\cdot,v)v||_0].$$

By assumption **(A)** we have with $\gamma_a = \sup_{t\in[0,a]} |A(0,u_0) - A(t,u_0)|_{\mathcal{B}(X_p,X_0)}$

$$\begin{aligned}
&||A(0,u_0)v - A(\cdot,v)v||_0 \le \\
&\quad \le \sup_{t\in[0,a]} [|A(0,u_0) - A(t,u_0)|_{\mathcal{B}(X_p,X_0)} + |A(t,u_0) - A(t,v(t)|_{\mathcal{B}(X_p,X_0)}]||v||_1 \\
&\quad \le [\gamma_a + L(R)||v - u_0||_{p,\infty}][||v - u^*||_1 + ||u^*||_1] \\
&\quad \le [\gamma_a + L(R)(C_1||v - u^*||_1 + ||u^* - u_0||_{p,\infty})][||v - u^*||_1 + ||u^*||_1] \\
&\quad \le [\gamma_a + L(R)(C_1 r + ||u^* - u_0||_{p,\infty})][r + ||u^*||_1],
\end{aligned}$$

and by assumption **(F)** in a similar way

$$\begin{aligned}||F(\cdot,u)-f(\cdot)||_0 &\leq ||F(\cdot,v)-F(\cdot,u^*)||_0+||F(\cdot,u^*)-F(\cdot,0)||_0\\ &\leq |\phi|_p[||v-u^*||_{p,\infty}+||u^*||_{p,\infty}]\\ &\leq |\phi|_p[C_1||v-u^*||_1+||u^*||_{p,\infty}]\\ &\leq |\phi|_p[C_1 r+||u^*||_{p,\infty}]\end{aligned}$$

Here $R=C_1+||u^*||_{p,\infty}$ is a fixed number; recall $r\leq 1$. This shows that we have

$$||Tv-u^*||_1\leq C_0[\gamma_a+L(R)(C_1r+||u^*-u_0||_{p,\infty})][r+||u^*||_1]+C_0|\phi|_p[C_1r+||u^*||_{p,\infty}].$$

In a similar way we obtain

$$\begin{aligned}&||Tv-T\bar{v}||_1\leq C_0[||F(\cdot,v)-F(\cdot,\bar{v}||_0+||(A(0,u_0)-A(\cdot,v))(v-\bar{v}||_0+\\ &\qquad +||(A(\cdot,v)-A(\cdot,\bar{v}))\bar{v}||_0\\ &\leq C_0[|\phi_R|_p||v-\bar{v}||_{p,\infty}+(\gamma_a+L(R)||u_0-v||_{p,\infty})||v-\bar{v}||_1+L(R)||v-\bar{v}||_{p,\infty}||\bar{v}||_1]\\ &\leq C_0[|\phi_R|_pC_1+\gamma_a+L(R)(||u_0-u^*||_{p,\infty}+C_1r)+L(R)C_1(r+||u^*||_1)]||v-\bar{v}||_1.\end{aligned}$$

Now fix $r\in(0,1]$ such that $r\leq[8C_0C_1L(R)]^{-1}$ and let $a\in(0,a_0]$ be such that

$$||u^*-u_0||_{p,\infty}\leq[8C_0L(R)]^{-1},\ ||u^*||_1\leq r,$$

as well as

$$\gamma_a\leq(8C_0)^{-1},\ |\phi_R|_p\leq[8C_0C_1]^{-1}.$$

This is possible since R is fixed, hence also $\phi_R\in L_p(J)$, and $u^*\in C(J;X_p)$ has initial value u_0.

Then there follows $||Tv-T\bar{v}||_1\leq\frac{3}{4}||v-\bar{v}||_1$, i.e. T is a contraction. If in addition a is chosen so small that $C_0|\phi_R|_p||u^*||_{p,\infty}\leq r/8$ then $||Tv-u^*||_1\leq r$, i.e. $T(\mathbb{B}_r)\subset\mathbb{B}_r$.

The contraction mapping principle therefore yields a unique fixed point of T in $\mathbb{B}_r$ which is the unique strong solution u on $J=[0,a]$ in the maximal regularity class $u\in H^1_p(J;X_0)\cap L_p(J;X_1)$. This is essentially the result of Clément and Li [4]

Theorem 3.1. *Suppose assumptions* **(A)** *and* **(F)** *are satisfied, and assume that* $A_0=A(0,u_0)$ *has the property of maximal* L_p*-regularity.*

Then there is $a\in(0,a_0]$ *such that (3.1) admits a unique solution* u *on* $J=[0,a]$ *in the maximal regularity class* $u\in H^1_p(J;X_0)\cap L_p(J;X_1)$.

Concerning continuation of the solution u, observe that $u\in C(J;X_p)$ holds. If $A(t,v)$ has the property of L_p-maximal regularity for each $t\in J_0$ and $v\in X_p=(X_0,X_1)_{1-1/p,p}$ then we may continue the solution and there will result a maximal interval of existence $J(u_0):=[0,t^+(u_0))\subset J_0$. This interval is characterized by the condition that $\lim_{t\to t^+(u_0)}u(t)$ does not exist in X_p, since otherwise we may apply Theorem 3.1 with initial value $u(t^*(u_0))=\lim_{t\to t^+(u_0)}u(t)$ to obtain a contradiction to maximality. In turn, this condition is equivalent to

$$\int_{J(u_0)}[|u(t)|_1^p+|\dot{u}(t)|_0^p]dt=\infty.$$

The natural phase space for the problem is the space X_p, and by uniqueness of the solutions the map $u_0 \mapsto u(t)$ defines a local semiflow on X_p. Thus we obtain

Corollary 3.2. *Suppose assumptions* **(A)** *and* **(F)** *are satisfied, and assume that $A(t,v)$ has maximal L_p-regularity for each $t \in J_0$, $v \in X_p$. Then the solution $u(t)$ of (3.1) has a maximal interval of existence $J(u_0) = [0, t^+(u_0))$, which is characterized by the equivalent conditions*

$$\int_{J(u_0)} [|u(t)|_1^p + |\dot{u}(t)|^p] dt = \infty,$$

and

$$\lim_{t \to t^+(u_0)} u(t) \quad \text{does not exist in} X_p.$$

The map $u_0 \mapsto u(t)$ defines a local semiflow on the natural phase space X_p.

We now give two abstract criteria for global existence.

Proposition 3.3. *Let the assumptions of Corollary 3.2 hold. Suppose that the solution u of (3.1) with maximal interval $J(u_0)$ satisfies one of the following conditions.*

(i) *u is uniformly continuous in X_p on $J(u_0)$;*

(ii) *$u(J(u_0)) \subset X_p$ is relatively compact.*

Then the solution $u(t)$ of (3.1) exists globally on J_0.

Proof. If u is uniformly continuous on $J(u_0)$ then obviously the limit $\lim_{t \to t^+(u_0)} u(t) = u(t^+(u_0))$ exists in X_p, hence Corollary 3.2 applies.

Suppose on the other hand that $u(J(u_0))$ is relatively compact in X_p. Then we may choose a sequence $t_n \to t^+(u_0) =: t_\infty$ such that $u_n := u(t_n) \to u_\infty$ in X_p. Let v_n denote the maximal solutions of the problems

$$\dot{v}(t) + A(t, v(t))v(t) = F(t, v(t)), \quad t \geq t_n, \ v(t_n) = u_n.$$

Since $u_n \to u_\infty$ and A is continuous we obtain $A_n = A(t_n, u_n) \to A(t_\infty, u_\infty)$ in $\mathcal{B}(X_1, X_0)$. By perturbation of maximal L_p-regularity the solutions v_n have a common existence intervall of length, say $\delta > 0$. Then choosing n so large that $t_n + \delta > t_\infty$ the corresponding function v_n yields an extension of u beyond $t^+(u_0)$ which gives a contradiction to maximality of $J(u_0)$. □

Specializing to the case $A(t,v) = A(t)$ and $F(t,v) = B(t)v + f(t)$, where $A : J_0 \to \mathcal{B}(X_1, X_0)$ is continuous, $B \in L_p(J_0; \mathcal{B}(X_p, X_0))$, and $f \in L_p(J_0; X_0)$, we obtain a result for the nonautonomous linear problem

$$\dot{u}(t) + A(t)u(t) = B(t)u(t) + f(t), \quad t \in J_0, \ u(0) = u_0, \tag{3.4}$$

which we state as

Corollary 3.4 *Let $A \in C(J_0; \mathcal{B}(X_1, X_0))$ be such that $A(t)$ has maximal L_p-regularity for each $t \in J_0$, and let $B \in L_p(J_0; \mathcal{B}(X_p, X_0))$.*

Then (3.4) admits a unique solution

$$u \in H^1_p(J_0; X_0) \cap L_p(J_0; X_1),$$

if and only if $f \in L_p(J_0; X_0)$ and $u_0 \in X_p$.

Proof. The necessity is clear by the embedding

$$H^1_p(J_0; X_0) \cap L_p(J_0; X_1) \hookrightarrow C(J_0; X_p).$$

Sufficiency follows from Theorem 3.1 since by the assumed continuity of $A(t)$ the maximal regularity constant C_0 is uniform in $t \in J$. Decomposing J_0 into finitely many intervals $J_n = [nh, (n+1)h]$, with $h > 0$ sufficiently small, Theorem 3.1 applied to each of these intervals yields a solution on all of J_0. □

4 Global Existence for Reaction Diffusion Problems

Let $G \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega \in C^2$, and let $J = (0, T)$. In this section we consider the reaction-diffusion problem

$$\begin{aligned} \partial_t u - D\Delta u = f(u), \quad x \in G,\ t \in J, \\ \partial_\nu u|_{\partial G} = 0, \quad u|_{t=0} = u_0. \end{aligned} \tag{4.1}$$

Here $D = \mathrm{diag}(d_i)$ is diagonal with constants $d_i > 0$ for each $i = 1, \ldots N$, and $f \in C^1(\mathbb{R}^N; \mathbb{R}^N)$ is such that the positivity condition

$$r_i \geq 0, \text{ for all } i,\ r_k = 0 \quad \Rightarrow \quad f_k(r) \geq 0 \tag{4.2}$$

is valid. It is well-known - and not difficult to prove - that (4.1) admits a unique classical solution $u(t, x)$ on some maximal time interval $J_m = (0, t_{max})$, which stays nonnegative if u_0 is nonnegative. The question we want to address here is whether the solutions exist globally in time, i.e. whether $t_{max} = T$.

Obviously, this cannot be true in general, since (4.1) contains as a special case the ode-problem

$$\dot{z} = f(z),\ t \in J, \quad z(0) = z_0. \tag{4.3}$$

However, in many cases of interest in applications, additional information is available in terms of a Lyapunov function for (4.3) which easily implies global existence for (4.3). We state this as an assumption.

(H1) *There is a function $H \in C^2(\mathbb{R}^N; \mathbb{R})$ with $H(z) \geq 0$ and $H''(z)$ positive semidefinite, for each $z \geq 0$, such that*

$$z \geq 0,\ H(z) \to \infty \quad \Rightarrow \quad |z| \to \infty,$$

and there constants $k_1, k_2 \in \mathbb{R}$ with

$$H'(z) \cdot f(z) \leq k_1 H(z) + k_2, \quad z \geq 0.$$

It is easy to see that **(H1)** implies global existence for the ode-problem (4.3) for nonegative initial values z_0. The same assertion also holds for (4.1) in case $D = dI$ is a multiple of the identity matrix; we normalize $d = 1$. In fact, it is a consequence of the maximum principle. To see this, let $u(t,x)$ be the solution of (4.1) on J_m and set $v(t,x) = H(u(t,x))$. Then we obtain

$$\begin{aligned}\partial_t v - \Delta v &= H'(u)(\partial_t u - \Delta u) - \nabla u \cdot H''(u)\nabla u \\ &\leq H'(u) \cdot f(u) \leq k_1 v + k_2,\end{aligned}$$

in $J_m \times G$, $v_0 := v(0) = H(u_0)$, and $\partial_\nu v = H'(u) \cdot \partial_\nu u = 0$ on ∂G. If u_0 is nonnegative and bounded, the maximum principle then implies

$$0 \leq v(t,x) = H(u(t,x)) \leq C, \quad t \in J_m,\ x \in G,$$

which by **(H1)** yields boundedness of $u(t,x)$ on $J_m \times G$, hence global existence, i.e. $t_{max} = T$.

However, for applications to biological and chemical problems the assumption $D = dI$ is highly unrealistic, diffusion coefficients of different species are unequal, in general. Unfortunately, in this case the simple argument given above does not work anymore, and it is a challenging, unsolved problem to prove global existence for (4.1), assuming only **(H1)** in the case $D \neq dI$.

Here we want to show how maximal L_p-regularity can be employed to prove global existence for (4.1) in some situations which are more special than **(H1)** but still applicable to many equations which are important in applications. For this purpose we introduce two extra assumptions on f and H; the first one is the following structural condition on the Lyapunov function H.

(H2) *The Lyapunov function H is of the form $H(z) = \sum_{i=1}^N h_i(z_i)$, where $h_i \in C^2(\mathbb{R})$ are nonnegative and convex.*

This assumption is motivated by laws like conservation of mass or decreasing entropy in chemical systems and is valid in many examples, some will be discussed below. The following a priori estimates resulting from **(H1)** and **(H2)** are due to Morgan [17]. We include here a proof which is simpler than that in Morgan's paper.

Theorem 4.1. *Assume* **(H1)** *and* **(H2)** *and let $u(t,x)$ be the solution of (4.1), with u_0 nonnegative, bounded, on the maximal time interval $J_m = (0, t_{max})$.*

Then there is a constant $C = C(u_0) > 0$ such that the following estimates are valid for each $i = 1, \ldots, N$.

$$\sup_{t\in J_m} \int_G h_i(u_i(t,x))dx \leq \sup_{t\in J_m} \int_G H(u(t,x))dx \leq C; \tag{4.4}$$

$$\sup_{x\in G} \int_{J_m} h_i(u_i(t,x))dt \leq \sup_{x\in G} \int_{J_m} H(u(t,x))dt \leq C; \tag{4.5}$$

$$\int_{J_m}\int_G h_i^2(u_i(t,x))dx \leq \int_{J_m}\int_G H^2(u(t,x))dx \leq C, \tag{4.6}$$

In particular, we have

$$H(u) \in L_\infty(J_m; L_1(G)) \cap L_2(J_m \times G) \cap L_\infty(G; L_1(J_m)), \quad i = 1, \dots, N.$$

Proof. Set $v_i = h_i(u_i)$; these functions satisfy the following problem.

$$\begin{aligned} &\partial_t v_i - d_i \Delta v_i = h_i'(u_i) f_i(u) - d_i h_i''(u_i)|\nabla u_i|^2, \quad t \in J_m,\ x \in G, \\ &\partial_\nu v_i|_{\partial G} = 0, \quad v_i|_{t=0} = v_i^0 := h_i(u_{0i}). \end{aligned} \tag{4.7}$$

To prove (4.4) we integrate over G and sum over i; using the divergence theorem, the boundary condition, as well as **(H1), (H2)** we obtain with $v = \sum_{i=1}^N v_i$

$$\partial_t \int_G v(t,x) = \int_G H'(u) \cdot f(u) dx - \int_G \sum_{i=1}^N d_i h_i''(u_i) |\nabla u_i|^2 dx \le \int_G (k_1 v + k_2) dx,$$

hence integrating over t yields estimate (4.4).

To prove (4.5), we integrate (4.7) with respect to t and set

$$w(t,x) = \sum_{i=1}^N d_i \int_0^t v_i(s,x) ds, \quad t \in J_m,\ x \in G.$$

This implies with $v_i \ge 0$ and **(H1), (H2)**

$$\begin{aligned} &\partial_t w = \sum_i d_i v_i \le d_M \sum_i v_i \\ &= d_M [v_0 + \Delta \sum_i d_i \int_0^t v_i(s,x) ds + \int_0^t H'(u) \cdot f(u) ds - \int_0^t \sum_i h_i''(u_i) |\nabla u_i|^2 ds] \\ \le\ & d_M \Delta w + [d_M/d_m] k_1 w + d_M k_2 + d_M v_0, \\ &\partial_\nu w|_{\partial G} = 0, \quad w|_{t=0} = 0, \end{aligned}$$

where $d_M = \max\{d_i\}$ and $d_m = \min\{d_i\}$. Using the maximum principle we obtain a constant $w_\infty \in \mathbb{R}$, such that

$$0 \le w(t,x) \le w_\infty, \quad t \in J_m,\ x \in G,$$

which implies with $w(t,x) \ge d_i \int_0^t v_i(s,x) ds$ estimate (4.5).

For the third estimate we multiply the differential inequality for w with $\partial_t w = \sum_i d_i v_i$ and integrate with respect to t and x. This yields after an integration by parts

$$\begin{aligned} &\int_0^\tau \int_G \partial_t w^2(t,x) dx dt + \frac{d_M}{2} \int_G |\nabla w(\tau,x)|^2 dx \le \\ &\le d_M k_2 \mathrm{mes}(G) + d_M \int_G v_0(x) w(\tau,x) dx + \frac{d_M k_1}{2 d_m} \int_G w^2(\tau,x) dx, \end{aligned}$$

for each $\tau < t_{max}$. Since we already know that w is bounded, the right hand side of this inequality is bounded hence by $v_i \ge 0$ we get

$$d_k^2 \int_0^\tau \int_G v_k^2(t,x) dx dt \le \int_0^\tau \int_G [\sum_i d_i v_i^2](t,x) dx dt = \int_0^\tau \int_G [\partial_t w]^2(t,x) dx dt \le C,$$

which yields the third estimate (4.6) with $\tau \to t_{max}$. This completes the proof. $\square$

There is another a priori estimate available in case the functions h_i are also logarithmic convex. A little more general, this is

Corollary 4.2. *Assume* **(H1)** *and* **(H2)** *and let $u(t,x)$ be the solution of (4.1), with u_0 nonnegative, bounded, on the maximal time interval $J_m = (0, t_{max})$. Assume in addition that there are $\alpha_i \geq 0$ such that*

$$h_i''(r)h_i(r) \geq \alpha_i (h_i'(r))^2, \quad r > 0.$$

Then there is a constant $C = C(u_0) > 0$ such that

$$\alpha_i \int_{J_m} \int_G |\nabla h_i(u_i(t,x))|^2 / h_i(u_i(t,x)) dx \leq C, \quad i = 1, \ldots, N. \tag{4.8}$$

In particular, if $\alpha_i > 0$ then $h_i(u_i)$ belongs to $L_1(J_m; L_{q_0}(G))$, with $q_0 = 1 + 2/(n-2)$ for $n > 2$, q_0 arbitrary for $n = 2$, and $h_i(u_i) \in L_1(J_m; C^{1/2}(\bar{G}))$ for $n = 1$. Further, $h_i(u_i)$ belongs to $L_{p_0}(J_m \times G)$, with $p_0 = 1 + 2/n$ for $n \geq 2$, $p_0 = 2$ for $n = 1$.

Proof. Observing that by the extra assumption on h_i we have

$$h_i''(u_i)|\nabla u_i|^2 \geq \alpha_i h_i(u_i)^{-1} |h_i'(u_i) \nabla u_i|^2 = \alpha_i |\nabla h_i(u_i)|^2 / h_i(u_i).$$

Hence from (4.7) we derive

$$\partial_t \int_G v(t,x) dx + \sum_{i=1}^N \alpha_i d_i \int_G |\nabla v_i(t,x)|^2 / v_i(t,x) dx \leq k_1 \int_G v(t,x) dx + k_2 \mathrm{mes}(G),$$

which implies after integration in time

$$\int_G v(\tau, x) dx + \sum_{i=1}^N \alpha_i d_i \int_0^\tau \int_G |\nabla v_i(t,x)|^2 / v_i(t,x) dx dt \leq C,$$

uniformly in τ by estimate (4.4). $v \geq 0$ then proves (4.8). With $\nabla \sqrt{v_i} = \nabla v_i / 2\sqrt{v_i}$, in case $\alpha_i > 0$, this implies further

$$\sqrt{v_i} \in L_2(J_m; H_2^1(G)) \hookrightarrow L_2(J_m; L_q(G))$$

by embedding, where $q = 2n/(n-2)$ for $n > 3$, $q > 1$ arbitrary for $n = 2$, and $\sqrt{v_i} \in L_2(J_m; C^{1/2}(\bar{G})$ for $n = 1$. Taking the square and interpolating with (4.4) this yields the last assertions of Corollary 4.2. $\square$

Let us compare the quality of the a priori estimates (4.4) $\sim$ (4.8) by means of the *regularity index* κ of the diffusion operator $\partial_t - \Delta$. This can be defined as $\kappa = 2\kappa_t + \kappa_x$, where κ_x denotes the Sobelev index in space and κ_t that in time. So if a function $v = v(x)$ belongs to $W_q^s(G)$ then $\kappa_x = s - n/q$, and if $\phi = \phi(t)$ is in $W_p^r(J)$ then $\kappa_t = r - 1/p$. Computing κ for the estimates in Theorem 4.1 and Corollary 4.2 we obtain $\kappa = -n$ for and (4.8), $\kappa = -2$ for (4.5), and $\kappa = -(n+2)/2$ for (4.6). the larger κ, the better the quality of the estimate. We see from this that (4.4) is the best for $n = 1$, for $n = 2$ all have the same value $\kappa = -2$, and for $n \geq 3$ (4.5) is better than (4.6) which in turn dominates (4.4) and (4.8). On the other hand, if $k_1, k_2 \leq 0$ then (4.4) and (4.8) are independent of the length of the interval, which is useful for the asymptotics of problem (4.1).

Remark 4.3. We may replace the homogeneous Neumann condition at the boundary ∂G by a Robin condition of the form $\partial_\nu u_i + \alpha_i(u_i - \gamma_i) = 0$, where $\alpha_i, \gamma_i \geq 0$. Then no extra assumption on $h_i(r)$ is needed, since

$$(\gamma_i - r)h_i'(r) \leq h_i(\gamma_i) - h_i(r) \leq h_i(\gamma_i) \quad \text{for all } r > 0,$$

by convexity and positivity of h_i. In the case of a Dirichlet condition $u_i = \gamma_i \geq 0$ we would have to require $h_i'(\gamma_i) = 0$, and we are not able to handle an inhomogeneous Neumann condition.

So far only elementary pde-techniques have been involved. Maximal regularity comes in to prove boundedness of the solutions, building upon a priori estimates like (4.4) $\sim$ (4.8). For this purpose we introduce another structural condition on f and H which is also due to Morgan [17].

(H3) *There are numbers $a_{ij} \geq 0$, $a_{ii} = 1$, and constants $r > 0$, $k_3, k_4 \geq 0$ such that*

$$\sum_{j=1}^{i} a_{ij} h_j'(z_j) f_j(z) \leq k_3 H(z)^r + k_4, \quad z \in \mathbb{R}_+^N,\ i = 1, \ldots, N.$$

The main theorem of this section extends a result due to Morgan [17]. It is based on maximal L_p-regularity and removes some restrictions in that paper, the most important is that we can cover the limiting case excluded there.

Theorem 4.4. *Assume* **(H2)** *and* **(H3)** *and let $u(t,x)$ be the solution of (4.1), with u_0 nonnegative, bounded, on the maximal time interval $J_m = (0, t_{max})$. In addition, suppose there is $a > 0$ such that*

$$\int_{J_m} \int_G |H(u(t,x))|^a dx dt < \infty,$$

with $r \leq 1 + 2a/(n+2)$.

Then $H(u(t,x)$ is bounded on $J_m \times G$. In particular, if also $H(z) \to \infty$ for $|z| \to \infty$ then $u(t,x)$ is bounded on $J_m \times G$, hence $t_{max} = T$, i.e. the solution exist globally.

Proof. Without loss of generality we may assume the critical case $r = 1 + 2a/(n+2)$. Fix $\tau < t_{max}$. By maximal L_p-regularity, for each $p \in (1, \infty)$, there is constant $M_p \geq 1$ such that the solutions v of the problem

$$\begin{aligned} &\partial_t v - d\Delta v = f, \quad t \in J_0,\ x \in G, \\ &\partial_\nu v|_{\partial G}, \quad v|_{t=0} = v_0, \end{aligned}$$

satisfy the estimate

$$|\dot{v}|_{L_p(J_0;L_p(G))} + |v|_{L_p(J_0;H_p^2(G))} \leq M_p[|f|_{L_p(J_0;L_p(G))} + |v_0|_{W_p^{2-2/p}(G))}]. \tag{4.9}$$

It is important to note that this estimate is independent of the length of the interval $J_0 = (0,\tau) \subset J = (0,T)$, for finite $T > 0$, at least.

Next we observe that with $1/p_1 = 1/p - 2/(n+2) > 0$, the embedding constant for

$$H_p^1(J_0; L_p(G)) \cap L_p(J_0; H_p^2(G)) \hookrightarrow L_{p_1}(J_0 \times G)$$

is also independent of the length of the interval J_0; we call it N_p.

(i) Now let $u(t,x)$ be the classical solution of (4.1) on $J_m = (0, t_{max})$, where the initial value u_0 is nonnegative and bounded. We set as before $v_i = h_i(u_i)$, $v = H(u)$, and remind on (4.7). By hypothesis **(H3)**, v_1 satisfies

$$\begin{aligned}
&\partial_t v_1 - d_1 \Delta v_1 \le h_1'(u_1) f_1(u) \le k_3 v^r + k_4, \quad t \in J_m,\ x \in G,\\
&\partial_\nu v_1|_{\partial G} = 0, \quad v_1|_{t=0} = v_{01} := h_1(u_{01}).
\end{aligned}$$

Let z_1 denote the solution of

$$\begin{aligned}
&\partial_t z_1 - d_1 \Delta z_1 = k_3 v^r + k_4, \quad t \in J_m,\ x \in G,\\
&\partial_\nu z_1|_{\partial G} = 0, \quad z_1|_{t=0} = v_{01}.
\end{aligned}$$

The maximum principle implies $0 \le v_1(t,x) \le z_1(t,x)$ in $J_m \times G$. On the other hand, by maximal L_p-regularity, we have with estimate (4.9)

$$|\dot z_1|_{L_p(J_0;L_p(G))} + |z_1|_{L_p(J_0;H_p^2(G))} \le M_p[c + k_3 |v|^r_{L_{pr}(J_0;L_{pr}(G))} + |v_{01}|_{W_p^{2-2/p}(G))}].$$

By embedding this yields

$$|z_1|_{L_{p_1}(J_0\times G} \le c + N_p M_p k_3 |v|^r_{L_{pr}(J_0\times G))},$$

hence v_1 belongs to the same space and

$$|v_1|_{L_{p_1}(J_0\times G} \le c + N_p M_p k_3 |v|^r_{L_{pr}(J_0\times G))}. \tag{4.10}$$

To gain something, we clearly must have $p_1 > pr$, which means that we choose p such that $(1-1/r)(n+2)/2 < p < (n+2)/2$, close to $(n+2)/2$.

(ii) To estimate v_2 multiply the first equation by a_{21} and and add it to the second one. This yields after a rearrangement

$$\begin{aligned}
&\partial_t[a_{21}v_1 + a_{22}v_2] - d_2\Delta[a_{21}v_1 + a_{22}v_2] \le a_{21}(d_1 - d_2)\Delta v_1 + k_3 v^r + k_4,\\
&\partial_\nu[a_{21}v_1 + a_{22}v_2]|_{\partial G} = 0, \quad [a_{21}v_1 + a_{22}v_2]|_{t=0} = a_{21}h_1(u_{01}) + a_{22}h_2(u_{02}).
\end{aligned}$$

Let z_{21} denote the solution of

$$\begin{aligned}
&\partial_t z_{21} - d_2 \Delta z_{21} = a_{21}(d_1 - d_2)\Delta v_1, \quad (t,x) \in J_m \times G,\\
&\partial_\nu z_{21}|_{\partial G} = 0, \quad z_{21}|_{t=0} = 0,
\end{aligned}$$

and let z_{22} denote that of

$$\begin{aligned}
&\partial_t z_{22} - d_2 \Delta z_{22} = k_3 v^r + k_4, \quad (t,x) \in J_m \times G,\\
&\partial_\nu z_{22}|_{\partial G} = 0, \quad z_{22}|_{t=0} = a_{21}h_1(u_{01}) + a_{22}h_2(u_{02}).
\end{aligned}$$

By the maximum principle we then obtain

$$0 \le v_2 \le [a_{21}v_1 + a_{22}v_2] \le z_{21} + z_{22}.$$

If A denotes the negative Neumann Laplacian, we have

$$z_{21} = e^{-d_2At} * a_{21}(d_1 - d_2)\Delta v_1 = a_{21}(d_1 - d_2)Ae^{-d_2At} * v_1,$$

hence $z_{21} \in L_{p_1}(J_0 \times G)$ by maximal L_{p_1}-regularity, since we already know $v_1 \in L_{p_1}(J_0 \times G)$. We may show as in (i) that z_{22} belongs to this space as well with essentially the same estimate. Therefore, $a_{21}v_1 + a_{22}v_2 \in L_{p_1}(J_0 \times G)$, and so v_2 belongs to this space. There results an estimate for v_2 of the form

$$|v_2|_{L_{p_1}(J_0 \times G)} \le c + c_2 N_p M_p |v|^r_{L_{pr}(J_0 \times G))}, \tag{4.11}$$

where $c_2 = 1 + M_{p_1}\frac{|d_2 - d_1|}{d_2}$.

(iii) We may now proceed by induction as in (ii) to obtain estimates for all v_k.

$$|v_k|_{L_{p_1}(J_0 \times G)} \le c + c_k N_p M_p |v|^r_{L_{pr}(J_0 \times G))}, \tag{4.12}$$

where c depends only on the data but not on τ, and c_k is depends only on M_{p_1} and on the diffusion coefficients d_j. Summing over k this yields the estimate

$$|v|_{L_{p_1}(J_0 \times G)} \le c + C|v|^r_{L_{pr}(J_0 \times G))}, \tag{4.13}$$

with a constant c depending on the data but independent of τ, and C depends only on M_{p_1} and the d_i.

(iv) We are going to show that $|v|_{L_{p_1}(J_0 \times G)}$ is bounded with bound independent of τ. Using (4.13) we estimate as follows.

$$\begin{aligned} |v|_{p_1} &\le c + C|v|^r_{pr} = c + C|v^a|^{r/a}_{pr/a} \\ &\le c + C|v^a|^{\gamma r/a}_{p_1/a}|v^a|^{(1-\gamma)r/a}_1 \\ &= c + C|H(u)|^{r(1-\gamma)}_a \cdot |v|^r_{p_1}\gamma, \end{aligned} \tag{4.14}$$

where $a/pr = \gamma a/p_1 + (1-\gamma)/1$, i.e. $\gamma = (1 - a/rp)/(1 - a/p_1)$. Since $p_1 \ge rp > a$ we obviously have $\gamma \in [0, 1]$. With

$$\gamma r = \frac{r - a/p}{1 - a/p_1} = \frac{1 + 2a/(n+2) - a/p}{1 - a/p + 2a/(n+2)} = 1,$$

we have

$$|v|_{p_1} \le c + C|H(u)|^{r-1}_a |v|_{p_1}.$$

By assumption $H(u) \in L_a(J_m \times G)$, hence if we in addition know that $|H(u)|^{r-1}_{L_a(J_m \times G)}$ is smaller than $1/2C$ we obtain $|v|_{p_1} \le 2c$.

This may seem to be a very strange assumption, however, by translation invariance of (4.1) we may always restrict to an interval $[t_{max} - \delta, t_{max})$ with δ so small that

$$|H(u)|^{r-1}_{L_a([t_{max} - \delta, t_{max}) \times G)} \le 1/2C.$$

Here it is important to note that C does not depend on u_0, the latter only enters c. Therefore, this proves $H(u) \in L_{p_1}(J_m \times G)$.

(v) To finish the proof, observe that $H(u) \in L_{p_1}(J_m \times G)$ by maximal L_p-regularity implies

$$z \in H^1_{p_1/r}(J_m; L_{p_1/r}(G)) \cap L_{p_1/r}(J_m; H^2_{p_1/r}(G)) \hookrightarrow L_\infty(J_m \times G),$$

provided $1/p_1 < 2/(n+2)$. Thus choosing p such that

$$(n+2)/2 > p > \max\{1, \frac{[(n+2)/2]a}{(n+2)/2+a}, \frac{[(n+2)/2]r}{r+1}\},$$

we see that $v = H(u) \in L_\infty(J_m \times G)$, hence in case $H(z) \to \infty$ if $|z| \to \infty$, we obtain $u \in L_\infty(J_m \times G)$, which gives $t_{max} = T$, i.e. the solution u exists globally. The proof is complete. $\square$

Example 4.5. Let us consider here some examples which can be handled via Theorems 4.1 and 4.4.

(a) Let $N = 3$ and consider the reversible reaction

$$A_1 + A_2 \leftrightarrow A_3.$$

If we impose mass-action action kinetics then

$$f_1(u) = f_2(u) = -f_3(u) = r(u) = -\kappa_1 u_1 u_2 + \kappa_2 u_3,$$

where $\kappa_1, \kappa_2 > 0$ are positive constants. Set $h_i(u_i) = u_i$, i.e. $H(u) = u_1 + u_2 + 2u_3$. Then **(H1)** and **(H2)** are valid with $k_1 = k_2 = 0$. Setting $a_{21} = 0$, $a_{31} = 1$, $a_{ii} = 1$ we see that also **(H3)** is valid with $k_3 = \kappa_2$, $k_4 = 0$, $r = 1$. Hence we may conclude from Theorems 4.1 and 4.2 global existence in any space dimension.

(b) Let $N = 4$ and consider this time the reaction

$$A_1 + A_2 \leftrightarrow A_3 + A_4.$$

This time mass-action kinetics yields

$$f_1(u) = f_2(u) = -f_3(u) = -f_4(u) = r(u) = -\kappa_1 u_1 u_2 + \kappa_2 u_3 u_4,$$

i.e. each term in f is second order. Choosing again $h_i(u_i) = u_i$, again **(H1)** and **(H2)** are valid with $k_1 = k_2 = 0$. Setting $a_{21} = 0$, $a_{31} = a_{41} = 1$, $a_{ii} = 1$ we see that also **(H3)** is valid with $k_3 = \kappa_2$, $k_4 = 0$, $r = 2$. Hence we may conclude from Theorems 4.1 and 4.2 global existence in space dimension $n \leq 2$. This result is new, however, global existence is not yet known for $n \geq 3$.

5 Regularity Theory

It is a folklore fact that parabolic equation have the property of smoothing. We want to show this by means of maximal L_p-regularity.

Let X_0 be a Banach space, $X_1 \hookrightarrow X_0$ densely, and let $1 < p < \infty$. In this section we consider the autonomous quasilinear problem

$$\dot{u}(t) + A(u(t))u(t) = F(u(t)), \quad t > 0, \; u(0) = u_0, \tag{5.1}$$

Here $u_0 \in X_p := (X_0, X_1)_{1-1/p,p}$, $A : X_p \to \mathcal{B}(X_1, X_0)$ and $F : X_p \to X_0$ are of class C^k with $k \in \mathbb{N} \cup \{\infty, \omega\}$; here ω means real analytic.

Suppose that we are given a solution $u \in Z_1(J) := H^1_p(J; X_0) \cap L_p(J; X_1)$, and assume that $A(u(t)) \in \mathcal{MR}_p(J, X_0)$ for each $t \in J$. Introduce a new function

$$u_\lambda(t) := u(\lambda t), \quad \lambda \in (1-\varepsilon, 1+\varepsilon), \; t \in J_\varepsilon,$$

where $J_\varepsilon = [0, T/(1+\varepsilon)]$, $\varepsilon > 0$ fixed but as small as we please. This new function satisfies $\dot{u}_\lambda(t) = \lambda \dot{u}(\lambda t)$, hence

$$\dot{u}_\lambda(t) + \lambda A(u_\lambda(t))u_\lambda(t) = \lambda F(u_\lambda(t)), \quad t \in J_\varepsilon, \; u_\lambda(0) = u_0.$$

now consider the map $H : (1-\varepsilon, 1+\varepsilon) \times Z_1(J_\varepsilon) \to Z_0(J_\varepsilon)$ defined by

$$H(\lambda, w)(t) = (\dot{w}(t) + \lambda A(w(t)w(t) - \lambda F(w(t)), w(0) - u_0), \quad t \in J_\varepsilon,$$

with $Z_0(J_\varepsilon) = L_p(J_\varepsilon; X_0) \times X_p$. Since A and F are of class C^k and H is polynomial in λ there follows also $H \in C^k$. Further we know $H(1, u) = 0$ and

$$D_\lambda H(\lambda, w) = (A(w)w - F(w), 0),$$

$$D_w H(\lambda, w)h = (\dot{h} + \lambda A(w)h + \lambda A'(w)hw - \lambda F'(w)h, h(0)).$$

In particular, $D_w H(1, u) : Z_1 \to Z_0$ is given by

$$D_w H(1, u)h = (\dot{h} + A(u)h + A'(u)hu - F'(u)h, h(0)).$$

Since $A(u(t))$ has maximal L_p-regularity for each $t \in J$, and $F'(u)$ as well as $A'(u)u$ are of lower order,

$$B(t) = A(u(t)) + A'(u(t))u(t) - F'(t), \quad t \in J,$$

has maximal L_p-regularity for each fixed $t \in J$ and is continuous on J since $u \in C(J; X_p)$. Corollary 3.4 shows that $D_w H(1, u) : Z_1(J_\varepsilon) \to Z_0(J_\varepsilon)$ is an isomorphism. Then we may apply the implicit function theorem to obtain a C^k-map $\phi : (1-\delta, 1+\delta) \to Z_1(J_\varepsilon)$ such that $H(\lambda, \phi(\lambda) = 0$ for each $\lambda \in (1-\delta, 1+\delta)$, and $\phi(1) = u$. By the definition of H and by uniqueness we then see that $\phi(\lambda) = u_\lambda$, i.e. $u_\lambda \in Z_1(J_\varepsilon)$ is C^k in λ. The embedding $Z_1(J_\varepsilon) \hookrightarrow C(J_\varepsilon; X_p)$ then shows that the map $\lambda \mapsto u_\lambda(t) = u(\lambda t)$ is of class C^k for each fixed t. But this implies $u \in C^k(J; X_p)$.

To extract more regularity from the map ϕ, note that

$$\frac{d}{d\lambda} u_\lambda(t)|_{\lambda=1} = t\dot{u}(t), \quad t \in J_\varepsilon,$$

hence $t\dot{u} \in Z_1(J_\varepsilon)$, and by induction $t^k u^{(k)} \in Z_\varepsilon(J_\varepsilon)$, which means

$$u \in H^{k+1}_p([\varepsilon, T]; X_0) \cap H^k_p([\varepsilon, T]; X_1),$$

for each $\varepsilon > 0$. By embedding we obtain from this

$$u \in C^{k+1-1/p}((0,T);X_0) \cup C^{k-1/p}((0,T);X_1).$$

In particular, if $k = \infty$ then $u \in C^\infty(0,T);X_1)$ and in case $k = \omega$ we get real analyticity of $u(t) \in X_1$ in $(0,T)$.

This method of proving regularity via scaling was first used by Angenent [1]. Let us summarize the result in

Theorem 5.1. *Let $1 < p < \infty$, $k \in \mathbb{N} \cup \{\infty, \omega\}$, and assume $A \in C^k(X_p;\mathcal{B}(X_1,X_0))$, $F \in C^k(X_p;X_0)$. Let $u \in H^1_p(J;X_0) \cap L_p(J;X_1)$ be a solution of (5.1) with initial value $u_0 \in X_p$. Suppose $A(u(t)) \in \mathcal{MR}_p(J;X_0)$ for each $t \in J$.*

Then $t^j u^{(j)} \in H^1_p(J;X_0) \cap L_p(J;X_1)$ for each $j \leq k$. In particular,

$$u \in H^{k+1}_p([\varepsilon,T);X_0) \cap H^k_p([\varepsilon,T];X_1),$$

for each $\varepsilon > 0$, and

$$u \in C^k((0,T);X_p) \cap C^{k+1-1/p}((0,T);X_0) \cap C^{k-1/p}((0,T);X_1).$$

If $k = \infty$ then $u \in C^\infty((0,T);X_1)$ and if $k = \omega$ then u is real analytic with values in X_1 on $(0,T)$.

Of course, this method also works in the nonautonomous case, we restricted to the autonomous situation just for convenience.

Using Angenent's method it is also possible to prove spatial regularity. To illustrate how this works, let us consider a parabolic quasilinear second order equation on $\mathbb{R}^n$.

$$\partial_t u(t,x) - a(u(t,x),\nabla u(t,x)) : \nabla^2 u(t,x) = f(u(t,x),\nabla u(t,x)), \quad t \in (0,T),\ x \in \mathbb{R}^n, \tag{5.2}$$

with initial condition $u(0,x) = u_0(x)$. Assume that $a : \mathbb{R}^{n+1} \to \mathbb{R}^{n^2}$ and $f : \mathbb{R}^{n+1} \to \mathbb{R}$ are of class C^k, $k \geq 1$, $a_{ij} = a_{ji}$ and $a(u,v)$ positive definite for each $u \in \mathbb{R}$, $v \in \mathbb{R}^n$.

Set $X_0 = L_p(\mathbb{R}^n)$, $X_1 = H^2_p(\mathbb{R}^n)$, we have

$$X_p = (X_0,X_1)_{1-1/p,p} = (L_p(\mathbb{R}^n), H^2_p(\mathbb{R}^n))_{1-1/p,p} = W^{2-2/p}_p(\mathbb{R}^n),$$

and choosing $p > n+2$ we have $W^{2-2/p}_p(\mathbb{R}^n) \hookrightarrow C^1_0(\mathbb{R}^n)$. Define $A : X_p \to \mathcal{B}(X_0,X_1)$ and $F : X_p \to X_0$ by means of

$$A(u)v(x) := a(u(x),\nabla u(x)) : \nabla^2 v(x), \quad x \in \mathbb{R}^n,\ u \in X_p,\ v \in X_1,$$

and

$$F(u)(x) := f(u(x),\nabla u(x)), \quad x \in \mathbb{R}^n,\ u \in X_p.$$

It is easy to see that A and F belong to the class C^k. Given $u \in X_p$, then $u \in C^1_0(\mathbb{R}^n)$, hence $\{(u(x),\nabla u(x)) :\ x \in \mathbb{R}^n$ is relatively compact. This implies that the functions $a_u(x) := a(u(x),\nabla u(x))$ are uniformly continuous and bounded on $\mathbb{R}^n$, and $\lim_{|x|\to\infty} a_u(x) = a(0,0)$. Moreover, $a_u(x)$ is positive definite, uniformly on $\mathbb{R}^n$. Therefore Theorem 2.1 shows that $A(u)$ belongs to the class $\mathcal{MR}_p(J;X_0)$, for each $u \in X_p$. Therefore, if $u_0 \in X_p$ we obtain by Theorem 4.1 a unique solution $u(t,x)$ of (5.2) on some maximal interval which we assume w.l.o.g. to be J. Thus the assumptions of Theorem 5.1 are satisfied and we may conclude e.g.

$$u \in C^{k+1-1/p}((0,T);L_p(\mathbb{R}^n)) \cap C^{k-1/p}((0,T);H^2_p(\mathbb{R}^n)) \cap C^k((0,T);W^{2-2/p}_p(\mathbb{R}^n)).$$

To derive more spatial regularity we may proceed as follows. Let $\lambda \in (1-\varepsilon, 1+\varepsilon)$ and $\mu \in \mathbb{R}^n$ and set $u_{\lambda,\mu}(t,x) = u(\lambda t, x+t\mu)$. Then

$$\begin{aligned}
\partial_t u_{\lambda,\mu}(t,x) &= \lambda \partial_t (u(\lambda t, x+t\mu) + \mu \cdot \nabla u(\lambda t, x+t\mu) \\
&= \lambda a(u_{\lambda,\mu}, \nabla u_{\lambda,\mu}) : \nabla^2 u_{\lambda,\mu} + f(u_{\lambda,\mu}, \nabla u_{\lambda,\mu}) + \mu \cdot \nabla u_{\lambda,\mu}, \\
&= \lambda a(u_{\lambda,\mu}, \nabla u_{\lambda,\mu}) : \nabla^2 u_{\lambda,\mu} + f_1(\mu, u_{\lambda,\mu}, \nabla u_{\lambda,\mu}).
\end{aligned}$$

Thus $u_{\lambda,\mu}$ solves a problem like (5.2) with a replaced by $\lambda a(u,\nabla u)$ and f by $f_1 = f(u,\nabla u) + \mu \cdot \nabla u$. Observe that the dependence on the parameters λ and μ is real analytic. Employing the implicit function theorem at $\lambda = 1$, $\mu = 0$, by uniqueness we may deduce that the map $(\lambda,\mu) \mapsto u_{\lambda,\mu}$ is of class C^k. By embedding and freezing (t,x) at a point $(t_0,x_0) \in (0,T) \times \mathbb{R}^n$ we see that u and ∇u is of class $C^k((0,T)\times\mathbb{R}^n)$. Thus the solutions are classical already in case $k=1$, they are C^∞ if the coefficients a and f are C^∞, and they are real analytic if a and f have this property.

We summarize these statements in

Theorem 5.2. *Let $p > n+2$, $k \in \mathbb{N} \cup \{\infty, \omega\}$, assume $a : \mathbb{R}^{n+1} \to \mathbb{R}^{n^2}$ and $f : \mathbb{R}^{n+1} \to \mathbb{R}$ are of class C^k. Suppose that $a_{i,j}(u,v) = a_{j,i}(u,v)$ and that $a(u,v)$ is positive definite for each $u \in \mathbb{R}$, $v \in \mathbb{R}^n$, and let $u_0 \in W^{2-2/p}_p(\mathbb{R}^n)$.*

Then there is a unique solution

$$u \in H^1_{p,loc}(J_0;L_p(\mathbb{R}^n)) \cap L_{p,loc}(J_0;H^2_p(\mathbb{R}^n)) \cap C(J_0;W^{2-2/p}_p(\mathbb{R}^n))$$

of (5.2) with initial condition $u(0,x) = u_0(x)$ on $J_0 = [0,t_{max})$. Moreover,

$$u, \nabla u \in C^k((0,t_{max})\times\mathbb{R}^n),$$

in particular u is a classical solution. If a, f are of class C^∞ then $u \in C^\infty$ as well, and if a, f are real analytic then the solution is real analytic jointly in t and x.

Let us emphasize the two ingredients which make this method of showing regularity work. The method is based on the *implicit function theorem*, for an equation $H((\mu),u) = 0$ in certain function spaces. To be able to apply it one must know that $D_u((1,0),u)$ is an isomorphism. That is precisely where maximal regularity is needed.

We point out that the method breaks down if we consider (5.2) on a domain $G \subset \mathbb{R}^n$ rather than $\mathbb{R}^n$. The reason for this is that the maps $x \mapsto x+t\mu$ is a diffeomorphism of $\mathbb{R}^n$ but it does not leave invariant any domain $G \neq \mathbb{R}^n$. However, recently it was shown that the method can be localized; see Escher, Prüss and Simonett [11]. To explain the basic idea, consider (5.2) on a domain $G \subset \mathbb{R}^n$ with compact C^2-boundary, and prescribe, say, Dirchlet conditions at ∂G. Actually it does not matter which boundary conditions we choose, as long as the Lopatinskii-Shapiro condition holds, which leads to maximal L_p-regularity. So we consider Dirichlet conditions for simplicity.

$$\begin{aligned}&\partial_t u(t,x) - a(u(t,x),\nabla u(t,x)) : \nabla^2 u(t,x) = f(u(t,x),\nabla u(t,x)), \quad t\in(0,T),\ x\in G,\\ &u(t,x)=0,\ t\in J,\ x\in\partial G, \quad u(0,x)=u_0(x),\ x\in G.\end{aligned} \tag{5.3}$$

Again we pick a solution u which exists on a maximal time interval J_0, by Theorems 2.1 and 3.1. Fix a point $(t_0,x_0)\in(0,t_{max})\times G$, where we want to show that u is of class C^k in a neighbourhood of this point. For this purpose we choose 2 cut-off functions $\chi(t)$ and $\psi(x)$ which are C_0^∞ with $\mathrm{supp}\chi\subset(t_0-3r,t_0+3r)\subset(0,t_{max})$ $\chi(t)=1$ for $|t-t_0|<r$, as well as $\mathrm{supp}\psi\subset B_{3r}(x_0)\subset G$, $\psi(x)=1$ for $|x-x_0|<r$. Then we define $u_{\lambda,\mu}$ by means of

$$u_{\lambda,\mu}(t,x) = u(t+\chi(t)\lambda, x+\mu\chi(t)\psi(x)), \quad t\in J_0, x\in G.$$

Since $u_{\lambda,\mu}$ and u differ only in $B_{3r}(t_0)\times B_{3r}(x_0)\subset(0,t_{max})\times G$, we obviously have $u_{\lambda,\mu}(0,x)=u_0(x)$ for $x\in G$, and $u_{\lambda,\mu}(t,x)=u(t,x)=0$ for $t\in J_0$, $x\in\partial G$. Provided $|\lambda|$ and $|\mu|$ are sufficiently small, the map

$$(t,x)\mapsto\theta_{\lambda,\mu}(t,x)=(t+\chi(t)\lambda, x+\mu\chi(t)\psi(x))$$

is a diffeomorphism of $(0,t_{max})\times G$ onto itself. Then $u_{\lambda,\mu}$ will satisfy an equation of the form

$$\partial_t w - a_{\lambda,\mu}(w,\nabla w):\nabla^2 w = f_{\lambda,\mu}(w,\nabla w), \quad t\in(0,t_{max}),\ x\in G.$$

For $\lambda=\mu=0$ we have $u_{0,0}=u$ and $a_{0,0}(w,v)=a(w,v)$ as well as $f_{\lambda,\mu}(w,v)=f(w,v)$ for all $w\in\mathbb{R}$, $v\in\mathbb{R}^n$. In particular, the linearization of this equation at $\lambda=\mu=0$ has again the maximal regularity property. The equation for $u_{\lambda,\mu}$ is considerably more difficult as before, however, it can be shown that the dependence on the parameters (λ,μ) is of class C^k if a and f belong to C^k. Hence we may employ the implicit function theorem once more to obtain that the map $(\lambda,\mu)\mapsto u_{\lambda,\mu}\in H_p^1(J_0,L_p(G))\cap L_p(J_0,H_p^2(G))$ is of class C^k in a neighbourhood of $(0,0)$. By embedding and restriction this then yields that $u,\nabla u$ are C^k in a neighbourhood of (t_0,x_0). Since $(t_0,x_0)\in(0,t_{max})\times G$ is arbitrary, this shows as in case $G=\mathbb{R}^n$ that $u,\nabla u\in C^k((0,t_{max})\times G)$.

Theorem 5.3. *Let $p>n+2$, $k\in\mathbb{N}\cup\{\infty,\omega\}$, assume $a:\mathbb{R}^{n+1}\to\mathbb{R}^{n^2}$ and $f:\mathbb{R}^{n+1}\to\mathbb{R}$ are of class C^k. Suppose that $a_{i,j}(u,v)=a_{j,i}(u,v)$ and that $a(u,v)$ is positive definite for each $u\in\mathbb{R}$, $v\in\mathbb{R}^n$, and let $u_0\in W_p^{2-2/p}(G)$, $u_0(x)=0$ on ∂G.*

Then there is a unique solution

$$u\in H^1_{p,loc}(J_0;L_p(G))\cap L_{p,loc}(J_0;H_p^2(G))\cap C(J_0;W_p^{2-2/p}(G))$$

of (5.3) on a maximal interval $J_0=[0,t_{max})$. Moreover,

$$u,\nabla u\in C^k((0,t_{max})\times G),$$

in particular u is a classical solution. If a,f are of class C^∞ then $u\in C^\infty$ as well, and if a,f are real analytic then the solution is real analytic jointly in t and x.

Note that the localized method explained above can be extended to the case where a and f depend on (t,x) too. For example, we can show this way that if a and f are real analytic on U where $U\subset(0,t_{max})\times G$ is open, then the solution u is real analytic in U as well; see Escher, Prüss and Simonett [11].

6 Linearized Stability

Let X_0 be a Banach space, $X_1 \hookrightarrow X_0$ densely, and let $1 < p < \infty$. In this section we consider the principle of linearized stability for the autonomous quasilinear problem

$$\dot{u}(t) + A(u(t))u(t) = F(u(t)), \quad t > 0, \; u(0) = u_0, \tag{6.1}$$

Here $u_0 \in X_p := (X_0, X_1)_{1-1/p,p}$, $A : X_p \to \mathcal{B}(X_1, X_0)$ and $F : X_p \to X_0$ are locally Lipschitz. We choose the norms in the spaces X_1, X_p and X_0 in such a way that $|v|_0 \le |v|_p \le |v|_1$ holds, for each $v \in X_1$.

Let $u^* \in X_1$ be an equilibrium solution of (6.1) which means that $A(u^*)u^* = F(u^*)$. In the sequel we suppose that the operator $A(u^*)$ has the property of maximal L_p-regularity. Introducing the deviation $v = u - u^*$ from equilibrium, the equation for v then reads as

$$\dot{v}(t) + A_0 v(t) = G(v(t)), \quad t > 0, \; v(0) = v_0, \tag{6.2}$$

where $v_0 = u_0 - u^*$, and $A_0 = A(u^*) + B_1 - B_2$ with $B_j \in \mathcal{B}(X_p, X_0)$ and $G(v) = G_1(v) + G_2(v, v)$, where

$$G_1(v) = [F(u^* + v) - F(v) - B_2 v] - [(A(u^* + v) - A(u^*))u^* - B_1 v], \quad v \in X_p,$$

and

$$G_2(v, w) = (A(u^*) - A(u^* + v))w, \quad v \in X_p, \; w \in X_1.$$

Note that the operators B_1 resp. B_2 resemble the derivatives of $A(\cdot)$ resp. $F(\cdot)$ at the equilibrium u^*. Therefore we assume

(S) There are constants $r > 0$ and $\varepsilon > 0$ such that

$$|A(u^* + v)u^* - A(u^*)u^* - B_1 v|_0 \le \varepsilon |v|_p, \quad |v|_p \le r,$$

as well as

$$|F(u^* + v) - F(u^*) - B_2 v|_0 \le \varepsilon |v|_p, \quad |v|_p \le r,$$

and there is $L > 0$ such that

$$|A(u^* + v) - A(u^*)|_{\mathcal{B}(X_1, X_0)} \le L|v|_p, \quad |v|_p \le r.$$

In particular, if $A(u)$ and $F(u)$ are Frechét differentiable at u^* in X_p then **(S)** holds with $B_1 = A'(u^*)u^*$ and $B_2 = F'(u^*)$, and $\varepsilon = \varepsilon(r) \to 0$ as $r \to 0$

We note that **(S)** implies

$$|G(v)|_0 \le 2\varepsilon |v|_p + L|v|_1 |v|_p \le (2\varepsilon + L\rho)|v|_1, \quad v \in X_1, \; |v|_p \le \rho \le r. \tag{6.3}$$

The stability result reads as follows.

Theorem 6.1. *Let $1 < p < \infty$, assume that $u^* \in X_1$ is an equilibrium for (6.1), and let $B_j \in \mathcal{B}(X_p, X_0)$ be such that* **(S)** *is satisfied. Suppose that $A(u^*)$ has maximal regularity of type L_p and that the spectrum $\sigma(A_0)$ of $A_0 := A(u^*) + B_1 - B_2$ is contained in the open right half-plane $\mathbb{C}_+ = \Sigma_{\pi/2}$. Assume that $\varepsilon > 0$ is sufficiently small.*

Then there is $\rho \in (0,r)$ such that for each $u_0 \in B_\rho(u^*) \subset X_p$ there is a unique global solution $u \in H^1_{p,loc}(R_+;X_0) \cap L_{p,loc}(\mathbb{R}_+;X_1)$ of (6.1). There is $\delta > 0$ such that

$$e^{\delta t} u \in H^1_p(R_+;X_0) \cap L_p(\mathbb{R}_+;X_1) \cap C_0(\mathbb{R}_+;X_p),$$

in particular, the equilibrium u^* of (6.1) is exponentially stable in the phase space X_p.

Proof. Since B_j are perturbations of $A(u^*)$ of lower order, A_0 has again the maximal regularity property, and so by the spectral condition on A_0, there is $\delta_0 > 0$ such that $A_\delta = A_0 - \delta$ has the maximal regularity property on $\mathbb{R}_+$, for each $\delta \in [0, \delta_0]$. Thus there is a constant $C_0 \geq 1$ such that the solution of

$$\dot{w}(t) + A_0 w(t) = f(t), \quad t > 0, \ w(0) = 0,$$

satisfies

$$||\dot{w}e^{\delta t}||_0 + ||we^{\delta t}||_1 \leq C_0 ||fe^{\delta t}||_0,$$

for each $f \in L_{p,loc}(\mathbb{R}_+;X_0)$ such that $||fe^{\delta t}||_0 < \infty$, and $\delta \in [0,\delta_0]$. By the embedding theorem this implies $we^{\delta_0 t} \in C_0(\mathbb{R}_+;X_p)$ and for all $\delta \in [0,\delta_0]$

$$|w(t)|_p \leq C_1[||we^{\delta t}||_1 + ||\dot{w}e^{\delta t}||_0]e^{-\delta t}, \ t \geq 0,$$

with some constant $C_1 \geq 1$ independent of f.

Fix $\rho < r$ and let $u_0 \in B_\rho(u^*) \subset X_p$, $v_0 = u_0 - u^*$. By Theorem 3.1, there is a unique solution v of (6.2) on some maximal interval. Let $t_0 > 0$ be such that $|v(t)|_p < \rho$ for $t < t_0$ and $|v(t_0)|_p = \rho$; put $t_0 = \infty$ if no such t_0 exists. By (6.3), on the interval $[0,t_0)$ we have the estimate

$$|G(v(t))|_0 \leq (2\varepsilon + L\rho)|v(t)|_1,$$

which implies with maximal regularity on $[0,t_0)$

$$\begin{aligned} ||v||_1 &\leq ||e^{-A_0 t}v_0||_1 + C_0||G(v)||_0 \\ &\leq C_2|v_0|_p + (2\varepsilon + L\rho)C_0||v||_1 \\ &\leq C_2\rho + C_0[2\varepsilon + L\rho]||v||_1. \end{aligned}$$

If $\rho > 0$ is chosen such that $L\rho C_0 < 1/4$ and $\varepsilon > 0$ is so small that $\varepsilon < (8C_0)^{-1}$ then we obtain the estimate $||v||_1 \leq 2C_2\rho$, which yields by (6.2)

$$\begin{aligned} ||\dot{v}||_0 &\leq C_3||v||_1 + ||G(v)||_0 \leq (C_3 + 2\varepsilon + L\rho)||v||_1 \\ &\leq (2C_3 + 1)C_2\rho, \end{aligned}$$

where $C_3 = |A_0|_{\mathcal{B}(X_1,X_0)}$ By embedding this implies

$$|v(t_0)| \leq C_1||v||_1 \leq C_1(2C_3+1)C_2\rho < r,$$

a contradiction if in addition we choose from the beginning $\rho < r/C_1(2C_3+1)C_2$. Therefore the solutions with initial value $u_0 \in B_\rho(u^*)$ exist globally.

To obtain the stability result, we use again the maximal regularity estimate, this time on the half-line, with $\delta > 0$

$$\begin{aligned} ||ve^{\delta t}||_1 &\leq ||e^{-A_\delta t}v_0||_1 + ||e^{-A_\delta t} * e^{\delta t}G(v)||_1 \\ &\leq C_2|v_0|_p + C_0[2\varepsilon + L\rho]||ve^{\delta t}||_1, \end{aligned}$$

hence as before

$$|v(t)|_p \leq C_1 e^{-\delta t}||ve^{\delta t}||_1 \leq 2C_1C_2e^{-\delta t}|v_0|_p.$$

This proves the theorem. □

For a corresponding result on instability, we need to assume that a part of the spectrum of A_0 in the negative half-plane is strictly separated from the rest of $\sigma(A_0)$.

Theorem 6.2. *Let $1 < p < \infty$, assume that $u^* \in X_1$ is an equilibrium for (6.1), and let $B_j \in \mathcal{B}(X_p, X_0)$ be such that* **(S)** *is satisfied. Suppose that $A(u^*)$ has maximal regularity of type L_p and that the spectrum $\sigma(A_0)$ of $A_0 := A(u^*) + B_1 - B_2$ decomposes into two sets $\sigma_\pm$ such that $\sigma_+ \subset -\mathbb{C}_+ - \delta$ and $\sigma_- \subset \mathbb{C}_+ - \delta$, for some $\delta > 0$. Assume that $\varepsilon > 0$ is sufficiently small.*

Then u^* is unstable in the phase space X_p. More precisely, there is $\rho > 0$ such that for $\eta > 0$ there is $u_\eta \in X_p$ with $|u_\eta - u^*|_p \leq \eta$, but the solution $u_\eta(t)$ of (6.1) satifies $|u_\eta(t_\eta) - u^*|_p > \rho$ for some positive time t_η.

Proof. Let $P_\pm$ denote the spectral projections of A_0 corresponding to the spectral sets $\sigma_\pm$. Since $A(u^*)$ has the property of L_p-maximal regularity, $A_0P_- + \delta$ has maximal regularity even on the half-line, hence there is constant $C_0 > 0$ such that

$$||e^{-\delta t}e^{-A_0 t}P_- f||_1 \leq C_0||e^{-\delta t}f||_0,$$

for each $f : \mathbb{R}_+ \to X_0$ such that $||e^{-\delta t}f||_0 < \infty$. On the other hand, $A_+ := A_0P_+$ is bounded since σ_+ is so, hence $e^{A_+ t}$ extend to a group on $\mathcal{R}(P_+)$. Furthermore, by the spectral condition we may change the norm on $\mathcal{R}(P_+)$ in such a way that $|e^{-A_+}P_+| \leq e^{(\eta+\delta)t}$ for $t \in \mathbb{R}$, for some sufficiently small $\eta > 0$. This simplies that $-A_+ - \delta - \eta$ is accretive on $\mathcal{R}(P_+)$. Note that, by boundedness of A_+, the norms $|P_+v|_0$ and $|P_+v|_1$ are equivalent, hence there is a constant $c \geq 1$ such that $|P_+v|_0 \leq |P_+v|_1 \leq c|P_+v|_0$.

Fix any $\rho < r$ and suppose that $v(t)$ is a solution of (6.2) which stays inside the ball $B_\rho(0) \subset X_p$, with initial value v_0 such that $P_-v_0 = 0$. Then from estimate (6.3) we obtain

$$||e^{-\delta t}P_-v||_1 \leq C_0||e^{-\delta t}P_-Gv||_0 \leq C_0(2\varepsilon + L\rho)[||e^{-\delta t}P_-v||_1 + c||e^{-\delta t}P_+v||_0],$$

hence in case $C_0(2\varepsilon + L\rho) < 1/2$ we obtain

$$||e^{-\delta t}P_-v||_1 \leq 2cC_0(2\varepsilon + L\rho)||e^{-\delta t}P_+v||_0. \tag{6.4}$$

Set $\phi(t) = e^{-\delta t}|P_+v(t)|_0$. Then we have by accretivity of $-A_+ - \delta - \eta$ and Young's inequality

$$\frac{d}{dt}\phi^p(t) \;=\; pe^{-\delta t}\phi^{p-1}[\frac{d}{dt}|P_+v|_0 - \delta|P_+v|_0]$$

$$
\begin{aligned}
&\geq p\eta\phi^p - p\phi^{p-1}e^{-\delta t}|G(v)|_0 \\
&\geq p\eta\phi^p - p\phi^{p-1}e^{-\delta t}(2\varepsilon + L\rho)|v(t)|_1 \\
&\geq p\eta\phi^p - (p/p')(2\varepsilon + L\rho)\phi^p - (2\varepsilon + L\rho)[|v(t)|_1 e^{-\delta t}]^p \\
&\geq p(\eta - (2\varepsilon + L\rho)/p')\phi^p - 2^p(2\varepsilon + L\rho)([|P_-v(t)|_1 e^{-\delta t}]^p + [|P_+v(t)|_1 e^{-\delta t}]^p) \\
&\geq [p(\eta - (2\varepsilon + L\rho)/p') - 2^p(2\varepsilon + L\rho))]\phi^p - 2^p(2\varepsilon + L\rho)[|P_-v(t)|_1 e^{-\delta t}]^p.
\end{aligned}
$$

Setting

$$
c_1 = p(\eta - (2\varepsilon + L\rho)/p') - 2^p(2\varepsilon + L\rho), \quad c_2 = 2^p(2\varepsilon + L\rho),
$$

we thus have shown

$$
\frac{d}{dt}\phi^p(t) \geq c_1\phi^p(t) - c_2[P_-v(t)|_1 e^{-\delta t}]^p.
$$

Integrating this inequality over $(0,t)$ and using (6.4), this leads to

$$
\begin{aligned}
\phi(t)^p &\geq \phi(0)^p + c_1\int_0^t \phi^p(s)ds - c_2\int_0^t [|P_-v(s)|_1 e^{-\delta t}]^p ds \\
&\geq \phi(0)^p + (c_1 - c_2[cC_0(2\varepsilon + L\rho)]^p)\int_0^t \phi^p(s)ds,
\end{aligned}
$$

Now if $\varepsilon > 0$ and $\rho > 0$ are chosen small enough such that

$$
p(\eta - (2\varepsilon + L\rho)/p') - 2^p(2\varepsilon + L\rho) - 4^p c^p C_0^p(2\varepsilon + L\rho)^{1+p} \geq 0,
$$

we obtain $\phi(t) \geq \phi(0)$ which means $|P_+v(t)|_0 \geq e^{\delta t}|P_+v_0|_0$. Since $\delta > 0$ this leads to a contradiction whenever $P_+v_0 \neq 0$, i.e. every such solution $v(t)$ of (6.2) must leave the ball $B_\rho(0) \subset X_p$ in finite time. This completes the proof. □

Observe that we have proved the stronger statement that the solutions with initial value $P_-v_0 = 0$ become unbounded in the norm of X_0 rather than X_p. It is not difficult to extend the assertion to initial conditions with $P_-v_0 \neq 0$ small enough.

7 The Saddle Point Property

Let X_0 be a Banach space, $X_1 \hookrightarrow X_0$ densely, and let $1 < p < \infty$. In this section we consider the so-called saddle-point property for the autonomous quasilinear problem

$$
\dot{u}(t) + A(u(t))u(t) = F(u(t)), \quad t > 0,\ u(0) = u_0, \tag{7.1}
$$

Here $u_0 \in X_p := (X_0, X_1)_{1-1/p,p}$, $A : X_p \to \mathcal{B}(X_1, X_0)$ and $F : X_p \to X_0$ are of class C^1.

Let $u^* \in X_1$ be an equilibrium solution of (6.1), i.e. $A(u^*)u^* = F(u^*)$. As in the previous section, we suppose that the operator $A(u^*)$ has the property of maximal L_p-regularity. Introducing the deviation $v = u - u^*$ from equilibrium, the equation for v then reads as

$$
\dot{v}(t) + A_0 v(t) = G(v(t)), \quad t > 0,\ v(0) = v_0, \tag{7.2}
$$

where $v_0 = u_0 - u^*$, and $A_0 = A(u^*) + A'(u^*)u^* + F'(u^*)$. G decomposes as $G(v) = G_1(v) + G_2(v,v)$, with

$$
G_1(v) = [F(u^* + v) - F(v) - F'(u^*)v] + [(A(u^* + v) - A(u^*) - A'(u^*)v]u^*, \quad v \in X_p,
$$

and

$$G_2(v, w) = (A(u^*) - A(u^* + v))w, \quad v \in X_p,\ w \in X_1.$$

By the differentiability assumtions on F and A, it follows that G_1 is of class C^1 from X_p to X_0 and $G_2 : X_p \times X_1 \to X_0$ is C^1 and linear in the second variable; moreover, $G_1(0) = G_2(0,0) = 0$ as well as $G_1'(0) = G_2'(0,0) = 0$.

The main assumption for the saddle-point property is the so-called **hyperbolicity** of the equilibrium $u^* \in X_1$. This means that the spectrum of A_0 does not intersect the imaginary axis. Specifically, we assume that $\sigma(A_0)$ decomposes into two disjoint nontrivial parts,

$$\sigma(A_0) = \sigma_+ \cup \sigma_-, \quad -\sigma_+ \cup \sigma_- \subset \mathbb{C}_+,$$

σ_- corresponds to the stable, σ_+ to the unstable part of the analytic C_0-semigroup $e^{-A_0 t}$, or equivalently of the Cauchy problem $\dot{w} + A_0 w = f$. Let P_+ resp. P_- denote the spectral projections according to these spectral sets, and let A_+ resp. A_- be defined as the restrictions of A_0 to the spaces $X_\pm := P_\pm X_0$. Observe that maximal regularity inherits to A_- and $\sigma(A_-) = \sigma_- \subset \mathbb{C}_+$ shows that A_- has maximal regularity on the half-line. On the other hand, A_+ is bounded since σ_+ has this property , hence $\mathcal{D}(A_+) = X_+ := P_+ X_0$, and $e^{-A_+ t}$ is entire. Moreover, there is a number $\delta > 0$ such that $A_- - \delta$ still has maximal regularity on $\mathbb{R}_+$, hence there is $C_0 \geq 1$ such that the solutions of $\dot{w} + A_- w = f$, $w(0) = 0$, satisfy

$$||e^{\sigma t} w||_1 \leq C_0 ||e^{\sigma t} f||_0, \quad \sigma \in [-\delta, \delta].$$

Further $|e^{-A_+ t}| \leq M e^{\delta t}$, for all $t \in \mathbb{R}$, and $|P_+ x|_1 \leq c|P_+ x|_0$ since the norms $|\cdot|_0$ and $|\cdot|_1$ are equivalent on $P_+ X_0$. Finally, let C_1 denote a constant such that

$$||e^{\delta t} e^{-A_0 t} P_- x||_1 \leq C_1 |x|_p.$$

After these preparations we can state the following result, which, in analogy to ode's, we call the **saddle point property** for quasilinear parabolic evolution equations.

Theorem 7.1. *Let $1 < p < \infty$, suppose $u^* \in X_1$ is an equilibrium of (7.1), and suppose that for some ball $B_R(0) \subset X_p$, $A : B_R(u^*) \to \mathcal{B}(X_1, X_0)$ and $F : B_R(u^*) \to X_0$ are of class C^1. Assume further that $A(u^*)$ has the property of maximal L_p-regularity, and that $A_0 := A(u^*) + A'(u^*)u^* - F'(u^*)$ is hyperbolic in the sense described above.*

Then there exist $0 < \eta \leq \rho$ and C^1-manifolds $\mathcal{M}_s$ and $\mathcal{M}_u$ near zero , $0 \in \mathcal{M}_s \cap \mathcal{M}_u$, with tangential spaces $\mathcal{R}(P_-)$ and $\mathcal{R}(P_+)$ in 0 such that for each solution $u(t; u_0)$ of (7.1) with $u_0 \in B_\eta(u^*)$ the following properties are valid.

(i) *If $u_0 \in \mathcal{M}_s$, then $|u(t; u_0)|_p \leq \rho$ for $t > 0$ and $|u(t, u_0)|_p \to 0$ exponentially as $t \to \infty$.*

(ii) *If $u_0 \notin \mathcal{M}_s$, then $|u(t; u_0)|_p > \rho$ for some $t > 0$.*

(iii) *If $u_0 \in \mathcal{M}_u$, then $u(t; u_0)$ exists on $(-\infty, 0]$ and is unique in the set $|u(t; u_0)|_p \leq \rho$; moreover $|u(t, u_0)|_p \to 0$ exponentially as $t \to -\infty$.*

(iv) *If $u_0 \notin \mathcal{M}_u$, any backward solution of $u(t; u_0)$ either ceases to exist or leaves the ball $B_\rho(u^*) \subset X_p)$ at some finite negative time.*

$\mathcal{M}_s$ *and* $\mathcal{M}_u$ *are called the* **stable** *resp.* **unstable** *manifolds for (7.1) at* u^**.*

Proof of Theorem 7.1.
Since A and F are asumed to be of class C^1 in some ball $B_R(u^*) \subset X_p$, for each $\varepsilon > 0$ and there are $r > 0$ and $L > 0$ such that

$$|F(u^* + v) - F(u^*) - F'(u^*)v|_0 \leq \varepsilon |v|_p, \quad v \in B_r(0),$$

$$|[A(u^* + v) - A(u^*) - A'(u^*)v]u^*|_0 \leq \varepsilon |v|_p, \quad v \in B_r(0),$$

and

$$|[A(u^* + v) - A(u^*)]v|_0 \leq L|v|_p|v|_1 \leq Lr|v|_1, \quad v \in B_r(0), v \in X_1.$$

Let $\varepsilon_0 > 0$ and $r_0 > 0$ as well as $L_0 > 0$ be fixed such that

$$(C_0 + cM/2\delta)(2\varepsilon + L_0 r_0) \leq 1/2.$$

(a) We first construct the stable manifold $\mathcal{M}_s$. For this purpose consider the integral equation

$$z(t) = e^{-A_0 t}x + \int_0^t e^{-A_0(t-s)}P_- G(z(s))ds - \int_t^\infty e^{-A_0(t-s)}P_+ G(z(s))ds, \quad t > 0, \tag{7.3}$$

where $x \in \mathcal{R}(P_-)$. To solve it, we define the space Z_1 by

$$Z_1 := \{z : (0, \infty) \to X_1 : e^{\delta t}z \in H_p^1(\mathbb{R}_+ X_0) \cap L_p(\mathbb{R}_+; X_1)\},$$

with norm

$$||z||_{Z_1} := ||e^{\delta t}\dot{z}||_0 + ||e^{\delta t}z||_1.$$

Let $\mathbb{B}_r(0) \subset Z_1$ denote the ball in Z_1 around 0 with radius $r > 0$, and consider the map $H : P_- X_0 \times \mathbb{B}_r(0) \to Z_1$ defined by

$$H(x, z) := z(t) - e^{-A_0 t}x - \int_0^t e^{-A_0(t-s)}P_- G(z(s))ds + \int_t^\infty e^{-A_0(t-s)}P_+ G(z(s))ds, \quad t > 0, \tag{7.4}$$

and set $Z_0 := \{u : (0, \infty) \to X_0 : \ e^{\delta t}u \in L_p(\mathbb{R}_+; X_0)\}$ with natural norm. Let $C_2 \geq 1$ denote the embedding constant for $Z_1 \hookrightarrow X_p$, i.e.

$$|z(t)|_p \leq C_2||z||_{Z_1}, \quad z \in Z_1.$$

We then choose $r > 0$ such that $C_2 r < r_0$; this ensures that each function in $\mathbb{B}_r(0)$ stays in the ball $B_{r_0}(0) \subset X_p$. Since G is of class C^1 then it easily follows that G extends as a C^1-map from $\mathbb{B}_r(0) \subset Z_1$ to Z_0. Since the remaining operators in the definition of H are linear, it is obvious that H belongs to C^1. Moreover, we have $H(0, 0) = 0$ as well as

$$D_x H(x, z)(t) = -e^{-A_0 t}P_-,$$

$$D_z H(x, z)(t) = I - \int_0^t e^{-A_0(t-s)}P_- G'(z(s) \cdot ds + \int_z^\infty e^{-A_0(t-s)}P_+ G'(z(s)) \cdot ds,$$

in particular $D_z H(0, 0) = I$ since $G'(0) = 0$.

The implicit function theorem thus yields balls $B_\eta(0) \subset P_-X_0$ and $\mathbb{B}_\rho(0) \subset \mathbb{B}_r(0) \subset Z_1$ and a C^1-function $\Phi : B_\eta(0) \to \mathbb{B}_\rho(0)$ with $\Phi(0) = 0$ and $H(x, \Phi(x)) = 0$ for each $x \in B_\eta(0)$, and there is no other solution of $H(x, z) = 0$ than $\Phi(x)$ in the ball $\mathbb{B}_\rho(0)$.

Define $z(t; x) := \Phi(x)(t)$ and set $\phi(x) = x - \Phi(x)(0)$; note that

$$\phi(x) = \int_0^\infty e^{A_+ s} P_+ G(z(s; x))ds, \quad x \in B_\eta(0).$$

The manifold $\mathcal{M}_s$ in X_p can now be defined according to

$$\mathcal{M}_s := \{x - \phi(x) : \ x \in B_\eta(0) \subset P_-X_0\}. \tag{7.5}$$

Since Z_1 embeds into $BUC(\mathbb{R}_+; X_p)$ it is obvious that ϕ is of class C^1, hence $\mathcal{M}_s$ is a C^1-manifold in X_p. Moreover, we have

$$D_x(0, 0) + D_z(0, 0)\Phi'(0) = 0,$$

hence $\phi'(0) = \Phi'(0)(0) = 0$, which shows that the tangential space of $\mathcal{M}_s$ at zero is given by P_-X_0.

Writing (7.3) for $z(t; x)$ as

$$z(t; x) = e^{-A_0 t}(x - \phi(x)) + \int_0^t e^{-A_0(t-s)} G(z(s; x))ds, \quad t > 0,$$

we see that $z(t; x)$ is the solution of (7.1) with initial value $x - \phi(x)$. This shows that the solutions of (7.1) with initial conditions in $\mathcal{M}_s$ exist globally, remain in the ball $B_{\rho_0}(0) \subset X_p$, $\rho_0 := C_2\rho$, and converge to zero exponentially in the norm of X_p. This proves assertion (i).

On the other hand, suppose that $u(t)$ is a solution of (7.1) which stays in the ball $B_{\rho_0}(u^*) \subset X_p$ for all $t > 0$, and satisfies $|P_-(u(0) - u^*)|_p < \eta$; let $v(t) = u(t) - u^*$. Then $|P_+v(t)|_1$ is bounded on $\mathbb{R}_+$ as well since the norms $|\cdot|_1$ and $|\cdot|_0$ are equivalent on P_+X_0. Next, we have on $J = \mathbb{R}_+$ with $\sigma \in (0, \delta)$,

$$||e^{-\sigma t} P_- v||_1 \le C_1|P_-v(0)|_p + (2\varepsilon_0 + L_0 r_0)C_0||e^{-\sigma t}P_-v||_1 + (2\varepsilon_0 + L_0 r_0)C_0 c||e^{-\sigma t}P_+v||_0,$$

which implies that $||e^{-\sigma t}P_-v(t)||_1 < \infty$. Thus the estimate

$$|\int_t^\infty e^{-A_+(t-s)} P_+ G(v(s)ds|_0 \le M(2\varepsilon_0 + L_0 r_0)\int_t^\infty e^{\delta(t-s)}|v(s)|_1 ds \le Ce^{\sigma t},$$

with some constant $C \ge 0$ shows that $\int_t^\infty e^{-A_+ t}P_+G(v(s)ds$ exists and is bounded by $Ce^{\sigma t}$. Therefore we may write

$$P_+v(t) = e^{-A_+ t}[P_+v(0) + \int_0^\infty e^{A_+ s}P_+G(v(s))ds] - \int_t^\infty e^{-A_+(t-s)}P_+G(v(s))ds.$$

Since $P_+v(t)$ is bounded and the second integral term is bounded by $Ce^{\sigma t}$, $\sigma < \delta$, we may conclude

$$P_+v(0) + \int_0^\infty e^{A_+ s} P_+ G(v(s))ds = 0,$$

since $|e^{-A_+t}x|_0 \geq me^{\delta t}$ for some $m > 0$, by the spectral properties of A_+. Therefore,

$$v(t) = e^{-A_-t}P_-v(0) + \int_0^t e^{-A_-(t-s)}P_-G(v(s))ds - \int_t^\infty e^{-A_+(t-s)}P_+G(v(s))ds,$$

which implies $v(t) = z(t; P_-v(0))$, in particular

$$v(0) = z(0; P_-v(0)) = P_-v(0) - \int_0^\infty e^{A_+s}P_+G(z(s, P_-v(0)))ds \in \mathcal{M}_s,$$

provided we know $||v||_1 < \rho$. To prove the latter we estimate as follows.

$$\begin{aligned}
||v||_{Z_1} &= ||e^{\delta t}\dot{v}||_0 + ||e^{\delta t}v||_1 \\
&\leq (1 + |A_0|_{\mathcal{B}(X_1,X_0)})||e^{\delta t}v||_1 + ||e^{\delta t}G(v)||_0 \\
&\leq (1 + |A_0|_{\mathcal{B}(X_1,X_0)} + 2\varepsilon_0 + L_0r_0)||e^{\delta t}v||_1,
\end{aligned}$$

and by the integral equation for $v(t)$

$$\begin{aligned}
||e^{\delta t}v||_1 &\leq ||e^{\delta t}e^{-A_-t}P_-v(0)||_1 + C_0||e^{\delta t}G(v)||_0 + Mc||e^{\delta t}\int_t^\infty e^{\delta(t-s)}|G(v(s))|_0 ds||_0 \\
&\leq C_1|P_-v(0)|_p + C_0(2\varepsilon_0 + L_0r_0)||e^{\delta t}v||_1 + (Mc/2\delta)(2\varepsilon_0 + L_0r_0)||e^{\delta t}v||_1 \\
&\leq C_1|P_-v(0)|_p + (C_0 + Mc/2\delta)(2\varepsilon_0 + L_0r_0)||e^{\delta t}v||_1 \\
&\leq C_1|P_-v(0)|_p + \frac{1}{2}||e^{\delta t}v||_1
\end{aligned}$$

which yields

$$||e^{\delta t}v||_1 \leq 2C_1|P_-v(0)|_p \leq 2C_1\eta,$$

and therefore

$$||v||_{Z_1} \leq (1 + |A_0|_{\mathcal{B}(X_1,X_0)} + 2\varepsilon_0 + L_0r_0)||e^{\delta t}v||_1 \leq (1 + |A_0|_{\mathcal{B}(X_1,X_0)} + 2\varepsilon_0 + L_0r_0)2C_1\eta < \rho,$$

where $\eta > 0$ may be decreased if necessary.

Note that this shows also the positive as well as negative invariance of $\mathcal{M}_s$. Thus the construction of $\mathcal{M}_s$ is complete and assertion (ii) is proved.

(b) The arguments to obtain $\mathcal{M}_u$ are similar, one only has to reverse the direction of time. However, here one must be careful since solutions in general do not extend to negative times. We consider the integral equation

$$z(t) = e^{-A_0t}x - \int_t^0 e^{-A_0(t-s)}P_+G(z(s))ds + \int_{-\infty}^t e^{-A_-(t-s)}P_-G(z(s))ds, \quad t < 0, \quad (7.6)$$

where $x \in \mathcal{R}(P_+)$. As before we find η, ρ, z. $\mathcal{M}_u$ has the representation

$$\mathcal{M}_u := \{x + \psi(x) : \ x \in \mathcal{R}(P_+), \ |x|_p < \eta\},$$

with $\psi(x) = \int_{-\infty}^0 e^{A_-s}P_-G(z(s;x))ds$. Obviously, $z(t;x)$ is a solution of (7.2) for all $t < 0$ with final value $z(0) = x - \psi(x) \in \mathcal{M}_u$, i.e. each initial value $v_0 \in \mathcal{M}_u$ admits a backwards solution v such that $v(t) \to 0$ exponentially in X_p as $t \to -\infty$.

Conversely suppose that $v(t)$ is a backward solution of (7.2) on $\mathbb{R}_-$ which stays in the ball $B_\rho(0) \subset X_p$, $|P_+v|_p < \eta$. Then we see as before that

$$P_-v(0) + \int_{-\infty}^{0} e^{A_- s} P_- G(v(s))ds = 0,$$

hence

$$v(t) = e^{-A_+ t} P_+ v(0) - \int_t^0 e^{A_+(t-s)} P_+ G(v(s))ds + \int_{-\infty}^{t} e^{-A_-(t-s)} P_- G(v(s))ds, \quad t < 0,$$

and so $v(t) = z(t; P_+v(0))$ by uniqueness, provided $\eta > 0$ is chosen small enough. This completes the construction of the unstable manifold. $\square$

We conclude with a simple example illustrating Theorems 6.1, 6.2, and 7.1.

Example 7.2. Let $G \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary, and consider the problem

$$\begin{aligned} \partial_t u - a(u, \nabla u)\Delta u = f(u, \nabla u), \quad t > 0,\ x \in G, \\ \partial_\nu u = 0, \quad t > 0,\ x \in \partial G, \\ u = u_0, \quad t = 0,\ x \in G, \end{aligned} \tag{7.7}$$

where $f \in C^1(\mathbb{R}^{n+1}; \mathbb{R})$ and $a \in C^1(\mathbb{R}^{n+1}; \mathbb{R})$, $a(u, v) > 0$ for all $u \in \mathbb{R}$, $v \in \mathbb{R}^n$. Then this problem is locally well-posed in $L_q(G)$, $n < q < \infty$, and solutions are classical by the results of Section 5. Because of the Neumann condition, each zero u^* of $f(\cdot, 0)$ defines an equilibrium of (7.7). Let us discuss its stability. The spectrum of the Laplacian consists of countably many eigenvalues

$$0 = \lambda_0 > \lambda_1 \geq \ldots \geq \lambda_j \geq \ldots, \quad j \in \mathbb{N}_0.$$

The linearization operator at such a constant steady state is given by

$$\begin{aligned} -Aw &= a(u^*, 0)\Delta w + (a_u(u^*, 0)w + a_v(u^*, 0)\nabla w)\Delta u^* + f'(u^*, 0)w = \\ &= a(u^*, 0)\Delta w + f'(u^*, 0)w, \end{aligned}$$

hence the spectrum of $-A$ consists of eigenvalues $\mu_j = a(u^*, 0)\lambda_j + f'(u^*, 0)$. Thus in case $f'(u^*, 0) < 0$ this equlibrium is asymptotically stable by Theorem 6.1, and it is unstable if $f'(u^*, 0) > 0$, by Theorem 6.2. u^* is hyperbolic iff $\mu_j \neq 0$ for all $j \in \mathbb{N}_0$. Assuming this, let $k \in \mathbb{N}$ be such that $\mu_{k+1} < 0 < \mu_k$; then by Theorem 7.1 the stable and unstable manifolds $\mathcal{M}_s$ and $\mathcal{M}_u$ at u^* exist, and the dimension of $\mathcal{M}_u$ is the sum of the algebraic multiplies of the eigenvalues $\lambda_0, \ldots \lambda_k$ of Δ.

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