ASYMPTOTIC BEHAVIOUR OF PARABOLIC NONAUTONOMOUS EVOLUTION EQUATIONS

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1. Introduction

The long term behaviour of autonomous linear Cauchy problems on a Banach space X has been studied systematically and quite successfully by means of spectral theory and transform methods, see e.g. [8], [21], [50]. These techniques fail (almost) completely in the nonautonomous case if one tries to generalize them directly. In fact, already nonautonomous ordinary differential equations lead to considerable difficulties and the available results are mostly restricted to perturbation type arguments, cf. [15], [16]. In view of the applications to inhomogeneous and nonlinear problems, we concentrate on the 'exponential dichotomy' of the linear homogeneous equation

$$u'(t) = A(t)u(t), \quad t > s, \qquad u(s) = x,$$

which is a fundamental concept in the study of asymptotic properties of evolution equations, compare [12], [27], [42], [54]. There are several ways to characterize this notion in abstract terms as described in [12], [27], [35], [58]. Here we prefer a characterization employing the spectra of an associated 'evolution semigroup' acting on $C_0(\mathbb{R}, X)$.

In a series of papers, [55], [56], [57], [59], we applied this characterization to establish the exponential dichotomy of parabolic evolution equations in certain situations. Here we generalized theorems from D. Henry's monograph [27] dealing with a special class of parabolic problems. We improve our previous results in the core Section 4, where we show robustness of dichotomy and treat asymptotically autonomous problems and equations with slowly oscillating coefficients. In contrast to our previous work, we allow for operators $A(t)$ with non–dense domains arising in the case of Dirichlet type boundary conditions in a sup–norm setting. This new aspect has forced us to change the arguments at several points, besides other simplifications and improvements. In particular we had to extend the spectral theory of evolution semigroups to certain evolution families (or, propagators) $U(t, s)$ not being strongly continuous at $t = s$. In Section 3 this theory is developed in detail and in a self–contained way. Here we also collect additional properties of exponential dichotomy.

Another major ingredient of our approach is the existence and regularity theory of parabolic nonautonomous evolution equations due to P. Acquistapace, B. Terreni, and others, which is discussed in the next section. Proofs of these powerful and difficult theorems cannot be given here, but we show a few additional regularity and approximation

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results needed later. We also give a brief introduction to interpolation spaces in the present context. Extrapolation spaces are needed in Section 4.3, where we recall the relevant properties.

In Sections 5 and 6 the main results of Section 4 are applied to inhomogeneous and quasilinear problems, respectively. Here we establish the existence of convergent solutions to asymptotically autonomous equations (extending an old result of Tanabe, see [62], on inhomogeneous problems) and of almost periodic solutions to almost periodic inhomogeneous equations. All results are illustrated by examples of parabolic partial differential equations of second order in divergence and non–divergence form formulated on $X = L^p(\Omega)$ or $X = C(\overline{\Omega})$.

It is supposed that the reader is familiar with basic spectral theory of semigroups and with regularity properties of analytic semigroups. Our arguments make use of standard results and methods of operator theory and the functional analytic treatment of (parabolic) partial differential equations (in the spirit of Henry's book, say). We strived for a rather self–contained and systematic presentation. Of course, these lecture notes do not exhaust the range of possible results and applications, but they should give a concise picture of the field. Thus we hope that they can serve as an introduction to the subject.

Notation. We denote by $D(A)$, $A\sigma(A)$, $\sigma(A)$, $\rho(A)$, and A^* the domain, (approximate point) spectrum, resolvent and adjoint of a closed (densely defined) linear operator A. We set $R(\lambda, A) := (\lambda - A)^{-1}$. X^* is the dual of the Banach space X, $\mathcal{L}(X, Y)$ is the space of bounded linear operators, $\mathcal{L}(X) := \mathcal{L}(X, X)$, and a subscript '$s$' indicates that $\mathcal{L}(X, Y)$ is endowed with the strong operator topology. Function spaces are designated as usual and endowed with their standard norms, where the subscript 'c' means 'compact support', $C_0(U, X)$ contains continuous functions vanishing at boundary points not belonging to U and at infinity, $C_b(U, X)$ is the space of bounded continuous functions, and $C_b^\alpha(U, X)$ the space of bounded, globally Hölder continuous functions, BUC means 'bounded and uniformly continuous'. Further, $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. By $c = c(\alpha, \beta, \cdots)$ we denote a generic constant only depending on the constants in the hypotheses involved and the quantities $\alpha, \beta, \cdots$.

2. Parabolic evolution equations

In this section we review several results on the existence and regularity of solutions to the parabolic evolution equation

$$\begin{aligned} \tfrac{d}{dt}\, u(t) &= A(t)u(t) + f(t), \qquad t > s, \\ u(s) &= x, \end{aligned} \tag{2.1}$$

for linear operators $A(t)$, $t \in J$, with domains $D(A(t))$ on a Banach space X, where $J \in \{[a, \infty), \mathbb{R}\}$ is the underlying time interval, $s \in J$, $x \in X$, and $f \in C(J, X)$. We concentrate on properties which will be used later on. Proofs are only given for a few facts which do not appear in the standard literature, but are important for our approach. We use the following notions of solutions.

Definition 2.1. *A* classical solution *of* (2.1) *is a map* $u \in C([s, \infty), X) \cap C^1((s, \infty), X)$ *such that* $u(t) \in D(A(t))$ *for* $t > s$ *and* (2.1) *holds. If additionally* $u \in C^1([s, \infty), X)$, $x \in D(A(s))$, *and* $A(t)u(t) \to A(s)x$ *as* $t \to s$, *then* u *is called a* strict solution.

There is no unified theory for nonautonomous linear evolution equations. We surveyed various well–posedness theorems and the corresponding methods in [58], where one can find further references. Many of the fundamental contributions are well documented in the monographs [7], [42], [47], [62], [63]. In these lecture notes we restrict ourselves to a class of parabolic problems[1] introduced by P. Acquistapace and B. Terreni in 1987, [2]; see also [1], [3], [7], [63], [64], [65].

(AT1) $A(t)$, $t \in J$, are linear operators on a Banach space X and there are constants $K \geq 0$, $w \in \mathbb{R}$, and $\phi \in (\pi/2, \pi)$ such that $\lambda \in \rho(A(t))$ and $\|R(\lambda, A(t))\| \leq \frac{K}{1+|\lambda - w|}$ for $\lambda \in \Sigma(\phi, w)$ and $t \in J$.

(AT2) There are constants $L \geq 0$ and $\mu, \nu \in (0,1]$ with $\mu + \nu > 1$ such that

$$|\lambda|^\nu \, \|A_w(t) R(\lambda, A_w(t)) \, (A_w(t)^{-1} - A_w(s)^{-1})\| \leq L \, |t-s|^\mu$$

for $A_w(t) := A(t) - w$, $t, s \in J$ and $|\arg \lambda| \leq \phi$.

Here we set $\Sigma(\phi, w) = \{w\} \cup \{\lambda \in \mathbb{C} \setminus \{w\} : |\arg(\lambda - w)| \leq \phi\}$. Observe that the domains are not required to be dense in X. If both (AT1) and (AT2) hold, then we say that (AT) is satisfied. Operators A fulfilling (AT1) are called *sectorial (of type* (ϕ, K, w)*)* (we always require that $\phi > \pi/2$). The equation (2.1) is called 'parabolic' because of the sectoriality of the operators $A(t)$.

Roughly speaking, the required degree of smoothness of the resolvents depends on the change of the domains $D(A(t))$. In the extreme case that $D(A(t))$, $t \in J$, equal a fixed Banach space X_1 with uniformly equivalent graph norms, (AT2) with $\nu = 1$ follows from the Hölder continuity of $A(t) : J \to \mathcal{L}(X_1, X)$ with any exponent $\mu > 0$. More variable domains than in (AT) are allowed in a different class of parabolic problems going back to work by T. Kato and H.Tanabe, see [62, §5.3], where it is assumed that $A(\cdot)^{-1} \in C^{1+\alpha}(J, \mathcal{L}(X))$ (besides another hypothesis). The Kato–Tanabe assumptions are logically independent of (AT), but the Acquistapace–Terreni conditions seem to be favourable since only a Hölder estimate is needed, see [2, §7] for a discussion of these matters and further literature.

It is well known that a sectorial operator A generates the analytic semigroup

$$e^{tA} = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} R(\lambda, A) \, d\lambda, \quad t > 0, \qquad e^{0A} = I, \tag{2.2}$$

and that its fractional powers are given by

$$(w - A)^{-\alpha} = \frac{1}{2\pi i} \int_\Gamma (w - \lambda)^{-\alpha} R(\lambda, A) \, d\lambda, \quad \alpha > 0, \tag{2.3}$$

where we may choose the path $\Gamma = \{\lambda : \arg(\lambda - w) = \pm\phi\}$ oriented counterclockwise, see [7], [21], [42], [47], [62]. Throughout we will make extensive use of standard regularity properties of analytic semigroups, often without mentioning it explicitely. Recall that $e^{tA}X \subseteq D(A)$ for $t > 0$, $t \mapsto e^{tA}$ is analytic on $(0, \infty)$ with derivative Ae^{tA}, and $e^{tA}x \to x$ as $t \to 0$ if and only if $x \in \overline{D(A)} =: X_0^A$.

[1] In fact, Acquistapace and Terreni used a somewhat more general version of condition (AT2).

Given a sectorial operator A and $0 < \alpha < 1$, we define the Banach spaces

$$\begin{aligned} &X^A_{\alpha,\infty} := \{x \in X : \|x\|^A_\alpha < \infty\} \quad \text{and} \quad X^A_\alpha := \overline{D(A)}^{\|\cdot\|^A_\alpha} \\ &\text{with the norm} \quad \|x\|^A_\alpha := \sup\{\|r^\alpha AR(r,A)x\| : r > w\}. \end{aligned} \tag{2.4}$$

These spaces coincide with the *real interpolation spaces* of exponent α and coefficient ∞ introduced by J.L. Lions and J. Peetre and with the *continuous interpolation spaces* due to G. Da Prato and P. Grisvard, respectively. We further set $\|x\|^A_1 := \|(w-A)x\|$ and $\|x\|^A_0 := \|x\|$. By X^A_1 we denote the domain of the part A_0 of A in X^A_0. The spaces X^A_1 and $D(A)$ are endowed with the norm $\|x\|^A_1$. For convenience, we also write $X^A_{0,\infty} := X$ and $X^A_{1,\infty} := D(A)$. One can verify the continuous embeddings

$$X^A_1 \subseteq D(A) \hookrightarrow X^A_\beta \subseteq X^A_{\beta,\infty} \hookrightarrow D((w-A)^\alpha) \hookrightarrow X^A_\alpha \hookrightarrow X^A_0 \subseteq X \tag{2.5}$$

for $0 < \alpha < \beta < 1$, where '$\subseteq$' means 'being a closed subspace' and the domain of the fractional power is equipped with the norm $\|(w-A)^\alpha x\|$. The norms of these embeddings depend only on α, β, and the type of A. Moreover, X^1_A is dense in $D((w-A)^\alpha)$ and X^A_α for $\alpha \in [0,1]$. The part A_α of A in X^A_α, $\alpha \in [0,1]$, generates an analytic strongly continuous semigroup being the restriction of e^{tA} (we mostly use the same symbol). We have the basic estimate

$$\|A^k e^{tA} x\|^A_\beta \leq C\, t^{\alpha-k-\beta}\, \|x\|^A_\alpha \tag{2.6}$$

for $k = 0, 1$, $0 \leq \alpha \leq \beta \leq 1$, $0 < t \leq 1$, $x \in X^A_{\alpha,\infty}$, and a constant C depending only on the type of A. We also need the moment inequality

$$\|(w-A)^\beta x\| \leq c(\alpha,\beta,\gamma)\, \|(w-A)^\alpha x\|^{\frac{\gamma-\beta}{\gamma-\alpha}}\, \|(w-A)^\gamma x\|^{\frac{\beta-\alpha}{\gamma-\alpha}}, \tag{2.7}$$

where $\alpha < \beta < \gamma$, $x \in D((w-A)^\gamma)$, and the constant depends only on the exponents and the type of A. These facts follow from, e.g., [21, §II.5] or [42, §1, 2.2], where the interpolation theory of semigroups is developed in detail and further references can be found. We assume now that (AT) holds. We set

$$X^t_{\alpha,\infty} := X^{A(t)}_{\alpha,\infty}, \quad X^t_\alpha := X^{A(t)}_\alpha, \quad \text{and} \quad \|\cdot\|^t_\alpha := \|\cdot\|^{A(t)}_\alpha$$

for $\alpha \in [0,1]$ and $t \in J$. Observing that

$$R(\lambda, A(t)) - R(\lambda, A(s)) = A_w(t)R(\lambda, A(t))\,(A_w(s)^{-1} - A_w(t)^{-1})\,A_w(s)R(\lambda, A(s)) \tag{2.8}$$

for $\lambda \in \rho(A(t)) \cap \rho(A(s))$, one easily deduces from (2.2) and (2.3) that

$$\|e^{\tau A(t)} - e^{\tau A(s)}\| \leq c(t_0)\, \tau^{\nu-1}\, |t-s|^\mu \tag{2.9}$$

$$\|(w-A(t))^{\alpha-1} - (w-A(s))^{\alpha-1}\| \leq c(\alpha)\, |t-s|^\mu \tag{2.10}$$

for $t, s \in J$, $0 < \tau \leq t_0$, $0 \leq \alpha < \nu$, and constants depending on the constants in (AT). Similarly,

$$\begin{aligned} &\|(w-A(t))^{\alpha-1} - (w-A(s))^{\alpha-1}\| \\ &\leq \frac{1}{2\pi}\int_\Gamma |w-\lambda|^{\alpha-1}\, \|R(\lambda,A(t)) - R(\lambda,A(s))\|^{1-\kappa}\, \|R(\lambda,A(t)) - R(\lambda,A(s))\|^\kappa\, |d\lambda| \\ &\leq c(\kappa)\, |t-s|^{\kappa\mu} \end{aligned} \tag{2.11}$$

for $t, s \in J$, $\alpha \in [\nu, 1)$, and $0 < \kappa < \frac{1-\alpha}{1-\nu}$.

Let E be a space of functions $f : J \to X$ and $B(t)$, $t \in J$, be linear operators on X with domains $D(B(t))$. Then the multiplication operator $B(\cdot)$ on E is defined by

$$\begin{aligned} (B(\cdot)f)(t) &:= B(t)f(t), \quad t \in J, \\ D(B(\cdot)) &:= \{f \in E : f(t) \in D(B(t)) \text{ for } t \in J,\ B(\cdot)f \in E\}, \end{aligned} \tag{2.12}$$

As stated in the next theorem, the solution of the homogeneous problem (2.1) with $f = 0$ is given by an *evolution family*, that is, by bounded linear operators $U(t,s)$, $t \geq s$, $s \in J$, satisfying

$$U(t,s) = U(t,r)U(r,s) \quad \text{and} \quad U(s,s) = I$$

for $t \geq r \geq s$ and $s \in J$. The evolution family is called *strongly continuous* if the mapping $(t,s) \mapsto U(t,s)$ is strongly continuous on the set $D_J := \{t, s \in J : t \geq s\}$. We say that $U(\cdot,\cdot)$ is *strongly continuous for* $t > s$ if this map is strongly continuous on $D_J^0 := \{t, s \in J : t > s\}$.

Theorem 2.2. *Assume that (AT) holds. Then there is an evolution family $U(\cdot,\cdot)$ on X with time interval J such that $D_J^0 \ni (t,s) \mapsto U(t,s) \in \mathcal{L}(X)$ is continuous, $U(t,s)X \subseteq D(A(t))$ for $t > s$, and*

$$\|U(t,s)x\|_\alpha^t \leq C\,(t-s)^{\beta-\alpha}\,\|x\|_\beta^s \tag{2.13}$$

$$\|U(t,s) - e^{(t-s)A(s)}\| \leq C\,(t-s)^{\mu+\nu-1}, \tag{2.14}$$

$$\|U(t,s)(w - A(s))^\theta y\| \leq C\,(\mu-\theta)^{-1}\,(t-s)^{-\theta}\,\|y\|, \tag{2.15}$$

for $0 < t-s \leq t_0$, $s \in J$, $0 \leq \beta \leq \alpha \leq 1$, $x \in X_{\beta,\infty}^s$, $0 \leq \theta < \mu$, $y \in D((w-A(s))^\theta)$, and a constant C only depending on the constants in (AT) and t_0. Further, $U(\cdot,s) \in C^1((s,\infty),\mathcal{L}(X))$, $\partial_t U(t,s) = A(t)U(t,s)$, and $\partial_s^+ U(t,s)x = -U(t,s)A(s)x$ for $x \in X_1^s$ and $t > s$. We have $A(t)^k U(t,s)x \to A(s)^k x$ as $t \to s$ if $x \in X_k^s$ and $k = 0,1$. Finally, the function

$$D_J \ni (t,s) \mapsto (w - A(t))^\alpha\, U(t,s)(w - A(s))^{-\alpha} f(s) \in X$$

is continuous for $\alpha \in [0,1]$ and $f \in \overline{D(A(\cdot))}$ (here $A(\cdot)$ is considered as an operator in $E = C_b(J,X)$ endowed with the sup–norm).

In particular, $u = U(\cdot,s)x$ is a classical solution of the inhomogeneous problem (2.1) if $x \in X_0^s$ and a strict one if $x \in X_1^s$. The evolution family $U(\cdot,\cdot)$ is strongly continuous if all domains $D(A(t))$ are dense in X. Estimate (2.13) implies that

$$\|U(t',s)x - U(t,s)x\| \leq \int_t^{t'} \|A(\tau)U(\tau,s)x\|\,d\tau \leq c\,|t'-t|^\alpha\,\|x\|_\alpha^s \tag{2.16}$$

for $t' \geq t \geq s$ and $\alpha \in (0,1]$.

Most assertions of the above theorem are established in [1, Thm.2.3]. There the evolution family $U(\cdot,\cdot)$ is defined by the formula

$$U(t,s) = e^{(t-s)A(s)} + \int_s^t Z(r,s)\,dr, \qquad t \geq s, \tag{2.17}$$

for certain operators $Z(r,s)$ with $\|Z(r,s)\| \leq c\,(r-s)^{\mu+\nu-2}$, see equations (2.6) and (2.7) and Lemma 2.2 in [1]. These results imply in particular (2.14). The estimate (2.13) for $\alpha = \beta \in \{0,1\}$ is proved in [1, Thm.2.3], and the other cases follow by interpolation and reiteration (see e.g. [42, Chap.2]). It can be seen by a (tedious) inspection of the proofs given in [1] that the constant C only depends on the constants in (AT), and not on s; see also [23]. The estimate (2.15) is proved in [65, Thm.2.3]. There it is also shown that

$$\|(w-A(t))^{\alpha}\,U(t,s)(w-A(s))^{-\alpha}-e^{(t-s)A(s)}\|\le c(t_0)\,(t-s)^{\mu+\nu-1} \tag{2.18}$$

for $\alpha\in(0,1]$, whereas for $\alpha=0$ the inequality (2.18) coincides with (2.14). Moreover, $\|e^{tA(s)}x-x\|\le ct\|A(s)x\|$ for $x\in D(A(s))$. This implies the final continuity assertion in Theorem 2.2 as $(t,s)\to(r,r)$. The other case easily follows from (2.11) and the norm continuity of $(t,s)\mapsto A(t)U(t,s)=A(t)U(t,\tau)U(\tau,s)$ for $t>s$. Clearly, (2.18) yields

$$\|(w-A(t))^{\alpha}\,U(t,s)(w-A(s))^{-\alpha}\|\le c(t_0). \tag{2.19}$$

for $0\le t-s\le t_0$ and $0\le\alpha\le 1$.

The representation formula for $U(\cdot,\cdot)$ given in [1] is rather implicit. H. Amann and A. Yagi developed different approaches to Theorem 2.2 in [7, Chap.IV], [64], [65], working in slightly different settings. These authors construct $U(t,s)$ as solutions of certain integral equations. Yagi's method is also employed in the proof of Proposition 2.4 below. It uses the Yosida approximations $A_n(t)=nA(t)R(n,A(t))$, $n>w$, which also satisfy (AT) as stated in the next lemma. Its elementary (but tedious) proof is omitted, cf. [2, Lem.4.2] or [64, Prop.2.1].

Lemma 2.3. *Assume that (AT) holds. Fix $w'>w$ and $\phi'\in(\frac{\pi}{2},\phi)$. Then there are constants $\overline{n}>w$, $L'\ge L$, and $K'\ge K$ (only depending on the constants in (AT), w', and ϕ') such that the operators $A_n(t)$, $t\in J$, satisfy (AT) with constants K',ϕ',w',L',μ,ν for all $n\ge\overline{n}$. Moreover, for $n\ge\overline{n}$ and $\lambda\in\Sigma(\phi',w')$, we have*

$$R(\lambda,A_n(t))=\tfrac{1}{\lambda+n}(n-A(t))R(\tfrac{\lambda n}{\lambda+n},A(t))=\tfrac{n^2}{(\lambda+n)^2}R(\tfrac{\lambda n}{\lambda+n},A(t))+\tfrac{1}{\lambda+n}\,, \tag{2.20}$$

$$R(\lambda,A_n(t))-R(\lambda,A(t))=\tfrac{1}{\lambda+n}A(t)R(\tfrac{\lambda n}{\lambda+n},A(t))A(t)R(\lambda,A(t)). \tag{2.21}$$

Thus the evolution families $U_n(\cdot,\cdot)$ generated by $A_n(\cdot)$ satisfy the same estimates as $U(\cdot,\cdot)$ with constants independent of $n\ge\overline{n}$. This fact is used later on without further notice.

The following refinements of the regularity statements of Theorem 2.2 are taken from [57], see also [42, Cor.6.1.8] for the case of constant domains.

Proposition 2.4. *Let (AT) hold and $0<\alpha<\mu+\nu-1$. Then*

$$\|A(t)U(t,s)x\|_{\alpha}^{t}\le C\,(t-s)^{-1-\alpha}\,\|x\|$$
$$\|A(t)U(t,s)-A(r)U(r,s)\|\le C\,(t-r)^{\alpha}\,(r-s)^{-1-\alpha}$$

for $0<t-s\le t_0$, $s<r<t$, $x\in X$, and a constant C depending only on α, t_0, and the constants in (AT).

Proof. By rescaling we may assume that $w<w'\le 0$ in (AT) and Lemma 2.3. We first prove the Hölder continuity. We infer from $A(t)U(t,s)=\partial_t U(t,s)$ and (2.17) that

$$\begin{aligned}A(t)U(t,s)-A(r)U(r,s)&=A(s)(e^{(t-s)A(s)}-e^{(r-s)A(s)})+Z(t,s)-Z(r,s)\\&=(e^{(t-r)A(s)}-I)\,A(s)e^{(r-s)A(s)}+Z(t,s)-Z(r,s)\end{aligned}$$

for $s<r<t$. So [42, Prop.2.2.4, 2.2.2] and [1, Lem.2.2] yield the second assertion. (It can be verified by an inspection of the proofs given there that C only depends on the mentioned quantities.)

Note that the fractional powers $(-A_n(t))^{-\beta}$, $\beta > 0$, are uniformly bounded in $n \in \mathbb{N}$ and $t \in J$ and converge in operator norm to $(-A(t))^{-\beta}$ as $n \to \infty$ due to (2.21). Fix $\alpha \in (0, \mu + \nu - 1)$ and set $\theta := \frac{1}{2}(\alpha + \nu + 1 - \mu) \in (1 - \mu, \nu)$. Then we have

$$U_n(t,s) = e^{(t-s)A_n(t)} + \int_s^t A_n(t) e^{(t-\tau)A_n(t)} \,(A_n(t)^{-1} - A_n(\tau)^{-1}) A_n(\tau) U_n(\tau,s)\, d\tau,$$

$$\begin{aligned}(-A_n(t))^{\alpha+1} U_n(t,s) = (-A_n(t))^{\alpha+1} e^{(t-s)A_n(t)} - \int_s^t (-A_n(t))^{2+\alpha-\theta} e^{(t-\tau)A_n(t)} \\ \cdot (-A_n(t))^{\theta} (A_n(t)^{-1} - A_n(\tau)^{-1})\, A_n(\tau) U_n(\tau,s)\, d\tau\end{aligned}$$

for $t \geq s$, $s \in J$, and $n \in \mathbb{N}$. We further define

$$W_n(t,s) := (-A_n(t))^{\alpha+1} U_n(t,s) - (-A_n(t))^{\alpha+1} e^{(t-s)A_n(t)} \quad \text{and} \tag{2.22}$$

$$R_n(t,s) := \int_s^t (-A_n(t))^{2+\alpha} e^{(t-\tau)A_n(t)} \,(A_n(t)^{-1} - A_n(\tau)^{-1})\, A_n(\tau) e^{(\tau-s)A_n(\tau)}\, d\tau.$$

This yields

$$\begin{aligned}W_n(t,s) = -R_n(t,s) + \int_s^t \big[(-A_n(t))^{2+\alpha-\theta} e^{(t-\tau)A_n(t)} \\ \cdot (-A_n(t))^{\theta} (A_n(t)^{-1} - A_n(\tau)^{-1})\, (-A_n(\tau))^{-\alpha}\big]\, W_n(\tau,s)\, d\tau\end{aligned} \tag{2.23}$$

By [47, Thm.2.6.13], (AT), and (2.5), the kernel $[\cdots]$ of this integral equation can be estimated by $c\,(t-\tau)^{\mu+\theta-\alpha-2}$ for $0 < t - \tau \leq t_0$. As in [64, p.144], one sees that $\|R_n(t,s)\| \leq c\,(t-s)^{\mu+\theta-\alpha-2}$ for $0 < t - s \leq t_0$. Here the constants c only depend on t_0, α, and the constants in (AT). Note that $\mu + \theta - \alpha - 2 > -1$. Thus the solution $W_n(t,s)$ can be estimated in the same way due to [7, Thm.II.3.2.2]. The kernel of (2.23) and the operators $R_n(t,s)$ converge strongly to the same expressions without the index n by [64, Prop.2.1]. Since (2.23) is solved by a Neumann series, see [7, §II.3.2], $W_n(t,s)$ also converges strongly as $n \to \infty$. Hence, $(-A_n(t))^{\alpha+1} U_n(t,s)$ has a strong limit $V(t,s)$ and

$$\|V(t,s)\| \leq c\,(t-s)^{-\alpha-1} \quad \text{for } \ 0 < t - s \leq t_0 \tag{2.24}$$

by (2.22) and $-\alpha - 1 < \mu + \theta - \alpha - 2$, where $c = c(t_0, \alpha, \mu, \nu, \phi, K, L)$. Since $A_n(t)U_n(t,s)$ tends strongly to $A(t)U(t,s)$ as $n \to \infty$ by [64, Prop.3.1.], we arrive at $A(t)U(t,s) = -(-A(t))^{-\alpha} V(t,s)$ for $t > s$. Thus the first assertion follows from (2.24) and (2.5). □

The following approximation result is quite useful and due to C.J.K. Batty and R. Chill, [11, Prop.4.4], who employed ideas from [2]. We give here a different, more elementary proof (taken from [43]) leading to a different rate of convergence. Observe that (2.2) and (2.21) imply

$$\|(e^{tA} - e^{tA_n})(w - A)^{-k}\| \leq C\, n^{-1} t^{k-1} \tag{2.25}$$

for $k = 0, 1$, $t \in (0, t_0]$, $n > \overline{n}$, cf. [2, Lem.4.3].

Proposition 2.5. *Let (AT) hold and $s \in J$. Fix $0 < t_0 < t_1$. Then*

$$\|U(t,s) - U_n(t,s)\| \leq c(t_1, \theta)\, n^{-\theta}$$

for $0 < t_0 \leq t - s \leq t_1$, $n \geq n_0(t_0) := \max\{\overline{n}, t_0^{-2/\mu}\}$, and any $0 < \theta < \min\{\mu/2, 1 - \mu/2, \mu(\mu + \nu - 1)/2\}$. Moreover,

$$\|(U(t,s) - U_n(t,s)) R(w, A(s))\| \leq c(t_1, \alpha)\, n^{-\alpha}$$

for $0 \le t-s \le t_1$*,* $n \ge \overline{n}$*, and* $\alpha \in (0,\mu)$*.*

Proof. Let $0 < h < t_0$, $n \ge \overline{n}$, $s \in J$, and $0 < t_0 \le t-s \le t_1$. Then we have

$$\begin{aligned} U(t,s) - U_n(t,s) &= (U(t,s+h) - U_n(t,s+h))U(s+h,s) - U_n(t,s+h) \\ &\qquad \cdot [U(s+h,s) - e^{hA(s)} + e^{hA(s)} - e^{hA_n(s)} + e^{hA_n(s)} - U_n(s+h,s)] \\ &=: S_1 - S_2. \end{aligned} \tag{2.26}$$

Due to (2.25) and (2.14), we obtain

$$\|S_2\| \le c(t_1)\,(h^{\mu+\nu-1} + (hn)^{-1}). \tag{2.27}$$

The other term can be transformed into

$$\begin{aligned} S_1 &= \int_{s+h}^{t} U_n(t,\sigma)(A(\sigma) - A_n(\sigma))U(\sigma,s)\,d\sigma \\ &= \int_{s+h}^{t} U_n(t,\sigma)(w' - A_n(\sigma))^{\alpha}\,(w' - A_n(\sigma))^{1-\alpha}[R(w',A(\sigma)) - R(w',A_n(\sigma))] \\ &\qquad \cdot (w' - A(\sigma))U(\sigma,s)\,d\sigma. \end{aligned}$$

where $\alpha \in (0,\mu)$ and $w' > w$ is fixed. The moment inequality (2.7) yields

$$\|(w' - A_n(\sigma))^{1-\alpha}\| \le c\,\|w' - A_n(\sigma)\|^{1-\alpha} \le cn^{1-\alpha} \tag{2.28}$$

This estimate as well as (2.15), (2.13), and (2.21) lead to

$$\|S_1\| \le c(t_1,\alpha)\,h^{-1}n^{-\alpha}. \tag{2.29}$$

Combining (2.26), (2.27), and (2.29), we deduce

$$\|U(t,s) - U_n(t,s)\| \le c(t_1,\alpha)\,((nh)^{-1} + h^{\mu+\nu-1} + n^{-\alpha}h^{-1}).$$

The first assertion now follows if we set $h := n^{-\mu/2}$. The second one can be shown using the formula

$$\begin{aligned} (U(t,s) - U_n(t,s))R(w,A(s)) &= \int_{s}^{t} U_n(t,\sigma)(w' - A_n(\sigma))^{\alpha}(w' - A_n(\sigma))^{1-\alpha} \\ &\qquad \cdot [R(w',A(\sigma)) - R(w',A_n(\sigma))]\,(w' - A(\sigma))U(\sigma,s)R(w,A(s))\,d\sigma, \end{aligned}$$

together with (2.15), (2.28), (2.21), and (2.13). $\square$

Batty and Chill used the above result to establish that $U(\cdot,\cdot)$ depends continuously on $A(\cdot)$ in [11, Thm.4.7]. A part of their theorem (and a slight variant of it) are stated and proved in the next proposition.

Proposition 2.6. *Assume that* $A(t)$ *and* $B(t)$*,* $t \in J$*, satisfy (AT) and generate the evolution families* $U(\cdot,\cdot)$ *and* $V(\cdot,\cdot)$*, respectively. Fix* $0 < t_0 < t_1$ *and let*

$$q(t,s) = \sup_{s \le \tau \le t} \|R(w,A(\tau)) - R(w,B(\tau))\|.$$

Then there are numbers $q_0(t_0), \beta > 0$ *(only depending on the constants in (AT)) such that*

$$\|U(t,s) - V(t,s)\| \le c(t_1)\,q(t,s)^{\beta}$$

for $0 < t_0 \le t-s \le t_1$ *and* $q(t,s) \le q_0(t_0)$*. Moreover,*

$$\|(U(t,s) - V(t,s))R(w,A(s))\| \le c(\alpha,t_1)\,q(t,s)^{\alpha}$$

for $0 \le t-s \le t_1$ *and* $\alpha \in (0,\mu)$.

Proof. In the following we use Lemma 2.3 with $w' = w+1$.

(1) Let $0 < h < t_0 \le t-s \le t_1$, $n \ge \overline{n}$, and set $q := q(t,s)$. We write

$$\begin{aligned} U(t,s) - V(t,s) = {} & [U(t,s) - U_n(t,s)] + [U_n(t,s+h) - V_n(t,s+h)]U_n(s+h,s) \\ & + V_n(t,s+h)\big[U_n(s+h,s) - e^{hA_n(s)} + e^{hA_n(s)} - e^{hB_n(s)} + e^{hB_n(s)} \\ & - V_n(s+h,s)\big] + [V_n(t,s) - V(t,s)] =: S_1 + S_2 + S_3 + S_4. \end{aligned}$$

Proposition 2.5 yields

$$\|S_1\| \le c(t_1,\theta)\, n^{-\theta} \quad \text{and} \quad \|S_4\| \le c(t_1,\theta)\, n^{-\theta}$$

with the number $\theta > 0$ given there. Formulas (2.2), (2.20), and (2.8) give

$$\begin{aligned} e^{hA_n(s)} - e^{hB_n(s)} &= \frac{1}{2\pi i}\int_\Gamma e^{\lambda h} \tfrac{n^2}{(n+\lambda)^2}\, [R(\tfrac{n\lambda}{n+\lambda}, A(s)) - R(\tfrac{n\lambda}{n+\lambda}, B(s))]\, d\lambda, \\ &= \frac{1}{2\pi i}\int_\Gamma e^{\lambda h} \tfrac{n^2}{(n+\lambda)^2}\, A_w(s) R(\tfrac{n\lambda}{n+\lambda}, A(s))\, (B_w(s)^{-1} - A_w(s)^{-1})\, B_w(s) R(\tfrac{n\lambda}{n+\lambda}, B(s))\, d\lambda \end{aligned}$$

for a suitable path $\Gamma \subseteq \Sigma(\phi', w')$. Thus $\|e^{hA_n(s)} - e^{hB_n(s)}\| \le cqh^{-1}$. Together with (2.14) this shows

$$\|S_3\| \le c(t_1)\,(h^{\mu+\nu-1} + qh^{-1}).$$

The remaining term can be transformed into

$$\begin{aligned} S_2 = \int_{s+h}^t & V_n(t,\sigma)(w' - B_n(\sigma))^\alpha (w' - B_n(\sigma))^{1-\alpha}((A_n(\sigma) - w')^{-1} - (B_n(\sigma) - w')^{-1}) \\ & \cdot (A_n(\sigma) - w')U_n(\sigma,s)\, d\sigma. \end{aligned} \tag{2.30}$$

where $\alpha \in (0,\mu)$. Due to (2.20) and (2.8) we have

$$\|R(w', A_n(\sigma)) - R(w', B_n(\sigma))\| = \|\tfrac{n^2}{(n+w')^2}\, [R(\tfrac{nw'}{n+w'}, A(\sigma)) - R(\tfrac{nw'}{n+w'}, B(\sigma))]\| \le c\, q$$

for $n > ww'/(w'-w) = w^2 + w$. So (2.15), (2.28), and (2.13) imply

$$\|S_2\| \le c(t_1,\alpha)\, h^{-1} n^{1-\alpha} q$$

for sufficiently large n. Putting everything together, we arrive at

$$\|U(t,s) - V(t,s)\| \le c(t_1,\theta,\alpha)\,(n^{-\theta} + h^{-1}n^{1-\alpha}q + h^{\mu+\nu-1} + qh^{-1}).$$

Now take, for instance, $n = q^{1/4(\alpha-1)}$ and $h = \sqrt{q}$ to deduce the first assertion.

(2) For the second assertion we write

$$\begin{aligned} & (U(t,s) - V(t,s))R(w,A(s)) \\ & = [U(t,s) - U_n(t,s)]R(w,A(s)) + [U_n(t,s) - V_n(t,s)]R(w', A_n(s))(w' - A_n(s))R(w,A(s)) \\ & \quad + [V_n(t,s) - V(t,s)]\,[R(w,A(s)) - R(w,B(s))] + [V_n(t,s) - V(t,s)]R(w,B(s)) \end{aligned}$$

Here we let $n = q^{-1}$ supposing that $q < \overline{n}$. (For $q \ge \overline{n}$, the result is trivially true.) So the claim follows from Proposition 2.5 and formula (2.30) with $h = 0$. $\square$

We come back to the inhomogeneous problem (2.1). If u is a classical solution with $x \in X_0^s$ and $f \in C(J,X)$, then u is given by

$$u(t) = U(t,s)x + \int_s^t U(t,\tau)f(\tau)\,d\tau, \qquad t \geq s, \tag{2.31}$$

by [1, Prop.3.2, 5.1]. This equality defines a function $u : [s,\infty) \to X$ for all $x \in X$ and $f \in C(J,X)$ which is called the *mild solution* of (2.1). The regularity of the first summand in (2.31) was discussed above. Using (2.13) and (2.16), one easily sees that the integral term is uniformly bounded in $X^t_{\alpha,\infty}$ for $\alpha < 1$ and Hölder continuous of exponent $\beta < 1$ on $[s, s+t_0]$. The mild solution becomes a classical one if one takes more regular data. This stated in the next theorem which recalls a part of Theorems 6.1–6.4 of [2] in a simplified way. The asserted uniformity of the constants can be verified inspecting the proofs.

Theorem 2.7. *Assume that (AT) holds and let* $s \in J$, $t_0 > 0$, $\alpha \in (0, \mu+\nu-1]$, $f \in C([s,s+t_0],X)$, *and* $x \in X_0^s$.

(a) If $f \in C^\alpha([s,s+t_0],X)$, *then the mild solution is the unique classical solution of* (2.1) *and*

$$\begin{aligned} \|u'\|_{C^\alpha([s+\varepsilon,s+t_0],X)} + \|A(\cdot)u\|_{C^\alpha([s+\varepsilon,s+t_0],X)} &+ \sup_{s+\varepsilon \leq t \leq s+t_0} \|u'(t)\|^t_\alpha \\ &\leq c(\varepsilon,t_0)\,[\|x\| + \|A(s)x\| + \|f\|_{C^\alpha([s,s+t_0],X)}] \end{aligned} \tag{2.32}$$

for $\varepsilon > 0$. *The solution is strict if* $x \in D(A(s))$ *and* $A(s)x + f(s) \in X_0^s$. *One can take* $\varepsilon = 0$ *in the above estimate if and only if* $A(s)x + f(s) \in X^s_{\alpha,\infty}$, *and then*

$$\begin{aligned} \|u'\|_{C^\alpha([s,s+t_0],X)} + \|A(\cdot)u\|_{C^\alpha([s,s+t_0],X)} &+ \sup_{s \leq t \leq s+t_0} \|u'(t)\|^t_\alpha \\ &\leq c(t_0)\,[\|x\| + \|A(s)x\| + \|A(s)x + f(s)\|^s_\alpha + \|f\|_{C^\alpha([s,s+t_0],X)}]. \end{aligned} \tag{2.33}$$

(b) If $\|f(t)\|^t_\alpha$ *is uniformly bounded for* $t \in [s,s+t_0]$, *then the mild solution is the unique classical solution of* (2.1) *and*

$$\begin{aligned} \|A(\cdot)u\|_{C^\alpha([s+\varepsilon,s+t_0],X)} + &\sup_{s+\varepsilon \leq t \leq s+t_0} (\|u'(t)\|^t_\alpha + \|A(t)u(t)\|^t_\alpha) \\ &\leq c(\varepsilon,t_0)\,[\|x\| + \|A(s)x\| + \sup_{s \leq t \leq s+t_0} \|f(t)\|^t_\alpha] \end{aligned} \tag{2.34}$$

for $\varepsilon > 0$. *The solution is strict if* $x \in D(A(s))$ *and* $A(s)x + f(s) \in X_0^s$. *One can take* $\varepsilon = 0$ *in the above estimate if and only if* $A(s)x \in X^s_{\alpha,\infty}$, *and then*

$$\begin{aligned} \|A(\cdot)u\|_{C^\alpha([s,s+t_0],X)} + &\sup_{s \leq t \leq s+t_0} (\|u'(t)\|^t_\alpha + \|A(t)u(t)\|^t_\alpha) \\ &\leq c(t_0)\,[\|x\| + \|A(s)x\| + \|A(s)x\|^s_\alpha + \sup_{s \leq t \leq s+t_0} \|f(t)\|^t_\alpha]. \end{aligned} \tag{2.35}$$

Here all constants only depend on the constants in (AT) and the indicated quantities.

Concluding this section, we present two different ways to verify that (AT) holds for parabolic partial differential equations of second order under suitable regularity and ellipticity assumptions. The following two examples are taken from [64] and [1], respectively; see also [5], [42], [45], [62], [63], [65]. More general problems could be treated analogously.

Example 2.8. Consider the initial–boundary value problem

$$\begin{aligned} D_t\, u(t,x) &= A(t,x,D)u(t,x) + h(t,x), \quad t > s \geq s,\ x \in \Omega, \\ B(t,x,D)u(t,x) &= 0, \quad t > s \geq 0,\ x \in \partial\Omega, \\ u(s,\cdot) &= u_0. \end{aligned} \tag{2.36}$$

Here we suppose that $D_t = \frac{\partial}{\partial t}$, $D_k = \frac{\partial}{\partial x_k}$, $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with boundary $\partial\Omega$ of class C^2 being locally on one side of Ω and outer unit normal vector $n(x)$, and

$$A(t,x,D) = \sum_{k,l=1}^{n} D_k\, a_{kl}(t,x)\, D_l + a_0(t,x), \qquad B(t,x,D) = \sum_{k,l=1}^{n} n_k(x)\, a_{kl}(t,x)\, D_l\,.$$

We assume that the coefficients satisfy

$$\begin{aligned} &a_{kl} \in C_b^{\mu}(\mathbb{R}_+, C(\overline{\Omega})), \quad a_{kl}(t,\cdot) \in C^1(\overline{\Omega}), \quad D_j\, a_{kl} \in BUC(\mathbb{R}_+ \times \overline{\Omega}) \\ &a_0 \in C_b^{\mu}(\mathbb{R}_+, L^n(\Omega)) \cap C_b(\mathbb{R}_+, C(\overline{\Omega})) \end{aligned} \tag{2.37}$$

for $k,l,j = 1,\cdots,n$, $t \geq 0$, and a constant $\frac{1}{2} < \mu \leq 1$ and that (a_{kl}) is symmetric, real, and uniformly elliptic, i.e.,

$$\sum_{k,l=1}^{n} a_{kl}(t,x)\, v_k\, v_l \geq \eta\, |v|^2 \tag{2.38}$$

for a constant $\eta > 0$ and each $x \in \overline{\Omega}$, $t \geq 0$, $v \in \mathbb{R}^n$.

On $X_p = L^p(\Omega)$, $1 < p < \infty$, and $X_\infty = C(\overline{\Omega})$ we introduce the realizations $A_p(t)$, $1 < p < \infty$, and $A_\infty(t)$ of the differential operator $A(t,x,D)$ with domains

$$\begin{aligned} D(A_p(t)) &= \{f \in W^{2,p}(\Omega) : B(t,\cdot,D)f = 0 \text{ on } \partial\Omega\}, \\ D(A_\infty(t)) &= \{f \in \textstyle\bigcap_{p>1} W^{2,p}(\Omega) : A(t,\cdot,D)f \in C(\overline{\Omega}),\ B(t,\cdot,D)f = 0 \text{ on } \partial\Omega\}, \end{aligned} \tag{2.39}$$

where the boundary condition is understood in the sense of traces if necessary. It is well known that these operators are sectorial of uniform type (ϕ, K, w), see e.g. [19], [42], [62], [63]. Moreover, the adjoint $A_p(t)^*$ of $A_p(t)$, $1 < p < \infty$, is given by $A_{p'}(t)$ on $W^{2,p'}(\Omega)$, where $1/p + 1/p' = 1$, and the resolvents of $A_p(t)$ and $A_q(t)$ coincide on $L^q(\Omega)$ if $q \in (p,\infty]$ and both resolvents exist. Therefore we will mostly omit the subscript p.

We verify (AT2) first for $p \in (1,\infty)$. Take $t,s \geq 0$, $f \in L^p(\Omega)$, and $g \in D((w - A(t)^*)^\nu)$ for some $\nu \in (1-\mu, 1/2)$. Set $u = (R(w,A(t)) - R(w,A(s)))f \in W^{2,p}(\Omega)$ and $v = (w - A(t)^*)^{\nu-1} g \in D(A_{p'}(t))$. Integrating by parts three times, we deduce

$$\begin{aligned} &\langle (R(w,A(t)) - R(w,A(s)))f, (w - A(t)^*)^\nu g\rangle = \int_\Omega u\,(w - A(t,\cdot,D))v\, dx \\ &\quad = \int_\Omega v\,(w - A(t,\cdot,D))u\, dx + \int_{\partial\Omega} v\, B(t,\cdot,D)u\, dS \\ &\quad = \int_\Omega v\,(A(t,\cdot,D) - A(s,\cdot,D))R(w,A(s))f\, dx - \int_{\partial\Omega} v\, B(t,\cdot,D)R(w,A(s))f\, dS \\ &\quad = \textstyle\sum_{kl} \displaystyle\int_\Omega (a_{kl}(s,x) - a_{kl}(t,x))\,(D_l R(w,A(s))f)(x)\,(D_k(w - A(t)^*)^{\nu-1} g)(x)\, dx \\ &\qquad + \int_\Omega (a_0(t,x) - a_0(s,x))(R(w,A(s))f)(x)\,((w - A(t)^*)^{\nu-1} g)(x)\, dx \end{aligned} \tag{2.40}$$

Observe that the operators $R(w, A(s)) : L^p(\Omega) \to W^{k,q}(\Omega)$ and $(w - A(t)^*)^{\nu-1} : L^{p'}(\Omega) \to W^{k,r}(\Omega)$ are bounded for $q \in [p, \infty]$, $r \in [p', \infty]$, and $k = 0, 1$ such that $\frac{1}{q} > \frac{1}{p} + \frac{k-2}{n}$ and $\frac{1}{r} > \frac{1}{p'} + \frac{k-2+2\nu}{n}$ due to e.g. [27, Thm.1.6.1] and $\nu < 1/2$. Using Hölder's inequality in (2.40), we then estimate

$$|\langle (R(w,A(t)) - R(w,A(s)))f, (w - A(t)^*)^\nu g\rangle| \le c_p \max\nolimits_{kl}\{[a_{kl}]_{\mu,\infty}, [a_0]_{\mu,n}\} \, \|f\|_p \, \|g\|_{p'} \, |t-s|^\mu$$

where $[u]_{\mu,q}$ is the Hölder constant of $u : \mathbb{R}_+ \to L^q(\Omega)$ for the Hölder exponent $\mu \in (0,1)$. This leads to

$$\|(w - A(t))^\nu (R(w,A(t)) - R(w,A(s)))\|_p \le c_p \max\nolimits_{kl}\{[a_{kl}]_{\mu,\infty}, \, [a_0]_{\mu,n}\} \, |t-s|^\mu. \tag{2.41}$$

We have verified (AT) for $p < \infty$ because of (2.5). In the case $p = \infty$ we fix $\nu \in (1-\mu, 1/2)$, $\theta \in (0, 1/2 - \nu)$, and $p > \frac{n}{2\theta}$. Using again [27, Thm.1.6.1], we see that $(w - A_p(t))^{-\theta} : L^p(\Omega) \to C(\overline{\Omega})$ is bounded. Thus (2.41) with ν replaced by $\nu + \theta$ yields

$$\begin{aligned} &\|(w - A(t))^\nu (R(w,A(t)) - R(w,A(s)))f\|_\infty \\ &\quad\le c \, \|(w - A_p(t))^{\nu+\theta} (R(w,A(t)) - R(w,A(s)))f\|_p \\ &\quad\le c \, |t-s|^\mu \max\nolimits_{kl}\{[a_{kl}]_{\mu,\infty}, \, [a_0]_{\mu,n}\} \|f\|_p \\ &\quad\le c \, |t-s|^\mu \max\nolimits_{kl}\{[a_{kl}]_{\mu,\infty}, \, [a_0]_{\mu,n}\} \|f\|_\infty \end{aligned} \tag{2.42}$$

for $f \in C(\overline{\Omega})$. So we deduce (AT) from (2.5) also for $p = \infty$. The above results now allow to solve the problem (2.36) under appropriate assumptions on $u_0 \in X$ and $h : \mathbb{R}_+ \to X$.

Example 2.9. We study again (2.36), but now with the differential expressions

$$\begin{aligned} A(t,x,D) &= \sum\nolimits_{k,l} a_{kl}(t,x) \, D_k D_l + \sum\nolimits_k a_k(t,x) \, D_k + a_0(t,x), \\ B(t,x,D) &= \sum\nolimits_k b_k(t,x) \, D_k + b_0(t,x). \end{aligned} \tag{2.43}$$

It assumed that $\partial\Omega$ is the disjoint union of two closed (possibly empty) subsets Γ_0 and Γ_1. We require that $a_{kl} = a_{lk} \in BUC(\mathbb{R}_+ \times \overline{\Omega})$, b_k, and b_0 are real, (2.38) holds, $a_{kl}, a_k, a_0 \in C_b^\mu(\mathbb{R}_+, C(\overline{\Omega}))$ and $b_k, b_0 \in C_b^\mu(\mathbb{R}_+, C^1(\overline{\Omega}))$ for some $\mu \in (1/2, 1)$, $b_0 = 1$ and $b_k = 0$ on Γ_0, and $\sum_{k=1}^n b_k(t,x) n_k(x) \ge \beta > 0$ for $x \in \Gamma_1$, $t \ge 0$, and $k, l = 1, \cdots, n$.[2]

As in the previous example, see (2.39), we define the operators $A_p(t)$ on X_p, $1 \le p \le \infty$. Observe that the closure of $D(A_\infty(t))$ is the space of the continuous functions on $\overline{\Omega}$ vanishing on Γ_0. Again it is well known that these operators are sectorial of uniform type (ϕ, K, w), see e.g. [19], [42], [62], [63]. For $f \in L^p(\Omega)$, $t, s \ge 0$, $|\arg \lambda| \le \phi$, we set

$$v = -R(w, A(s))f \quad \text{and} \quad u = R(\lambda + w, A(t))(\lambda + w - A(s))v.$$

Then $u - v = (A(t) - w)R(\lambda + w, A(t))(R(w,A(t)) - R(w,A(s)))f$ and

$$\begin{aligned} (\lambda + w)u - A(t,\cdot,D)u &= \lambda v - f, & (w - A(s,\cdot,D))v &= -f, & &\text{on } \Omega, \\ B(t,\cdot,D)u &= 0, & B(s,\cdot,D)v &= 0 & &\text{on } \partial\Omega. \end{aligned}$$

This shows that

$$\begin{aligned} (\lambda + w)(u - v) - A(t,\cdot,D)(u-v) &= (A(t,\cdot,D) - A(s,\cdot,D))v, \qquad &&\text{on } \Omega, \\ B(t,\cdot,D)(u-v) &= (B(s,\cdot,D) - B(t,\cdot,D))v, &&\text{on } \partial\Omega. \end{aligned}$$

[2] If $\Gamma_1 = \emptyset$, it suffices that $\mu > 0$.

The Agmon–Douglas–Nirenberg estimate and our regularity assumptions now imply

$$\begin{aligned}\|u-v\|_p &\le \frac{c_p}{|\lambda+w|}\big(\|(A(t,\cdot,D)-A(s,\cdot,D))v\|_{L^p(\Omega)}+\|(B(s,\cdot,D)-B(t,\cdot,D)v\|_{W^{1,p}(\Omega)}\\ &\qquad\qquad +|\lambda+w|^{1/2}\|((B(s,\cdot,D)-B(t,\cdot,D))v\|_{L^p(\Omega)}\big) \qquad (2.44)\\ &\le c_p\,|\lambda+w|^{-1/2}\,|t-s|^{\mu}\,\|f\|_{L^p(\Omega)}\end{aligned}$$

see e.g. [19], [42], [63, Thm.5.5], and the references therein. Thus (AT2) holds with μ and $\nu=1/2$ on $L^p(\Omega)$ for $p\in(1,\infty)$. One can deal with the case $X=C(\overline{\Omega})$ as in (2.42) obtaining $\nu\in(1-\mu,1/2)$.[3] Theorems 2.2 and 2.7 again allow to solve (2.36).

Remark 2.10. Inspecting the proofs given above and in, e.g., [63], it can be seen in both examples that all constants c and c_p and the resulting constants in (AT) only depend on Ω, the ellipticity constants $\eta,\beta>0$, the modulus of continuity w.r.t. the space variables of $D_j a_{kl}$ (in Example 2.8) and a_{kl} (in Example 2.9), $k,l,j=1,\cdots,n$, and the norms of the coefficients in the spaces indicated above.[4] Unfortunately, it is hard to control this dependence explicitely. In Example 4.16 we will modify the approach of Example 2.8 for $p=2$ in order to obtain an explicit bound for the constants in (AT) which is important for some of our applications.

3. Exponential dichotomy

Having discussed the solutions of parabolic evolution equations, we now turn our attention to their long term behaviour. In this section we introduce and characterize *exponential dichotomy* which is a fundamental concept in the asymptotic theory of evolution families. We formulate most of our results for general evolution families since the proofs do not use properties peculiar to parabolic problems and in later sections we occasionally employ evolution families which do not quite satisfy the Aquistapace–Terreni conditions.

3.1. **Basic observations.** Let $U(\cdot,\cdot)$ be an evolution family with time interval $J\in\{\mathbb{R},[a,\infty)\}$ on a Banach space X. Its *(exponential) growth bound* is defined by

$$\omega(U)=\inf\{\gamma\in\mathbb{R}:\exists M\ge 1 \text{ such that } \|U(t,s)\|\le Me^{\gamma(t-s)} \text{ for } t\ge s,\ s\in J\}.$$

If $\omega(U)<\infty$, then $U(\cdot,\cdot)$ is called *exponentially bounded*; if $\omega(U)<0$, then $U(\cdot,\cdot)$ is *exponentially stable.* For instance, the evolution families $U(t,s)=e^{\pm(t^2-s^2)}I$ have growth bound $\pm\infty$. In contrast to the semigroup case, the location of $\sigma(U(t,s))$, $t\ge s$, has no influence on the asymptotic behaviour, in general, see [58, §3]. However, the implication

$$\|U(t,s)\|\le C \text{ for } 0\le t-s\le t_0 \quad\Longrightarrow\quad \|U(t,s)\|\le C\exp(\tfrac{\log C}{t_0}(t-s)) \text{ for } t\ge s$$

holds because of the estimate

$$\|U(t,s)\|=\|U(t,s+nt_0)\cdots U(s+t_0,s)\|\le C^{n+1}\le C\exp(\tfrac{\log C}{t_0}(t-s))$$

for $t=s+nt_0+\tau$ with $n\in\mathbb{N}_0$ and $\tau\in[0,t_0)$. In a similar way one sees that

$$\omega(U)<\infty \text{ and } \|U(s+t_1,s)\|\le C_1 \text{ for } s\in J \quad\Longrightarrow\quad \|U(t,s)\|\le C_2\exp(\tfrac{\log C_1}{t_1}(t-s))$$

for $t\ge s$ and a suitable constant $C_2\ge 1$. In particular, the evolution family obtained in Theorem 2.2 is exponentially bounded.

[3] A modified argument also gives $\nu=1/2$, see [1, §6].

[4] In fact, it suffices to consider the norms of b_k, $k=0,\cdots,n$, in $C_b^{\mu}(\mathbb{R}_+,C^1(\partial\Omega))$.

The following notion combines forward exponential stability on certain subspaces with backward exponential stability on their complements. Here and below we set $Q = I - P$ for a projection P.

Definition 3.1. *An evolution family $U(\cdot,\cdot)$ on a Banach space X (with time interval $J = \mathbb{R}$ or $[a,\infty)$) has an* exponential dichotomy *(or is called* hyperbolic*) if there are projections $P(t)$, $t \in J$, and constants $N, \delta > 0$ such that, for $t \geq s$, $s \in J$,*

(a) $U(t,s)P(s) = P(t)U(t,s)$,

(b) *the restriction $U_Q(t,s) : Q(s)X \to Q(t)X$ of $U(t,s)$ is invertible (and we set $U_Q(s,t) := U_Q(t,s)^{-1}$),*

(c) $\|U(t,s)P(s)\| \leq Ne^{-\delta(t-s)}$, *and* $\|U_Q(s,t)Q(t)\| \leq Ne^{-\delta(t-s)}$.

The operator family $\Gamma(t,s)$, $t,s \in J$, given by

$$\Gamma(t,s) = \begin{cases} U(t,s)P(s), & t \geq s, \\ -U_Q(t,s)Q(s), & t < s, \end{cases}$$

is called Green's function *corresponding to $U(\cdot,\cdot)$ and $P(\cdot)$.*

Note that $N \geq 1$ and $U(\cdot,\cdot)$ is exponentially stable if $P(t) = I$. Among the vast literature on exponential dichotomy, we refer to [12], [15], [16], [17], [21], [27], [42], [54]. The importance of this concept relies in particular on its robustness under small perturbations and on its impact on inhomogeneous and nonlinear problems as we will see later.

Let $U(\cdot,\cdot)$ be hyperbolic on J with projections $P(t)$. In computations involving Green's function it is useful to observe that

$$U_Q(t,s)Q(s) = U_Q(t,r)U_Q(r,s)Q(s) \quad \text{for} \quad t,r,s \in J.$$

If $U(\cdot,\cdot)$ is strongly continuous for $t \geq s$, then $P(\cdot)$ is strongly continuous since then the expressions

$$P(t)x - P(s)x = P(t)(x - U(t,s)x) + U(t,s)P(s)x - P(s)x, \quad t \geq s, \tag{3.1}$$
$$Q(t)x - Q(s)x = U_Q(t,s)Q(s)(U(s,t)x - x) + (U(t,a) - U(s,a))U_Q(a,s)Q(s)x, \; t \in [a,s],$$

tend to 0 as $t \to s$. In the parabolic case, the dichotomy projections are in fact Hölder continuous as shown in Proposition 3.18 below. For $t \leq s$ and $t' \leq s'$, we have

$$\begin{aligned} &U_Q(t',s')Q(s')x - U_Q(t,s)Q(s)x \\ &\quad = U_Q(t',s')Q(s')\,[Q(s') - Q(s)]x + U_Q(t',s')Q(s')[U(s,t) - U(s',t')]U_Q(t,s)Q(s)x \\ &\qquad + [Q(t') - Q(t)]U_Q(t,s)Q(s)x, \end{aligned} \tag{3.2}$$

so that $J^2 \ni (t,s) \mapsto U_Q(t,s)Q(s)$ is strongly continuous for $t \geq s$ if $U(\cdot,\cdot)$ has the same property.

The dichotomy projections are not uniquely determined if $J = [a,\infty)$, in general, see [15, p.16]. However, the estimate

$$N^{-1}e^{\delta(t-s)}\|Q(s)x\| \leq \|U(t,s)Q(s)x\| \leq \|U(t,s)x\| + Ne^{-\delta(t-s)}\|P(s)x\|, \quad t \geq s,$$

implies that

$$P(s)X = \{x \in X : \lim_{t\to\infty} U(t,s)x = 0\} \tag{3.3}$$

for $s \in J$. In the case $J = \mathbb{R}$, it is proved in Corollary 3.14 that the projections are unique and that the projections do not depend on t if the evolution family $U(t,s) = T(t-s)$ is given by a semigroup. Definition 3.1 thus agrees with the usual one for semigroups.

Recall that a semigroup $T(\cdot)$ is hyperbolic if and only if the unit circle $\mathbb{T}$ belongs to $\rho(T(t_0))$ for some/all $t_0 > 0$. Then the dichotomy projection P coincides with the *spectral projection*

$$P = \frac{1}{2\pi i} \int_{\mathbb{T}} R(\lambda, T(t_0))\, d\lambda, \tag{3.4}$$

see e.g. [21, §V.1.c]. We denote by A_P the restriction of the generator A of $T(\cdot)$ to PX and by A_Q the restriction of A to QX. Observe that A_P and A_Q generate the restriction of $T(\cdot)$ to PX and QX, respectively. Similar results hold for *periodic* evolution families $U(\cdot,\cdot)$, i.e., $U(t+p, s+p) = U(t,s)$ for $t \geq s$, $s \in \mathbb{R}$, and some $p > 0$. In this case, one has the equalities

$$\sigma(U(s+p,s)) = \sigma(U(p,0)), \quad U(t,s) = U(t,t-p)^n U(s+\tau,s) = U(t,t-\tau)U(s+p,s)^n$$

where $t = s + np + \tau$, $n \in \mathbb{N}_0$, and $\tau \in [0,p)$. A periodic evolution family $U(\cdot,\cdot)$ is hyperbolic if and only if $\sigma(U(p,0)) \cap \mathbb{T} = \emptyset$ because of these relations. The dichotomy projections are then given by

$$P(s) = \frac{1}{2\pi i} \int_{\mathbb{T}} R(\lambda, U(s+p,s))\, d\lambda$$

for $s \in \mathbb{R}$, cf. [16], [17], [27], [42].

In the autonomous case the exponential dichotomy of a semigroup $T(\cdot)$ generated by A always implies the spectral condition $\sigma(A) \cap i\mathbb{R} = \emptyset$. In particular, the exponential stability of $T(\cdot)$ yields $s(A) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\} < 0$. The converse implications fail for general C_0–semigroups, but can be verified if the spectral mapping theorem

$$\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)}, \qquad t \geq 0, \tag{3.5}$$

holds. Formula (3.5) is fulfilled by eventually norm continuous semigroups, and hence by analytic or eventually compact semigroups. We refer to [21] and the references therein for these and related results.

In many cases it is thus possible to characterize the exponential dichotomy of a semigroup by the spectrum of its generator. This is an important fact since in applications usually the generator A is the given object. Unfortunately, in the nonautonomous case there is no hope to relate the location of $\sigma(A(t))$ to the asymptotic behaviour of the evolution family $U(\cdot,\cdot)$ generated by $A(\cdot)$ as is shown by the following example, cf. [15, p.3]. We point out that it deals with periodic evolution families $U_k(\cdot,\cdot)$ on $X = \mathbb{C}^2$ satisfying

$$s(A_1(t)) = -1 < \omega(U_1) \quad \text{and} \quad s(A_2(t)) = 1 > \omega(U_2) \quad \text{for } t \in \mathbb{R},$$

whereas $s(A) \leq \omega(T)$ for each semigroup $T(\cdot)$ with generator A.

Example 3.2. Let $A_k(t) = D(-t)A_k D(t)$ for $t \in \mathbb{R}$, where

$$D(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \qquad A_1 = \begin{pmatrix} -1 & -5 \\ 0 & -1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The operators $A_1(t)$ and $A_2(t)$, $t \in \mathbb{R}$, generate the evolution families

$$U_1(t,s) = D(-t)\ \exp\left[(t-s)\begin{pmatrix} -1 & -4 \\ -1 & -1 \end{pmatrix}\right]\ D(s) \quad \text{and}$$

$$U_2(t,s) = D(-t)\ \exp\left[(t-s)\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\right]\ D(s),$$

respectively. Thus, $\omega(U_1) = 1$ and $\omega(U_2) = 0 = \omega(U_2^{-1})$, but $\sigma(A_1(t)) = \{-1\}$ and $\sigma(A_2(t)) = \{-1, 1\}$ for $t \in \mathbb{R}$. In other words, the exponential stability of $(e^{\tau A_1(t)})_{\tau \geq 0}$ and the exponential dichotomy of $(e^{\tau A_2(t)})_{\tau \geq 0}$ (with constants independent of t) are lost when passing to the nonautonomous problem.

In Example 3.2 we also have $\|R(\lambda, A_k(t))\|_2 = \|R(\lambda, A_k)\|_2$ for $t \in \mathbb{R}$, $\lambda \in \rho(A_k)$, and $k = 1, 2$. This shows that we cannot expect to deduce asymptotic properties of an evolution family from estimates on the resolvent of $A(t)$ along vertical lines as it is possible for a semigroup on a Hilbert space by virtue of the theorem of Gearhart–Howland–Prüss, see e.g. [12, Thm.2.16]. Positivity in the sense of order theory does not help, either: In [55, §5] we constructed a positive evolution family $U(\cdot,\cdot)$ such that $\omega(U) = +\infty$ and $U(\cdot,\cdot)$ solves the Cauchy problem corresponding to generators $A(t)$, $t \geq 0$, of uniformly bounded, positive semigroups on an L^1–space with $s(A(t)) = -\infty$ except for a sequence t_n where $A(t_n) = 0$.

3.2. **Characterizations of exponential dichotomy.** The above mentioned examples indicate that it is quite difficult to establish the exponential dichotomy of a given problem. In fact, the available results for ordinary differential equations are usually restricted to perturbation type arguments, see [15], [16]. Several of these theorems are extended to parabolic evolution equations in the next section. Our approach relies on characterizations of exponential dichotomy discussed below.

Let $U(\cdot,\cdot)$ be an exponentially bounded, strongly continuous evolution family with time interval $J = \mathbb{R}$ acting on a Banach space X and let $E = C_0(\mathbb{R}, X)$ be the space of continuous functions $f : \mathbb{R} \to X$ vanishing at infinity endowed with the sup–norm $\|\cdot\|_\infty$. We then introduce

$$(T_U(t)f)(s) := U(s, s-t)f(s-t), \quad s \in \mathbb{R},\ t \geq 0,\ f \in E. \tag{3.6}$$

It is easy to see that this definition yields a C_0–semigroup $T_U(\cdot)$ on E with $\omega(T_U) = \omega(U)$, which is called the *evolution semigroup* corresponding to $U(\cdot,\cdot)$, cf. [21, Lem.VI.9.10]. We denote its generator by G_U and omit the subscript if the underlying evolution family $U(\cdot,\cdot)$ is clear from the context.

Evolution semigroups were invented by J.S. Howland in 1974 for applications in scattering theory. They have further been used to study perturbation theory and well–posedness of evolution equations. In the nineties their relationship with the asymptotic behaviour of evolution equations was realized by several authors. The recent monograph [12] by C. Chicone and Y. Latushkin is entirely devoted to these matters (with some emphasis on spectral theory and applications to dynamical systems). The bibliographical notes of [12] and the survey article [58] give a detailed account of the field and provide plenty of relevant references. In the following we thus restrict ourselves to a sample of citations directly connected with subjects discussed here.

Evolution semigroups can be defined on a large variety of function spaces. In the present lecture notes we only need the space $E = C_0(\mathbb{R}, X)$ (and some slight variants). Therefore our treatment can be based on the short exposition given in [21, §VI.9], but extending it in several respects. We start with a simple, but important observation.

Lemma 3.3. *Let $U(\cdot,\cdot)$ be a strongly continuous, exponentially bounded evolution family with $J = \mathbb{R}$ on X and $T(\cdot)$ be the corresponding evolution semigroup on $E = C_0(\mathbb{R}, X)$ with generator G. Then $\sigma(T(t))$ is rotationally invariant for $t > 0$ and $\sigma(G)$ is invariant under translations along the imaginary axis. Moreover, $\|R(\lambda, T(t))\| = \|R(|\lambda|, T(t))\|$ and $\|R(\lambda, G)\| = \|R(\operatorname{Re}\lambda, G)\|$ for λ in the respective resolvent sets.*

Proof. We define an isometric isomorphism on E by setting $M_\mu f(s) := e^{i\mu s} f(s)$ for $\mu \in \mathbb{R}$. Clearly, $M_\mu T(t) M_{-\mu} = e^{i\mu t} T(t)$ for $t \geq 0$ so that the assertions follow from standard spectral and semigroup theory, see e.g. [21, §II.2.a]. $\square$

In order to relate the exponential dichotomy of the evolution family $U(\cdot,\cdot)$ and its evolution semigroup we need two preliminary results.

Lemma 3.4. *Let $U(\cdot,\cdot)$ be a strongly continuous, exponentially bounded evolution family with $J = \mathbb{R}$ on X and $T(\cdot)$ be the corresponding evolution semigroup on $E = C_0(\mathbb{R}, X)$. Assume that $T(\cdot)$ is hyperbolic with dichotomy projection $\mathcal{P}$. Then $\varphi\mathcal{P}f = \mathcal{P}(\varphi f)$ for $\varphi \in C_b(\mathbb{R})$ and $f \in E$.*

Proof. Let $\varphi \in C_b(\mathbb{R})$ and $f \in E$. Since $T(t)\varphi\mathcal{P}f = \varphi(\cdot - t)T(t)\mathcal{P}f$, the identity (3.3) yields $\varphi\mathcal{P}f \in \mathcal{P}E$. On the other hand, we have

$$\begin{aligned}\|\mathcal{P}(\varphi\mathcal{Q}f)\|_\infty &= \|\mathcal{P}\varphi T(t)T_Q^{-1}(t)\mathcal{Q}f\|_\infty = \|T(t)\mathcal{P}\varphi(\cdot + t)T_Q^{-1}(t)\mathcal{Q}f\|_\infty \\ &\leq N^2 e^{-2\delta t}\,\|\varphi\|_\infty\,\|f\|_\infty\end{aligned}$$

for $t \geq 0$, and thus $\mathcal{P}(\varphi\mathcal{Q}f) = 0$. As a result, $\mathcal{P}(\varphi f) = \mathcal{P}(\varphi\mathcal{P}f) + \mathcal{P}(\varphi\mathcal{Q}f) = \varphi\mathcal{P}f$. $\square$

Proposition 3.5. *A bounded operator $\mathcal{M}$ on $E = C_0(\mathbb{R}, X)$ is of the form $(\mathcal{M}f)(s) = M(s)f(s)$ for $M(\cdot) \in C_b(\mathbb{R}, \mathcal{L}_s(X))$ if (and only if) $\mathcal{M}(\varphi f) = \varphi\mathcal{M}f$ for $f \in E$ and $\varphi \in C_c(\mathbb{R})$. Moreover, $\|\mathcal{M}\|_{\mathcal{L}(E)} = \sup_{t\in\mathbb{R}} \|M(t)\|_{\mathcal{L}(X)}$.*

Proof. For $\varepsilon > 0$ and $t \in \mathbb{R}$, we choose a continuous function $\varphi_\varepsilon : \mathbb{R} \to [0, 1]$ with $\varphi_\varepsilon(t) = 1$ and $\operatorname{supp}\varphi_\varepsilon \subseteq [t - \varepsilon, t + \varepsilon]$. Let $f \in E$. Our assumption yields

$$\|(\mathcal{M}f)(t)\| = \|(\varphi_\varepsilon\mathcal{M}f)(t)\| = \|(\mathcal{M}\varphi_\varepsilon f)(t)\| \leq \|\mathcal{M}\|\,\|\varphi_\varepsilon f\|_\infty \leq \|\mathcal{M}\| \sup_{|t-s|\leq\varepsilon} \|f(s)\|.$$

Therefore, $f(t) = 0$ implies $(\mathcal{M}f)(t) = 0$. So we can define linear operators $M(t)$ on X by setting $M(t)x := (\mathcal{M}f)(t)$ for some $f \in E$ with $f(t) = x$. Clearly, $\sup_{t\in\mathbb{R}} \|M(t)\| = \|\mathcal{M}\|$ and $\mathcal{M} = M(\cdot)$. For $x \in X$ and $t \in \mathbb{R}$, take $f \in E$ being equal to x in a neighbourhood V of t. Then $V \ni t \mapsto M(t)x = (\mathcal{M}f)(t)$ is continuous. $\square$

Theorem 3.6. *Let $U(\cdot,\cdot)$ be an exponentially bounded evolution family with time interval $J = \mathbb{R}$ on a Banach space X being strongly continuous for $t \geq s$ and $T(\cdot)$ be the corresponding evolution semigroup on $E = C_0(\mathbb{R}, X)$. Then $U(\cdot,\cdot)$ has an exponential* *dichotomy on X with projections $P(\cdot) \in C_b(\mathbb{R}, \mathcal{L}_s(X))$ if and only if $T(\cdot)$ is hyperbolic on E with projection $\mathcal{P}$. If this is the case, then the formulas*

$$P(\cdot) = \mathcal{P} = \frac{1}{2\pi i} \int_{\mathbb{T}} R(\lambda, T(t))\, d\lambda \qquad \text{and} \tag{3.7}$$

$$R(\lambda, T(t))f = \sum_{n=-\infty}^{\infty} \lambda^{-(n+1)} \Gamma(\cdot, \cdot - nt) f(\cdot - nt) \tag{3.8}$$

hold for $t > 0$, $\lambda \in \mathbb{T}$, $f \in E$, and Green's function $\Gamma(\cdot, \cdot)$ of $U(\cdot, \cdot)$.

Proof. (1) Assume that $U(\cdot, \cdot)$ is hyperbolic with strongly continuous projections $P(\cdot)$. Then $\mathcal{P}f := P(\cdot)f$ defines a bounded projection $\mathcal{P}$ on E commuting with $T(t)$ for $t \geq 0$ due to Definition 3.1(a). Using Definition 3.1(b) and (3.2), we see that the operator $T_Q(t) : \mathcal{Q}E \to \mathcal{Q}E$ has the inverse given by

$$\left(T_Q^{-1}(t)\mathcal{Q}f\right)(s) = U_Q(s, s+t)Q(s+t)f(s+t). \tag{3.9}$$

Finally, Definition 3.1(c) implies $\|T(t)\mathcal{P}\|, \|T_Q^{-1}(t)\mathcal{Q}\| \leq Ne^{-\delta t}$ for $t \geq 0$. As a result, $T(\cdot)$ is hyperbolic.

(2) Assume that $T(\cdot)$ is hyperbolic with spectral projection $\mathcal{P}$. By Lemma 3.4 and Proposition 3.5, we have $\mathcal{P} = P(\cdot) \in C_b(\mathbb{R}, \mathcal{L}_s(X))$. The operators $P(t)$ are projections and satisfy Definition 3.1(a) since $\mathcal{P}$ is a projection commuting with $T(t)$. Because of $T(t)\mathcal{Q}E = \mathcal{Q}E$, we obtain

$$Q(s)X = \{(\mathcal{Q}f)(s) : f \in E\} = \{(T(t)\mathcal{Q}g)(s) : g \in E\} = U(s, s-t)Q(s-t)X$$

for $s \in \mathbb{R}$ and $t \geq 0$. Let $s \in \mathbb{R}$, $t \geq 0$, $x \in Q(s-t)X$, and $\varepsilon > 0$. Choose $\varphi_\varepsilon \in C_c(\mathbb{R})$ with $\varphi_\varepsilon(s-t) = x$ and $0 \leq \varphi_\varepsilon \leq 1$ such that $f := \varphi_\varepsilon Q(\cdot)x$ satisfies $\|T(t)f\|_\infty \leq \|T(t)f(s)\| + \varepsilon$. Since $f = \mathcal{Q}f$ and $T(\cdot)$ is hyperbolic, there are constants $N, \delta > 0$ such that

$$N^{-1}e^{\delta t}\,\|x\| \leq N^{-1}e^{\delta t}\,\|f\|_\infty \leq \|T(t)f\|_\infty \leq \|U(s, s-t)x\| + \varepsilon. \tag{3.10}$$

This yields condition (b) and the second estimate in (c) of Definition 3.1. For the other estimate we take $f = \varphi_\varepsilon x$, where $\varphi_\varepsilon \in C_c(\mathbb{R})$ with $\varphi_\varepsilon(s-t) = 1$ and $0 \leq \varphi_\varepsilon \leq 1$. Then

$$\|U(s, s-t)P(s-t)x\| \leq \|T(t)\mathcal{P}f\|_\infty \leq Ne^{-\delta t}\|f\|_\infty = Ne^{-\delta t}\|x\|.$$

(3) Formula (3.7) follows from (3.4), and (3.8) can easily be checked using (3.9). $\square$

In our applications it is necessary to control the dichotomy constants of $U(\cdot, \cdot)$ by given quantities. This is rather simple for the exponent δ, but quite complicated for the constant N. We establish two lemmas which allow to overcome this problem. The first one is taken from [43, Lem.2.2]. To simplify notation, we set

$$U(t, s) := 0 \qquad \text{for} \quad t < s. \tag{3.11}$$

Lemma 3.7. *Let $U(\cdot, \cdot)$ be a strongly continuous evolution family with $J = \mathbb{R}$ on X such that $\|U(t, s)\| \leq Me^{\gamma(t-s)}$ and $\|R(1, T(1))\| \leq C$ for the corresponding evolution semigroup on $E = C_0(\mathbb{R}, X)$. Then $U(\cdot, \cdot)$ has an exponential dichotomy with exponent $\delta \in (0, \log(1 + \frac{1}{C}))$ and a constant $N \geq 1$ depending only on C, M, γ, and δ.*

Proof. Lemma 3.3 and standard spectral theory, see e.g. [21, Prop.IV.1.3], imply

$$\|R(\lambda, T(1))\| \leq \tilde{C} := \frac{C}{1-(e^{\delta}-1)C} \tag{3.12}$$

for $|\lambda| = e^{\alpha}$ and $0 \leq |\alpha| \leq \delta < \log(1+\frac{1}{C})$. By Theorem 3.6 and a simple rescaling argument, we obtain the exponential dichotomy of $U(\cdot,\cdot)$ for every exponent $0 < \delta < \log(1+\frac{1}{C})$. Fix such a δ. If the result were false, then there would exist evolution families $U_n(\cdot,\cdot)$ on Banach spaces X_n satisfying the asumptions (and thus (3.12)), real numbers t_n and s_n, and elements $x_n \in X_n$ such that $\|x_n\| = 1$ and

$$e^{\delta|t_n-s_n|}\,\|\Gamma_n(t_n,s_n)x_n\| \longrightarrow \infty \qquad \text{as} \;\; n\to\infty, \tag{3.13}$$

where $\Gamma_n(\cdot,\cdot)$ is Green's function of $U_n(\cdot,\cdot)$. By Lemma 3.3 and (3.7), the projections $P_n(t)$ are uniformly bounded for $t \in \mathbb{R}$ and $n \in \mathbb{N}$. Hence the operators $\Gamma_n(t,s)$, $s \leq t \leq s+1$, $n \in \mathbb{N}$, are also uniformly bounded. Thus we have either $t_n > s_n+1$ or $t_n < s_n$ in (3.13).

In the first case, we write $t_n = s_n + k_n + \tau_n$ for $k_n \in \mathbb{N}$ and $\tau_n \in (0,1]$. Otherwise, $t_n = s_n - k_n + \tau_n$ for $k_n \in \mathbb{N}$ and $\tau_n \in (0,1]$. Take continuous functions φ_n with $0 \leq \varphi_n \leq 1$, $\operatorname{supp}\varphi_n \subset (s_n + \frac{\tau_n}{2}, s_n + \frac{3\tau_n}{2})$, and $\varphi_n(t_n \mp k_n) = 1$ (here $t_n - k_n$ is used in the first case, and $t_n + k_n$ in the second). Set $\lambda = e^{\mp\delta}$ and $f_n(s) = e^{\pm\delta(s-s_n)}\,\varphi_n(s)\,U_n(s,s_n)x_n$ for $s \in \mathbb{R}$. Using (3.8) and a rescaling, we obtain

$$\begin{aligned}
[R(\lambda, T_{U_n}(1))f_n](t_n) &= \sum_{k=0}^{\infty} \lambda^{-(k+1)} U_n(t_n, t_n-k)P_n(t_n-k)f_n(t_n-k) \\
&\quad - \sum_{k=1}^{\infty} \lambda^{k-1} U_{n,Q}(t_n,t_n+k)Q_n(t_n+k)f(t_n+k) \\
&= \sum_{k=0}^{\infty} e^{\pm\delta(k+1)} e^{\pm\delta(t_n-k-s_n)} \varphi_n(t_n-k)U_n(t_n,s_n)P_n(s_n)x_n \\
&\quad - \sum_{k=1}^{\infty} e^{\mp\delta(k-1)} e^{\pm\delta(t_n+k-s_n)} \varphi_n(t_n+k)U_{n,Q}(t_n,s_n)Q_n(s_n)x_n.
\end{aligned}$$

Here exactly one term does not vanish, namely $k = k_n$ in the first sum if $t_n > s_n$ and $k = k_n$ in the second sum if $t_n < s_n$, so that

$$[R(\lambda, T_{U_n}(1))f_n](t_n) = e^{\pm\delta} e^{\delta|t_n-s_n|}\,\Gamma_n(t_n,s_n)x_n\,.$$

Thus (3.12) yields

$$\|e^{-\delta}e^{\delta|t_n-s_n|}\Gamma_n(t_n,s_n)x_n\| \leq \|R(\lambda, T_{U_n}(1))f_n\|_{\infty} \leq \tilde{C}Me^{2(\gamma+\delta)}.$$

This estimate contradicts (3.13). □

In a second step we involve the generator G of the evolution semigroup $T(\cdot)$ and establish the spectral mapping theorem for $T(\cdot)$ and G. This fact is somewhat astonishing since the standard results from semigroup theory can not be applied here, cf. [21, §IV.3]. Instead one has to construct approximative eigenfunctions of G directly using formula (3.6).

Proposition 3.8. *Let $U(\cdot,\cdot)$ be an exponentially bounded evolution family with $J = \mathbb{R}$ on X being strongly continuous for $t \geq s$.Then the corresponding evolution semigroup $T(\cdot)$ on $E = C_0(\mathbb{R}, X)$ with generator G satisfies $\sigma(T(t)) \setminus \{0\} = \exp(t\sigma(G))$ for $t \geq 0$.*

Proof. By standard spectral theory of semigroups, see [21, Thm.IV.3.6., IV.3.7], and a rescaling argument, it suffices to show that $0 \in \sigma(G)$ if $1 \in A\sigma(T(t_0))$ for some $t_0 > 0$. So assume that $1 \in A\sigma(T(t_0))$. For each $n \in \mathbb{N}$, there exists $f_n \in C_0(\mathbb{R}, X)$ such that $\|f_n\|_\infty = 1$ and $\|f_n - T(kt_0)f_n\|_\infty < \frac{1}{2}$ for all $k = 0, 1, \ldots, 2n$. Hence,

$$\frac{1}{2} < \sup_{s\in\mathbb{R}} \|U(s, s - kt_0)f_n(s - kt_0)\| \leq 2 \tag{3.14}$$

for $k = 0, 1, \ldots, 2n$. Take $s_n \in \mathbb{R}$ such that $\|U(s_n, s_n - nt_0)x_n\| \geq \frac{1}{2}$ where $x_n := f_n(s_n - nt_0)$. Let $I_n = (s_n - nt_0, s_n + nt_0)$. Choose $\alpha_n \in C^1(\mathbb{R})$ such that $\alpha_n(s_n) = 1$, $0 \leq \alpha_n \leq 1$, $\operatorname{supp} \alpha_n \subseteq I_n$, and $\|\alpha_n'\|_\infty \leq \frac{2}{nt_0}$. Define $g_n(s) = \alpha_n(s)U(s, s_n - nt_0)x_n$ for $n \in \mathbb{N}$ and $s \in \mathbb{R}$ (recall (3.11)). Then $g_n \in E$, $\|g_n\|_\infty \geq \|g_n(s_n)\| \geq \frac{1}{2}$, and

$$T(t)g_n(s) = \alpha_n(s-t)U(s, s-t)U(s-t, s_n - nt_0)x_n = \alpha_n(s-t)U(s, s_n - nt_0)x_n$$

for $s - t \geq s_n - nt_0$ and $T(t)g_n(s) = 0$ for $s - t < s_n - nt_0$. Therefore, $g_n \in D(G)$ and $Gg_n(s) = -\alpha_n'(s)U(s, s_n - nt_0)x_n$. Each $s \in I_n$ can be written as $s = s_n + (k + \sigma - n)t_0$ for $k \in \{0, 1, \ldots, 2n\}$ and $\sigma \in [0, 1)$. Using $M := \sup\{\|U(t, s)\| : 0 \leq t - s \leq t_0\} < \infty$ and (3.14), we estimate

$$\begin{aligned}
\|Gg_n(s)\| &\leq \tfrac{2M}{nt_0} \|U(s_n + (k-n)t_0, s_n - nt_0)\, x_n\| \\
&= \tfrac{2M}{nt_0} \|U(s_n + (k-n)t_0, s_n + (k-n)t_0 - kt_0)f_n(s_n + (k-n)t_0 - kt_0)\| \\
&\leq \tfrac{4M}{nt_0}
\end{aligned}$$

for $s \in I_n$. Consequently, 0 is an approximate eigenvalue of G. □

We have thus established our main characterization theorem.

Theorem 3.9. *Let $U(\cdot,\cdot)$ be an exponentially bounded evolution family with time interval $J = \mathbb{R}$ on a Banach space X being strongly continuous for $t \geq s$ and $T(\cdot)$ be the corresponding evolution semigroup on $E = C_0(\mathbb{R}, X)$ with generator G. Then the follwing assertions are equivalent.*

(a) *$U(\cdot,\cdot)$ is hyperbolic with projections $P(t)$.*
(b) *$T(\cdot)$ is hyperbolic with projection $\mathcal{P}$.*
(c) *$\rho(T(t)) \cap \mathbb{T} \neq \emptyset$ for one/all $t > 0$.*
(d) *$\rho(G) \cap i\mathbb{R} \neq \emptyset$.*

In (a) the projections are automatically strongly continuous. If (a)–(d) are true, then (3.7) and (3.8) hold and we have, for $\lambda \in i\mathbb{R}$,

$$(R(\lambda, G)f)(t) = \int_{\mathbb{R}} e^{-\lambda(t-s)}\Gamma(t, s)f(s)\, ds, \qquad t \in \mathbb{R}, \quad f \in E. \tag{3.15}$$

Proof. In view of Theorem 3.6, Lemma 3.3, Proposition 3.8, (3.1), and a rescaling argument, it remains to verify (3.15) for $\lambda = 0$. Set $u = \int_{\mathbb{R}} \Gamma(\cdot, s)f(s)\, ds$ for $f \in E$. Then $u \in E$ and

$$(T(h)u)(t) = \int_{-\infty}^{t-h} U(t, s)P(s)f(s)\, ds - \int_{t-h}^{\infty} U_Q(t, s)Q(s)f(s)\, ds,$$

$$\frac{1}{h}\left[(T(h)u)(t) - u(t)\right] = -\frac{1}{h}\int_{t-h}^{t} U(t, s)f(s)\, ds$$

for $h > 0$. Therefore $u \in D(G)$ and $Gu = -f$. □

The next lemma (from [56]) allows to control the dichotomy constants by $\|G^{-1}\|$, too.

Lemma 3.10. *Let $U(\cdot,\cdot)$ be a strongly continuous evolution family with $J=\mathbb{R}$ on X such that $\|U(t,s)\| \le M^{\gamma(t-s)}$ and $\|G^{-1}\| \le 1/\eta$ for the generator of the corresponding evolution semigroup on $E = C_0(\mathbb{R},X)$ and some $\eta > 0$. Then $U(\cdot,\cdot)$ has exponential dichotomy with exponent $\delta \in (0,\eta)$ and constant N depending only on $\eta, M, \gamma,$ and δ.*

Proof. Since $(-\eta,\eta) \subseteq \rho(G)$, the evolution family $U(\cdot,\cdot)$ is hyperbolic with exponent $\delta \in (0,\eta)$ by Theorem 3.9 and a simple rescaling argument. Fix some $\delta \in (0,\eta)$. Suppose the dichotomy constant $N = N(\delta)$ were not uniform in η, M, γ. Then there would exist evolution families $U_n(\cdot,\cdot)$ on Banach spaces X_n satisfying the assumptions, but there are $x_n \in X_n$ and $t_n, s_n \in \mathbb{R}$ with $\|x_n\| = 1$ and

$$e^{\delta|t_n-s_n|}\,\|\Gamma_n(t_n,s_n)x_n\| \longrightarrow \infty \qquad \text{as } \ n\to\infty, \tag{3.16}$$

where $\Gamma_n(\cdot,\cdot)$ is Green's function of $U_n(\cdot,\cdot)$. Set $f_n(s) = \varphi_n(s)e^{\pm\delta(s-s_n)}\,U_n(s,s_n)x_n$ for $s\in\mathbb{R}$, where $\varphi_n \in C(\mathbb{R})$ has compact support in (s_n, s_n+2), $0 \le \varphi_n \le 1$, $\int \varphi_n(s)\,ds = 1$, and we take $+\delta$ if $t_n \ge s_n$ and $-\delta$ if $t_n < s_n$. Using (3.15) and a rescaling, we obtain

$$\begin{aligned} R(\mp\delta, G_{U_n})f_n(t_n) &= \int_{\mathbb{R}} e^{\pm\delta(t_n-s)}\Gamma_n(t_n,s)f_n(s)\,ds \\ &= e^{\delta|t_n-s_n|}\int_{s_n}^{s_n+2} \varphi_n(s)\,\Gamma_n(t_n,s)U_n(s,s_n)x_n\,ds. \end{aligned}$$

The right hand side is equal to $e^{\delta|t_n-s_n|}\,\Gamma_n(t_n,s_n)x_n$ if $t_n \notin [s_n, s_n+2]$ and to

$$\begin{aligned} &e^{\delta|t_n-s_n|}\,U_n(t_n,s_n)P_n(s_n)x_n\int_{s_n}^{t_n}\varphi_n(s)\,ds - e^{\delta|t_n-s_n|}\,U_n(t_n,s_n)Q_n(s_n)x_n\int_{t_n}^{s_n+2}\varphi_n(s)\,ds \\ &\quad = e^{\delta|t_n-s_n|}\,\Gamma_n(t_n,s_n)x_n - e^{\delta|t_n-s_n|}\,U_n(t_n,s_n)x_n\int_{t_n}^{s_n+2}\varphi_n(s)\,ds \end{aligned}$$

if $t_n \in [s_n, s_n+2]$. In both cases, (3.16) implies that $\|R(\mp\delta, G_{U_n})f(t_n)\| \to \infty$ as $n\to\infty$. This contradicts the estimate

$$\|R(\mp\delta, G_{U_n})f_n\|_\infty \le \|G_{U_n}^{-1}\,(\mp\delta G_{U_n}^{-1} - I)^{-1}\|\,\|f_n\|_\infty \le \frac{M\,e^{2(\delta+\gamma)}}{\eta-\delta}. \qquad \square$$

Theorem 3.6 is essentially due to R. Rau, [52]. Y. Latushkin and S. Montgomery-Smith showed Proposition 3.8 in [33]. Above we presented a somewhat simpler proof taken from [51]. Using different methods, Theorem 3.6 was also proved in [33] and [34]. Theorem 3.9 remains valid if $E = C_0(\mathbb{R},X)$ is replaced by $L^p(\mathbb{R},X)$, $1\le p<\infty$, see [12], [33], [34], [51]. For an alternative approach using 'mild solutions' of the inhomogeneous Cauchy problem, we refer to [10], [12], [15, §3], [16, §IV.3], [35], [44], and the references therein.

3.3. **Extensions and the parabolic case.** Theorem 3.9 is not quite sufficient for the applications in Section 4. First, the evolution family in the parabolic case is strongly continuous only for $t > s$ if the domains of $A(t)$ are not dense in X. Second, in some proofs we 'glue together' evolution families $U(\cdot,\cdot)$ and $V(\cdot,\cdot)$ at a point $a\in\mathbb{R}$; i.e., we set

$$W(t,s)=\begin{cases}U(t,s), & t\ge s\ge a,\\ U(t,a)V(a,s), & t\ge a\ge s,\\ V(t,s), & a\ge t\ge s.\end{cases} \tag{3.17}$$

This clearly defines an evolution family which is strongly continuous for $t\ge s$ if the same holds for $U(\cdot,\cdot)$ and $V(\cdot,\cdot)$. However, if $U(\cdot,\cdot)$ is only strongly continuous for $t>s$, then $t\mapsto W(t,a-1)$ becomes discontinuous at $t=a$. Fortunately, in our setting we still obtain one–sided strong continuity of $s\mapsto W(s,s-t)$, see e.g. the proof of Theorem 4.1, and this property suffices to save Theorem 3.6.

We start with the second problem. Let $U(\cdot,\cdot)$ be an exponentially bounded evolution family with $J=\mathbb{R}$ such that $\mathbb{R}\ni s\mapsto U(s,s-t)$ is strongly continuous for each $t\ge 0$. Then (3.6) still defines an exponentially bounded semigroup on $E=C_0(\mathbb{R},X)$. An inspection of the proof of Theorem 3.6 and of the auxiliary results leading to it shows that the arguments given there remain valid in the present setting. If we additionally assume that $t\mapsto U(t,s)$ is strongly continuous for $t>s$ and each $s\in\mathbb{R}$, then also Lemma 3.7 can be extended to this situation. Next, assume that $U(\cdot,\cdot)$ is an exponentially bounded evolution family with $J=\mathbb{R}$ such that $\mathbb{R}\ni s\mapsto U(s,s-t)$ is strongly continuous from the left for each $t\ge 0$. In this case (3.6) defines an exponentially bounded semigroup $T_U(\cdot)=T(\cdot)$ on the space

$$E_l=E_l(X):=\{f:\mathbb{R}\to X : f \text{ is bounded and left continuous}\}$$

endowed with the supremum norm. Observe that E_l is a Banach space and that the assertions of Lemmas 3.3 and 3.4 concerning $T(t)$ still hold. The other arguments need some modifications. In Proposition 3.5 one only obtains that $\mathcal{M}=M(\cdot)$ is strongly continuous from the left if $\mathcal{M}(\varphi f)=\varphi\mathcal{M}f$ for $f\in E_l(X)$ and $\varphi\in E_l(\mathbb{C})$. In the proof we have to replace the function φ_ε by, e.g., $\mathbb{1}_{(t-\varepsilon,t]}$. Analogously, we can only require strong continuity from the left of the dichotomy projection $P(\cdot)$ of $U(\cdot,\cdot)$ in Theorem 3.6. In the proof of this theorem one further has to change the function φ_ε in (3.10) into $\mathbb{1}_{(s-t-\eta,s-t]}$, where $\eta=\eta(\varepsilon)$ is sufficiently small. Lemma 3.7 still holds if we also assume that $t\mapsto U(t,s)$ is strongly continuous from the left for $t>s$ and each $s\in\mathbb{R}$. Of course, analogous arguments work for 'right continuity' if we use the space

$$E_r:=\{f:\mathbb{R}\to X : f \text{ is bounded and right continuous}\}.$$

Thus we have shown the following facts.

Theorem 3.11. *Let $U(\cdot,\cdot)$ be an exponentially bounded evolution family with time interval $J=\mathbb{R}$ on a Banach space X such that the map $\mathbb{R}\ni s\mapsto U(s,s-t)$ is strongly continuous (from the left, resp. right) for each $t\ge 0$. Let $T(\cdot)$ be the corresponding evolution semigroup on E (on E_l, resp. E_r). Then $U(\cdot,\cdot)$ has an exponential dichotomy on X with projections $P(t)$ being strongly continuous in $t\in\mathbb{R}$ (from the left, resp. right) if and only if $T(\cdot)$ is hyperbolic on E (on E_l, resp. E_r) with projection $\mathcal{P}$ if and only if $\rho(T(t))\cap\mathbb{T}\ne\emptyset$ for some/all $t>0$. If this is the case, then the formulas* (3.7) *and* (3.8) *hold. Let $\|R(1,T(1))\|\le C$. Then the dichotomy exponent δ can be chosen from the interval $(0,\log(1+\frac{1}{C}))$. If, in addition, $(s,\infty)\ni t\mapsto U(t,s)$ is strongly continuous (from* *the left, resp. right) for each $s \in \mathbb{R}$, then the dichotomy constant N only depends on δ, C, and the exponential estimate of $U(\cdot,\cdot)$.*

We state two more variants of the above theorem as remarks which can be proved by the same reasoning.

Remark 3.12. One can replace E by $\tilde{E} = C_b(\mathbb{R}, X)$ in Theorem 3.11 in the case that $\mathbb{R} \ni s \mapsto U(s, s-t)$ is strongly continuous for $t \geq 0$.

Remark 3.13. One implication in Theorem 3.11 holds without any continuity assumption: Let $U(\cdot,\cdot)$ be an exponentially bounded evolution family with $J = \mathbb{R}$ on X which has an exponential dichotomy with projections $P(t)$. As in (3.6) we define the evolution semigroup $T(\cdot)$ on the space $B(\mathbb{R}, X)$ of bounded functions $f : \mathbb{R} \to X$ endowed with the sup–norm. Then $T(\cdot)$ is hyperbolic and (3.7) and (3.8) hold for $f \in B(\mathbb{R}, X)$.

The latter remark implies the uniqueness of the dichotomy projections $P(\cdot)$ if $J = \mathbb{R}$. For an invertible evolution family this fact can be proved directly, cf. [16, p.164].

Corollary 3.14. *Let $U(\cdot,\cdot)$ be an hyperbolic evolution family with $J = \mathbb{R}$. Then the dichotomy projections $P(s)$, $s \in \mathbb{R}$, of $U(\cdot,\cdot)$ are uniquely determined. If $U(s+t, s) = U(t)$ for $s \in \mathbb{R}$, $t \geq 0$, and an exponentially bounded semigroup $U(\cdot)$, then the projections $P(s)$, $s \in \mathbb{R}$, equal the spectral projection P of $U(\cdot)$.*

Proof. The first claim follows from formula (3.7) which holds due to Remark 3.13. If $U(s+t, s) = U(t)$, then we have $\sigma(U(t)) \cap \mathbb{T} = \emptyset$ by (the proof of) [12, Lem.2.38]. Thus $U(\cdot)$ has a spectral projection P, and $P \equiv P(t)$ due to the first assertion. $\square$

As a first application we show a result on the robustness of exponential dichotomy. The first part is well known for strongly continuous evolution families, see e.g. [12, Thm.5.23], [13, Thm.4.3], [27, Thm.7.6.10]. Equation (3.19) is taken from [57, Prop.2.3] and the last assertion from [43, Prop.2.1]. In particular, (3.19) implies that $V(\cdot,\cdot)$ inherits the exponential stability of $U(\cdot,\cdot)$.

Theorem 3.15. *Let $U(\cdot,\cdot)$ and $V(\cdot,\cdot)$ be exponentially bounded evolution families with $J = \mathbb{R}$ such that $\mathbb{R} \ni s \mapsto U(s+t, s)$ and $\mathbb{R} \ni s \mapsto V(s+t, s)$ are strongly continuous (from the left, resp. right) for each $t \geq 0$. Assume that $U(\cdot,\cdot)$ has an exponential dichotomy with projections $P_U(s)$ being strongly continuous (from the left, resp. right) and constants $N, \delta > 0$ and that*

$$q(\tau) := \sup_{s \in \mathbb{R}} \|U(s+\tau, s) - V(s+\tau, s)\| \leq \frac{(1 - e^{-\delta\tau})^2}{8N^2} \tag{3.18}$$

for some $\tau > 0$. Then $V(\cdot,\cdot)$ has an exponential dichotomy with exponent $0 < \delta' < -\frac{1}{\tau}\log(2q(\tau)N + e^{-\delta\tau})$ and projections $P_V(s)$ which are strongly continuous (from the left, resp. right) and satisfy

$$\dim P_V(s)X = \dim P_U(s)X \quad \textit{and} \quad \dim\ker P_V(s) = \dim\ker P_U(s), \quad s \in \mathbb{R}. \tag{3.19}$$

If also $(s, \infty) \ni t \mapsto V(t, s)$ is strongly continuous (from the left, resp. right) for each $s \in \mathbb{R}$, then the dichotomy constant N' of $V(\cdot,\cdot)$ depends only on N, δ, δ', and the exponential estimate of $V(\cdot,\cdot)$.

Proof. For simplicity we consider $\tau = 1$, the proof for arbitrary $\tau > 0$ is the same. We set $q := q(1)$. Formula (3.8) allows us to estimate

$$\|R(\lambda, T_U(1))\| \le \frac{2N}{1 - e^{-\delta}} \tag{3.20}$$

for $\lambda \in \mathbb{T}$ and the corresponding evolution semigroup $T_U(\cdot)$ on E (on E_l, resp. E_r). Therefore condition (3.18) implies that $\lambda - T_V(1)$ has the inverse

$$R(\lambda, T_V(1)) = R(\lambda, T_U(1)) \sum_{n=0}^{\infty} [(T_V(1) - T_U(1))R(\lambda, T_U(1))]^n \quad \text{and}$$

$$\|R(\lambda, T_V(1))\| \le \frac{2N}{(1 - e^{-\delta})} \Big(1 - \frac{2Nq}{(1 - e^{-\delta})}\Big)^{-1} \le \frac{8N}{3(1 - e^{-\delta})} \,. \tag{3.21}$$

Thus $V(\cdot, \cdot)$ is hyperbolic by Theorem 3.11. Considering $e^{\pm\mu(t-s)}U(t, s)$ and $e^{\pm\mu(t-s)}V(t, s)$ for $\mu \in (0, \delta)$, we see that $I - e^{\pm\mu}T_V(1)$ is invertible if

$$e^{\mu} \frac{2qN}{1 - e^{(\mu-\delta)}} < 1.$$

This yields the asserted estimate for the exponent δ' because of Theorem 3.11. We further deduce from (3.7) that

$$\begin{aligned} P_V(\cdot) - P_U(\cdot) &= \frac{1}{2\pi i} \int_{\mathbb{T}} [R(\lambda, T_V(1)) - R(\lambda, T_U(1))] \, d\lambda \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} R(\lambda, T_V(1)) \, [T_V(1) - T_U(1)] \, R(\lambda, T_U(1)) \, d\lambda. \end{aligned}$$

Together with (3.18), (3.20), (3.21), this identity yields

$$\|P_V(s) - P_U(s)\| \le q \, \frac{16 \, N^2}{3(1 - e^{-\delta})^2} \le \frac{2}{3} \tag{3.22}$$

for $s \in \mathbb{R}$. Assertion (3.19) now follows from [32, p.298]. The last claim is a consequence of Theorem 3.11 and (3.21). ☐

If $U(\cdot, \cdot)$ and $V(\cdot, \cdot)$ are strongly continuous, then the above result can be extended to time intervals $J = [a, \infty)$ by a suitable extension as in (3.17). This is formulated in a different setting in Theorem 4.1, where we also give further references and discuss the smallness assumption (3.18).

In a second step we want to extend the full Theorem 3.9 to evolution families being strongly continuous for $t > s$. As the following arguments indicate, probably this objective cannot be achieved without further assumptions. In view of the scope of the present lecture notes, we assume that the Acquistapace–Terreni conditions (AT) from Section 2 hold with $J = \mathbb{R}$. Thus we have operators $A(t)$, $t \in \mathbb{R}$, generating an exponentially bounded evolution family $U(\cdot, \cdot)$ on X being strongly continuous for $t > s$. It is easy to see that the multiplication operator $A(\cdot)$ (see (2.12)) generates the analytic semigroup $e^{tA(\cdot)}$ on $E = C_0(\mathbb{R}, X)$. On E we also define the first derivative $\frac{d}{dt} f = f'$ with domain $C_0^1(\mathbb{R}, X) = \{f \in C^1(\mathbb{R}, X) : f, f' \in C_0(\mathbb{R}, X)\}$.

Formula (3.6) defines an exponentially bounded evolution semigroup $T_U(\cdot) = T(\cdot)$ on $E = C_0(\mathbb{R}, X)$ being strongly continuous on $(0, \infty)$. Hence, we can apply Theorem 3.11

to this semigroup. We further introduce the space of strong continuity of $T(\cdot)$:

$$E_0 = \{f \in E : T(t)f \to f \ \text{ as } \ t \to 0\}.$$

Clearly, E_0 is closed subspace of E, $T(t)E_0 \subseteq E_0$, and the restriction $T_0(t) : E_0 \to E_0$ of $T(t)$ yields a C_0–semigroup. We denote its generator by G_0. Let $f \in E$. Observe that

$$\begin{aligned} T(t)f(s) - f(s) &= U(s,s-t)[f(s-t) - f(s)] + [U(s,s-t) - e^{tA(s-t)}]f(s) \\ &\quad + [e^{tA(s-t)} - e^{tA(s)}]f(s) + [e^{tA(s)} - I]f(s) \end{aligned}$$

for $t \ge 0$ and $s \in \mathbb{R}$. Using (2.14) and (2.9), we see that

$$f \in E_0 \iff \|e^{tA(\cdot)}f - f\|_\infty \to 0 \ \text{ as } \ t \to 0 \iff f \in \overline{D(A(\cdot))}.$$

In other words, $T(t)$ and $e^{tA(\cdot)}$ possess the same space of strong continuity $E_0 = \overline{D(A(\cdot))}$. We also have $T(t)E \subseteq D(A(\cdot)) \subseteq E_0$ for $t > 0$ because of Theorem 2.2. Hence, $T(t)$ induces the 0 operator on the quotient E/E_0 and $T_0(t)$ is not surjective for $t > 0$ (unless $D(A(s)) = X$ for all s, but in this case $T_0(t) = T(t)$). So we deduce from [21, Prop.IV.2.15] that $\sigma(T_0(t)) = \sigma(T(t))$ for $t \ge 0$.

In order to associate an operator G with $T(\cdot)$, we take the Laplace transform of $T(\cdot)$. This yields the bounded operator R_λ on E given by

$$(R_\lambda f)(t) := \Big(\int_0^\infty e^{-\lambda\tau} T(\tau) f \, d\tau\Big)(t) \tag{3.23}$$

$$= \int_{-\infty}^t e^{-\lambda(t-s)} U(t,s) f(s) \, ds \tag{3.24}$$

for $t \in \mathbb{R}$, $f \in E$, and $\operatorname{Re}\lambda > \omega(U)$. As in [8, Thm.3.1.7] one sees that R_λ fulfils the resolvent equation. In view of (3.24), the function $u = R_\lambda f$ satisfies

$$u(t) = e^{-\lambda(t-s)} U(t,s) u(s) + \int_s^t e^{-\lambda(t-\tau)} U(t,\tau) f(\tau) \, d\tau, \quad t \ge s. \tag{3.25}$$

Conversely, if (3.25) holds for $u, f \in E$, then one obtains $u = R_\lambda f$ by letting $s \to -\infty$ in (3.25). If $R_\lambda f = 0$ for some $\operatorname{Re}\lambda > \omega(U)$, then $R_\mu f = 0$ for each $\operatorname{Re}\mu > \omega(U)$ by the resolvent equation, and thus $T(t)f = 0$ for $t > 0$ due to the uniqueness of the Laplace transform and the continuity of $T(\cdot)f$. This means that $U(s+t,s)f(s) = 0$ for $t > 0$ and $s \in \mathbb{R}$. Estimate (2.14) then implies that $e^{tA(s)}f(s) \to 0$ as $t \to 0$, so that $f = 0$ because of [42, Prop.2.1.4]. Since the operators R_λ are injective, there exists a closed operator G with $D(G) = R_\lambda E$ such that $R_\lambda = R(\lambda, G)$ for $\operatorname{Re}\lambda > \omega(U)$, see the proof of e.g. [21, Prop.III.4.6].

For $u \in D(A(\cdot)) \cap C_0^1(\mathbb{R}, X)$ and $\operatorname{Re}\lambda > \omega(U)$, we set $f := \lambda u + u' - A(\cdot)u \in E$. Then u is a strict solution of (2.1) for the operators $A(t) - \lambda$, the initial value $u(s)$, and the inhomogeneity f. Therefore u satisfies (3.25) by Theorem 2.7, and hence $u = R_\lambda f$. This means that G extends the operator $-\frac{d}{dt} + A(\cdot)$ defined on $D(A(\cdot)) \cap C_0^1(\mathbb{R}, X)$. Moreover, (3.25) (with $s = t-1$), (2.13), and (2.32) (with $\varepsilon = 1/2$) imply that $u \in R(\lambda, G)[E \cap C_b^\alpha(\mathbb{R}, X)]$ is differentiable, $u(t) \in D(A(t))$, and $u', A(\cdot)u \in C_b^\alpha(\mathbb{R}, X)$ for $\alpha \in (0, \mu + \nu - 1)$. We thus deduce

$$R(\lambda, G)[E \cap C_b^\alpha(\mathbb{R}, X)] \subseteq D(A(\cdot)) \cap C_0^1(\mathbb{R}, X)$$

by an interpolation argument using [42, Prop.0.2.2, 1.2.19], cf. (5.13). Consequently, $D(A(\cdot)) \cap C_0^1(\mathbb{R}, X)$ is a core of G and, hence, $D(G) \subseteq E_0$. Since $R(\lambda, G)f = R(\lambda, G_0)f$ for $f \in E_0$ and $\operatorname{Re}\lambda > \omega(U)$ due to (3.23), the operator G_0 is the part of G in E_0. This fact and the inclusion $D(G) \subseteq E_0$ yield $\sigma(G_0) = \sigma(G)$ thanks to [21, Prop.IV.2.17]. We summarize our observations in the next proposition.

Proposition 3.16. *Assume that $A(t)$, $t \in \mathbb{R}$, satisfy the Acquistapace–Terreni conditions (AT) from Section 2. Let $U(\cdot,\cdot)$ be the evolution family on X generated by $A(\cdot)$ and $T(\cdot)$ be the induced evolution semigroup on $E = C_0(\mathbb{R}, X)$. Then the operator $-\frac{d}{dt} + A(\cdot)$ defined on $D(A(\cdot)) \cap C_0^1(\mathbb{R}, X)$ has a closure G in E whose resolvent is given the Laplace transform of $T(\cdot)$. The restriction $T_0(\cdot)$ of $T(\cdot)$ to $E_0 = \overline{D(A(\cdot))}$ is a C_0–semigroup generated by the part G_0 of G in E_0. Moreover, $\sigma(G) = \sigma(G_0)$ and $\sigma(T(t)) = \sigma(T_0(t))$ for $t \geq 0$.*

The desired extension of Theorem 3.9 is now a consequence of the previous results.

Theorem 3.17. *In the situation of Proposition 3.16 the following are equivalent.*

(a) *$U(\cdot,\cdot)$ is hyperbolic on X with projections $P(t)$.*
(b) *$T(\cdot)$ is hyperbolic on E with projection $\mathcal{P}$.*
(c) *$T_0(\cdot)$ is hyperbolic on E_0.*
(d) *$\rho(T(t)) \cap \mathbb{T} \neq \emptyset$ for one/all $t > 0$.*
(e) *$\rho(T_0(t)) \cap \mathbb{T} \neq \emptyset$ for one/all $t > 0$.*
(f) *$\rho(G) \cap i\mathbb{R} \neq \emptyset$.*
(g) *$\rho(G_0) \cap i\mathbb{R} \neq \emptyset$.*

If this is the case, then formulas (3.7), (3.8), and (3.15) hold for $T(\cdot)$ and G and, after restriction to E_0, also for $T_0(\cdot)$ and G_0. The dichotomy constants of $U(\cdot,\cdot)$ only depend on its exponential estimate and either $\|R(1, T(1))\|$ or $\|R(1, T_0(1))\|$ or $\|G^{-1}\|$ or $\|G_0^{-1}\|$.

Proof. By Theorem 3.11 assertions (a), (b), and (d) are equivalent and the concluding claims involving $T(\cdot)$ hold. Observe that Lemma 3.3 is valid for $T_0(\cdot)$. Then Proposition 3.16 yields the equivalence of (c), (d), and (e) as well as of (f) and (g). The proof of Proposition 3.8 can directly be extended to the semigroup $T_0(\cdot)$ so that also (e) and (g) are equivalent. Let $u = \int_{\mathbb{R}} \Gamma(\cdot, \tau) f(\tau)\, d\tau$ for $f \in E$. Then one has $u \in E$ and

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\tau) f(\tau)\, d\tau, \quad t \geq s.$$

As above we see that $u \in D(G)$ and $Gu = -f$ if $f \in C_b^\alpha(\mathbb{R}, X) \cap E$. By approximation and rescaling we deduce (3.15) for all $f \in E$. One easily checks that $R(\lambda, T(t))f = R(\lambda, T_0(t))f$ and $R(\mu, G)f = R(\mu, G_0)f$ for $\lambda \in \mathbb{T}$, $\mu \in i\mathbb{R}$, and $f \in E_0$. Thus, (3.7), (3.8), and (3.15) are true for $T_0(t)$ and G_0. Finally, the proofs of Lemmas 3.7 and 3.10 work also for $T_0(\cdot)$, G_0, and G. □

A related characterization of exponential dichotomy in terms of bounded strong solutions to the inhomogeneous problem was shown in [44, Thm.3.2].

Concluding this section we establish additional regularity properties of Green's function in the parabolic case. Refinements of the first part of the next proposition are (in principle) well known, compare [27, Lem.7.6.2] and [42, §6.3]. Parts (b) and (c), however, seem to be new for non–periodic parabolic problems. Their proofs are based on Theorem 3.17.

Proposition 3.18. *Assume that $A(t)$, $t \in J$, satisfy (AT) and that the generated evolution family $U(\cdot,\cdot)$ has an exponential dichotomy with constants $N, \delta > 0$ and projections $P(t)$ on $J \in \{\mathbb{R}, [a,\infty)\}$.*
(a) Let $s \geq t \geq a + \eta > a$ if $J = [a,\infty)$ and $s \geq t$ if $J = \mathbb{R}$. Then $Q(t)X \subseteq D(A(t))$, $P(t)D(A(t)) \subseteq D(A(t))$,

$$\|A(t)Q(t)\| \leq c, \quad \|P(t)\|_{\mathcal{L}(D(A(t)))} \leq c, \quad \text{and} \quad \|A(t)U_Q(t,s)Q(s)\| \leq ce^{-\delta(s-t)}. \tag{3.26}$$

Moreover, $U_Q(\cdot,s)Q(s) \in C^1((a,\infty),\mathcal{L}(X))$ and $\frac{d}{dt}U_Q(t,s)Q(s) = A(t)U_Q(t,s)Q(s)$ (where $a := -\infty$ if $J = \mathbb{R}$). Let $t > s$. Then

$$\|A(t)U(t,s)P(s)\| \leq c\max\{1,(t-s)^{-1}\}e^{-\delta(t-s)}. \tag{3.27}$$

(b) If $J = \mathbb{R}$, then $P(\cdot) \in C_b^\kappa(\mathbb{R},\mathcal{L}(X))$ for $\kappa = \mu + \nu - 1 > 0$.
(c) If $J = [a,\infty)$, then $P(\cdot) \in C_b^\kappa([a+\eta,\infty),\mathcal{L}(X))$ for each $\eta > 0$.
The constants c in (3.26) if $J = \mathbb{R}$ and (3.27) and the Hölder constants in (b) only depend on N, δ, and the constants in (AT). The constants in (3.26) if $J = [a,\infty)$ and (c) depend additionally on $\eta > 0$.

Proof. (a) We deduce (3.26) from $Q(t) = U(t,t')U_Q(t',t)Q(t)$ and (2.13), where $t' := \max\{t-1,a\}$. For $t,s > a$ we take $r \in (a,t)$ and write $U_Q(t,s)Q(s) = U(t,r)U_Q(r,s)Q(s)$. This yields the asserted differential equation. Estimate (3.27) is clear for $s < t \leq s+1$ and follows from $A(t)U(t,s)P(s) = A(t)U(t,t-1)U(t-1,s)P(s)$ for $t \geq s+1$.

(b) Let $\tau \in \mathbb{R}$. The evolution family $U_\tau(t,s) = U(t+\tau,s+\tau)$, $t \geq s$, (generated by $A(\cdot+\tau)$) has an exponential dichotomy with constants N,δ and projections $P(\cdot+\tau)$. Fix $t > 0$, $s \in \mathbb{R}$, and $x \in X$. Take $\varphi \in C(\mathbb{R})$ with $\varphi(s) = 1$ and $\operatorname{supp}\varphi \subseteq (s-t/2, s+t/2)$. Set $f = \varphi(\cdot)x$. Using (3.7) and (3.8) in Remark 3.13, we obtain

$$\begin{aligned}
&P(s+\tau)x - P(s)x \\
&= (P(\cdot+\tau)f)(s) - (P(\cdot)f)(s) \\
&= \frac{1}{2\pi i}\int_{\mathbb{T}} \Big(R(\lambda,T_{U_\tau}(t))[T_{U_\tau}(t) - T_U(t)]R(\lambda,T_U(t))f\Big)(s)\,d\lambda \\
&= \frac{1}{2\pi i}\int_{\mathbb{T}} \Big(\sum_{k,l=-\infty}^{\infty} \lambda^{-k-1}\Gamma(s+\tau,s+\tau-kt)[U(s+\tau-kt,s+\tau-(k+1)t) \\
&\quad - U(s-kt,s-(k+1)t)]\,\lambda^{-l-1}\Gamma(s-(k+1)t,s-(k+1+l)t)\varphi(s-(k+1+l)t)x\Big)d\lambda.
\end{aligned}$$

Observing that $\varphi(s-(k+1+l)t) = 1$ for $l = -k-1$ and 0 otherwise, we arrive at

$$\begin{aligned}
P(s+\tau)x - P(s)x = \sum_{k=-\infty}^{\infty} &\Gamma(s+\tau,s+\tau-kt)\,[U(s+\tau-kt,s+\tau-(k+1)t) \\
&- U(s-kt,s-(k+1)t)]\,\Gamma(s-(k+1)t,s)x.
\end{aligned} \tag{3.28}$$

Taking $t = 1$, this formula yields

$$\|P(s+\tau) - P(s)\| \leq \frac{2N^2e^{-\delta}}{1-e^{-2\delta}} \sup_{r\in\mathbb{R}} \|U(r+\tau,r+\tau-1) - U(r,r-1)\|. \tag{3.29}$$

If $|\tau| \le 1/2$, we deduce from (2.13) and [1, Thm.2.3]

$$\begin{aligned}\|U(r+\tau,&r+\tau-1) - U(r,r-1)\| \\ &\le \int_r^{r+\tau} \|A(\sigma)U(\sigma,r-1+\tau)\|\,d\sigma + \|U(r,r-1+\tau) - U(r,r-1)\| \\ &\le c\,(|\tau| + |\tau|^{\mu+\nu-1}).\end{aligned} \tag{3.30}$$

Assertion (b) is a consequence of (3.29) and (3.30).

(c) We define $R = Q(a) - P(a)$ and

$$V(t,s) = \begin{cases} U(t,s), & t \ge s \ge a, \\ U(t,a)e^{\delta(a-s)R}, & t \ge a \ge s, \\ e^{\delta(t-s)R}, & a \ge t \ge s. \end{cases}$$

Since $e^{t\delta R} = e^{-t\delta}P(a) + e^{t\delta}Q(a)$, the evolution family $V(\cdot,\cdot)$ is exponentially dichotomic with constants N, δ and projections $P_V(t) = P(t)$ for $t \ge a$ and $P_V(t) = P(a)$ for $t \le a$.

Let $s \ge a + \eta$ for some $\eta \in (0, 1/2]$. If $s - a \in [\eta, 3/2)$, we set $t := 2(s-a) \in [2\eta, 3]$. If $s - a = j + 1/2 + \tau$ for some $j \in \mathbb{N}$ and $\tau \in [0,1)$, we set $t := 1 + \tau(j+1/2)^{-1} \in [1, 5/3]$. As a result, we have fixed $j \in \mathbb{N}_0$ and $t \in [2\eta, 3]$ such that $s - a = jt + t/2$. Take $|\tau| \le t/4$. Replacing $U(\cdot,\cdot)$ by $V(\cdot,\cdot)$ in (3.28) (which is possible by Remark 3.13), we arrive at

$$\|P(s+\tau) - P(s)\| \le \frac{2N^2 e^{-\delta t}}{1 - e^{-2\delta t}} \sup_{k\in\mathbb{Z}} \|V(s_k+\tau, s_k+\tau-t) - V(s_k, s_k-t)\|,$$

where $s_k = s - kt$. If $k \ge j+1$, then $s_k \le a - t/2$ and thus

$$V(s_k+\tau, s_k+\tau-t) - V(s_k, s_k-t) = e^{-\delta tR} - e^{-\delta tR} = 0.$$

In the case $k \le j-1$, we obtain $s_k - t \ge a + t/2$ and

$$V(s_k+\tau, s_k+\tau-t) - V(s_k, s_k-t) = U(s_k+\tau, s_k+\tau-t) - U(s_k, s_k-t).$$

As in (3.30) we estimate

$$\|V(s_k+\tau, s_k+\tau-t) - V(s_k, s_k-t)\| \le c\,|\tau|^{\mu+\nu-1}, \qquad k \le j-1. \tag{3.31}$$

For $k = j$, we have $s_k = a + t/2$ and

$$\begin{aligned}&\|V(s_k+\tau, s_k+\tau-t) - V(s_k, s_k-t)\| \\ &\quad\le \|(U(a+\tau+t/2, a) - U(a+t/2, a))e^{(\frac{t}{2}-\tau)\delta R}\| + \|U(a+t/2, a)(e^{(\frac{t}{2}-\tau)\delta R} - e^{\frac{t}{2}\delta R})\| \\ &\quad\le c\,|\tau|\end{aligned}$$

as in (3.30). Here the constants may depend on η. Assertion (c) follows by combining the above estimates. ☐

Remark 3.19. In assertions (b) and (c) the projections are Lipschitz continuous in t if $A(t)$ is densely defined and their adjoints $A^*(t)$ also satisfy (AT). This follows if one uses in (3.30) and (3.31) the estimate $\|\frac{d}{ds}U(t,s)\| \le c(t-s)^{-1}$ from [3, Thm.6.4].

4. Exponential dichotomy of parabolic evolution equations

Employing the theory presented in the previous two sections, we now derive sufficient conditions for exponential dichotomy of $U(\cdot,\cdot)$ in terms of the given operators $A(t)$. Our treatment is inspired by the monographs [15] by W.A. Coppel on ordinary differential equations and [27] by D. Henry on the case that $A(t) = A + B(t)$, where A is sectorial and densely defined and $B(\cdot) \in C_b^\alpha(J, D((w-A)^\beta))$ for some $\alpha, \beta \in (0,1)$.

4.1. **Robustness.** Assume that $A(\cdot)$ and $B(\cdot)$ generate the evolution families $U(\cdot,\cdot)$ and $V(\cdot,\cdot)$, respectively, and that $U(\cdot,\cdot)$ is hyperbolic. It is one of the main features of exponential dichotomy that $V(\cdot,\cdot)$ is also hyperbolic if $B(t)$ is 'close' to $A(t)$. The trivial example $A(t) = -\delta I$ shows that one really needs a smallness condition. One can formulate such results on the level of the evolution families as we did in Theorem 3.15. But, of course, this should only be a preliminary step in order to derive conditions on $B(t)$ itself. This was done in, e.g., [12], [13], [15], [16] for bounded perturbations, in [14], [27], [55] for unbounded perturbations of 'lower order', and in [37], [48] for the case of constant domains $D(A(t)) = D(B(t)) = Y$. (Here [13], [14], [48] are fomulated in a different context and in [37] further conditions are required.) In the next theorem we allow for all parabolic problems in the framework of Section 2 and only assume that the resolvents of $A(t)$ and $B(t)$ are close in operator norm. In [59] our result was partially extended to the case of partial functional differential equations, see also [36] for ordinary functional differential equations.

Theorem 4.1. *Let $A(t)$ and $B(t)$ satisfy (AT) from Section 2 for $t \in J$, $J \in \{[a,\infty), \mathbb{R}\}$. Let $U(\cdot,\cdot)$ and $V(\cdot,\cdot)$ be the generated evolution families. Assume that $U(\cdot,\cdot)$ has an exponential dichotomy on J with constants $N, \delta > 0$ and projections $P_U(\cdot)$. Then there is a number $\tilde{q}$ depending only on N, δ, and the constants in (AT) such that if*

$$q := \sup_{t\in J} \|R(w, A(t)) - R(w, B(t))\| \le \tilde{q}, \tag{4.1}$$

then $V(\cdot,\cdot)$ has an exponential dichotomy on $\mathbb{R}$ (if $J = \mathbb{R}$), resp. on $[a+\eta, \infty)$ for every $\eta > 0$ (if $J = [a,\infty)$), with Hölder continuous projections $P_V(s)$ and constants only depending on N, δ, and the constants in (AT) (and η if $J = [a,\infty)$). Moreover,

$$\dim P_V(s)X = \dim P_U(s)X \quad \text{and} \quad \dim\ker P_V(s) = \dim\ker P_U(s) \tag{4.2}$$

for $s \in \mathbb{R}$, resp. $s \in [a+\eta, \infty)$.

Proof. Recall that in the parabolic case the dichotomy projections are automatically Hölder continuous due to Proposition 3.18.

At first, let $J = [a,\infty)$. In order to apply Theorem 3.15 we extend $U(\cdot,\cdot)$ and $V(\cdot,\cdot)$ to the time interval $J = \mathbb{R}$ by setting

$$\tilde{U}(t,s) = \begin{cases} U(t,s), & t \ge s \ge b, \\ U(t,b)e^{(b-s)R}, & t \ge b \ge s, \\ e^{(t-s)R}, & b \ge t \ge s, \end{cases} \qquad \tilde{V}(t,s) = \begin{cases} V(t,s), & t \ge s \ge b, \\ V(t,b)e^{(b-s)R}, & t \ge b \ge s, \\ e^{(t-s)R}, & b \ge t \ge s, \end{cases}$$

where $R = \delta Q(b) - dP(b)$, $d \ge \delta$, and $b = a + \eta$ for a fixed $\eta > 0$. Observe that $\mathbb{R} \ni s \mapsto W(s, s-t)$ and $(\sigma, \infty) \ni \tau \mapsto W(\tau, \sigma)$ are continuous from the left for each $t \ge 0$ and $\sigma \in \mathbb{R}$ and $W = \tilde{U}$ and $W = \tilde{V}$. Since $e^{tR} = e^{-td}P(a) + e^{t\delta}Q(a)$, the evolution family

$\tilde{U}(\cdot,\cdot)$ is exponentially dichotomic with constants N,δ and projections $\tilde{P}(t) = P_U(t)$ for $t \geq b$ and $\tilde{P}(t) = P_U(b)$ for $t \leq b$. Moreover, both evolution families possess exponential bounds not depending on d. Set $\overline{q} = (1-e^{-\delta})^2/(8N^2)$. For $b-1 < s < b-1/2$, we have

$$\tilde{U}(s+1,s) - \tilde{V}(s+1,s) = (U(s+1,b) - V(s+1,b))\,(P(b)e^{(s-b)d} + Q(b)e^{(b-s)\delta}).$$

We fix $d \geq \delta$ such that

$$\|(U(s+1,b) - V(s+1,b))P(b)e^{(s-b)d}\| \leq 2NM\,e^{-d/2} \leq \tfrac{\overline{q}}{2}\,,$$

where M is a uniform estimate for $U(t,r)$ and $V(t,r)$ and $0 \leq t-r \leq 1/2$. Propositions 2.6 and 3.18 further show that

$$\|(U(s+1,b) - V(s+1,b))Q(b)e^{(b-s)\delta}\| \leq c\,\|(U(s+1,b) - V(s+1,b))R(w,A(b))\| \leq c\,q^{\alpha}$$

where $\alpha \in (0,\mu)$ is fixed and c depends only on N,δ,η, and the constants in (AT). Proposition 2.6 also yields

$$\|\tilde{U}(s+1,s) - \tilde{V}(s+1,s)\| \leq N(1+e^{\delta})\,\|(U(s+1,b) - V(s+1,b)\| \leq c\,q^{\beta}$$

for $b-1/2 \leq s < b$ and $q \leq q_0(1/2)$, where $\beta > 0$ and $q_0(1/2) > 0$ are given by Proposition 2.6. Similarly,

$$\|\tilde{U}(s+1,s) - \tilde{V}(s+1,s)\| = \|U(s+1,s) - V(s+1,s)\| \leq c\,q^{\beta}$$

for $s \geq b$. Further $\tilde{U}(s+1,s) = \tilde{V}(s+1,s)$ for $s \leq b-1$. So we find a number $\tilde{q} > 0$ such that $q \leq \tilde{q}$ implies $\|\tilde{U}(s+1,s) - \tilde{V}(s+1,s)\| \leq \overline{q}$ for each $s \in \mathbb{R}$. The assertions for $J = [a,\infty)$ now follow from Theorem 3.15.

In the case $J = \mathbb{R}$ one can directly estimate $U(s+1,s) - V(s+1,s)$ using Proposition 2.6 and then apply Theorem 3.15. □

In the above arguments we used Theorem 3.15, where $T_U(1) - T_V(1)$ was estimated. One can also try to involve the difference $G_U - G_V$ of the corresponding generators, cf. [55]. It seems that this approach does not work in the same generality as the above theorem, but it yields better values of $\tilde{q}$. For instance, if $A(t)$ and $B(t)$, $t \in \mathbb{R}$, are bounded, then the smallness condition becomes $\|A(\cdot) - B(\cdot)\|_\infty < \frac{\delta}{2N}$, see [55, §4]. Refinements of this assumption are close to be optimal, see [46]. The next example indicates how the estimate (4.1) can be translated into conditions on the coefficients of a partial differential equation.

Example 4.2. Let the operators $A(t)$, $t \geq 0$, be given on $X = L^p(\Omega)$, $1 < p < \infty$, or $X = C(\overline{\Omega})$ as in Example 2.9. We assume that there are time independent coefficients $\tilde{a}_{kl}, \tilde{a}_k, \tilde{a}_0 \in C(\overline{\Omega})$, and $\tilde{b}_k, \tilde{b}_0 \in C^1(\overline{\Omega})$, $k,l = 1,\cdots,n$, satisfying the assumptions of Example 2.9 and define the operator A on X as in (2.43) and (2.39) for these coefficients. Arguing as in (2.44), one sees that

$$\sup_{t\geq 0}\|R(w,A(t)) - R(w,A)\| \leq c\sup_{t\geq 0}\max_{kl}\{\|a_{kl}(t,\cdot) - \tilde{a}_{kl}(\cdot)\|_{C(\overline{\Omega})}\,, \|a_k(t,\cdot) - \tilde{a}_k(\cdot)\|_{C(\overline{\Omega})}\,,$$
$$\|b_k(t,\cdot) - \tilde{b}_k(\cdot)\|_{C^1(\partial\Omega)}\}.$$

We also assume that $\sigma(A) \cap i\mathbb{R} = \emptyset$. Due to Theorem 4.3, the evolution family generated by $A(\cdot)$ has an exponential dichotomy if the quantity on the right hand side of the above estimate is sufficiently small, cf. Remark 2.10.

4.2. **Asymptotically autonomous equations.** We suppose that (AT) holds and that the resolvents $R(w,A(t))$ converge in operator norm as $t\to\infty$ to the resolvent $R(w,A)$ of a sectorial operator with $\sigma(A)\cap i\mathbb{R}=\emptyset$. It is reasonable to expect that then the evolution family $U(\cdot,\cdot)$ generated by $A(\cdot)$ has an exponential dichotomy, at least on an interval $[a,\infty)\subseteq\mathbb{R}_+$. In the case of constant domains (cf. Remark 4.5) and exponential stability, this was in fact shown by H. Tanabe already in 1961, see [62, §5.6]. Tanabe's result was extended by D. Guidetti, [26], to the case of Kato–Tanabe conditions. In [57], we have treated the case of exponential dichotomy under Acquistapace–Terreni conditions assuming the density of the domains and requiring convergence of $R(w,A(\cdot))$ in a slightly stronger norm. C.J.K. Batty and R. Chill, [11], then generalized this result in several directions: They also considered Kato–Tanabe conditions and asymptotically periodic problems, they managed to deal with resolvents converging in $\mathcal{L}(X)$, and they gave sufficient conditions for the almost periodicity of $U(s+\cdot,s)$. In the next theorem the case of non–dense domains is covered for the first time. In [59] we have partially extended our result to the case of partial functional differential equations, see also [36].

Theorem 4.3. *Let $A(t)$, $t\ge 0$, satisfy (AT) from Section 2 and let A satisfy (AT1). Assume that $\sigma(A)\cap i\mathbb{R}=\emptyset$ and $R(w,A(t))\to R(w,A)$ in $\mathcal{L}(X)$ as $t\to\infty$. Then the evolution family $U(\cdot,\cdot)$ generated by $A(\cdot)$ has an exponential dichotomy on an interval $[a,\infty)\subseteq\mathbb{R}_+$ with projections $P(s)$ being globally Hölder continuous for $s\ge a$. Moreover, $P(s)\to P$ strongly and $U(s+\tau,s)\to e^{\tau A}$ in $\mathcal{L}(X)$ for $\tau\ge 0$ as $s\to\infty$,* $\dim P(t)X=\dim PX$*, and* $\dim\ker P(t)=\dim\ker P$ *for $t\ge a$, where P is the dichotomy projection of $e^{\tau A}$. The dichotomy constants of $U(\cdot,\cdot)$ only depend on the dichotomy constants of A, the type of A, and the constants in (AT). We can take the same number a for all $A(t)$ and A subject to the same constants and satisfying* $\sup_{t\ge s}\|R(w,A(t))-R(w,A)\|\le q(s)$*, $s\ge 0$, for a fixed function $q(s)$ converging to 0 as $s\to\infty$.*

Proof. Recall that e^{tA} has an exponential dichotomy with exponent $\delta>0$ due to, e.g., [21, Thm.IV.3.12, V.1.17] or [42, §2.3]. For a given $a\ge 0$, we set

$$A_a(t)=\begin{cases}A(t), & t\ge a,\\ A(a), & t<a,\end{cases}\qquad U_a(t,s)=\begin{cases}U(t,s), & t\ge s\ge a,\\ U(t,a)e^{(a-s)A(a)}, & t\ge a\ge s,\\ e^{(t-s)A(a)}, & a\ge t\ge s.\end{cases}$$

Observe that $A_a(t)$, $t\in\mathbb{R}$, satisfy (AT) and generate $U_a(\cdot,\cdot)$. Moreover,

$$\sup_{\tau\in\mathbb{R}}\|R(w,A_a(\tau))-R(w,A)\|\le q(a)$$

for the function q from the statement. Proposition 2.6 thus yields

$$\|U_a(s+t,s)-e^{tA}\|\le c(t)\,q(a)^\beta$$

for $s\in\mathbb{R}$, $t>0$, $q(a)\le q_0(t)$, and constants $\beta>0$, $q_0(t)>0$, $c(t)>0$. Thanks to Theorem 3.15 the evolution family $U_a(\cdot,\cdot)$ has an exponential dichotomy and its projections have the same rank as those of e^{tA} if we choose a large enough. The projections $P_a(\cdot)$ are Hölder continuous by Proposition 3.18. Restricting to the time interval $[a,\infty)$, we establish the theorem except for asserted convergence. The convergence of $U(s+t,s)$ follows directly from Proposition 2.6, whereas the strong convergence of $P(\cdot)$ is a consequence of the next lemma.[5] $\square$

Lemma 4.4. *Let $U(\cdot,\cdot)$ be an exponentially bounded evolution family on a Banach space X with $J=\mathbb{R}$ such that $s\mapsto U(s,s-t)$ is strongly continuous for $t\ge 0$ and let $S(\cdot)$ be a semigroup. Assume that $U(\cdot,\cdot)$ and $S(\cdot)$ have exponential dichotomies with projections $P(t)$ and P, respectively, and that $U(s+t,s)\to S(t)$ strongly for $t\ge 0$ as $s\to\infty$. Then $P(s)\to P$ strongly as $s\to\infty$.*

Proof. Let $\tilde{E} := \{f\in C(\mathbb{R},X) : f(t)\to 0 \text{ as } t\to-\infty,\ f(t)\to f_\infty \text{ as } t\to\infty\}$ be endowed with the sup–norm. Clearly,

$$(\tilde{T}_U(t)f)\,(s) := U(s,s-t)f(s-t), \quad t\ge 0,\ s\in\mathbb{R},\ f\in\tilde{E}, \tag{4.3}$$

defines an exponentially bounded semigroup $\tilde{T}_U(\cdot)$ on $\tilde{E}$. Its restriction to $E=C_0(\mathbb{R},X)$ is the evolution semigroup $T_U(\cdot)$. The spaces $\tilde{E}$ and $E\times X =: \hat{E}$ are isomorphic via

$$\Phi:\tilde{F}\to\hat{F},\ f\mapsto (f-Mf_\infty, f_\infty),$$

where $Mx:=\varphi(\cdot)x$ for a fixed function $\varphi\in C(\mathbb{R})$ with support in $\mathbb{R}_+$ and $\lim_{t\to\infty}\varphi(t)=1$. We consider the induced semigroup $\hat{T}_U(\cdot)$ on $\hat{E}$ given by

$$\hat{T}_U(t) := \Phi\tilde{T}_U(t)\Phi^{-1} = \begin{pmatrix} T_U(t) & \tilde{T}_U(t)M - MS(t) \\ 0 & S(t)\end{pmatrix}, \quad t\ge 0.$$

By the assumption and Theorem 3.11, $\lambda - T_U(1)$ and $\lambda - S(1)$ are invertible for $\lambda\in\mathbb{T}$; hence $\lambda\in\rho(\hat{T}_U(1))$. This means that $\hat{T}_U(\cdot)$ and $\tilde{T}_U(\cdot)$ are hyperbolic with spectral projection $\hat{\mathcal{P}}$ and $\tilde{\mathcal{P}}$, respectively. On the other hand, formula (4.3) defines a semigroup $T_U^b(\cdot)$ on $C_b(\mathbb{R},X)$ which is hyperbolic with projection

$$P(\cdot) = \mathcal{P}^b = \frac{1}{2\pi i}\int_{\mathbb{T}} R(\lambda, T_U^b(1))\,d\lambda$$

by Remark 3.12 and the exponential dichotomy of $U(\cdot,\cdot)$. Since $R(\lambda,T_U^b(1))f = R(\lambda,\tilde{T}_U(1))f$ for $f\in\tilde{E}$, we obtain $\tilde{\mathcal{P}}f = \mathcal{P}^b f = P(\cdot)f$ for $f\in\tilde{E}$. Therefore $P(t)$ converges strongly to a projection P' as $t\to\infty$. Finally,

$$\begin{aligned}\hat{\mathcal{P}} &= \frac{1}{2\pi i}\int_{\mathbb{T}} R(\lambda,\hat{T}_U(1))\,d\lambda = \begin{pmatrix} P(\cdot) & * \\ 0 & P\end{pmatrix} \\ &= \frac{1}{2\pi i}\int_{\mathbb{T}} \Phi R(\lambda,\tilde{T}_U(1))\Phi^{-1}\,d\lambda = \Phi P(\cdot)\Phi^{-1} = \begin{pmatrix} P(\cdot) & * \\ 0 & P'\end{pmatrix}\end{aligned}$$

so that $P=P'$. $\square$

Remark 4.5. Assume that the operators $A(t)$ satisfy (AT1), that $D(A(t)) = D(A(0)) =: Y$, $t\ge 0$, with uniformly equivalent graph norms, and that $A(\cdot):\mathbb{R}_+\to\mathcal{L}(Y,X)$ is globally Hölder continuous of exponent $\mu>0$. Let $A(t)$ converge in $\mathcal{L}(Y,X)$ to a closed operator A with domain Y as $t\to\infty$. If $\sigma(A)\cap i\mathbb{R}=\emptyset$, then the assumptions of Theorem 4.3 hold. Moreover, if $w=0$ then $s(A)<0$. We omit the straightforward proof.

The following simple example shows that strong convergence of the resolvents would not suffice in Theorem 4.3.

[5] In [57, Thm.3.3] we gave a direct proof of the convergence of $P(\cdot)$ if $D(A(t))$ are dense.

Example 4.6. Let $X = \ell^2$, $A = -I$, and

$$A(t)(x_k) = -(x_1, \cdots, x_{n-1}, (t-n+1)x_n, 0, \cdots) \quad \text{for} \quad n-1 \le t \le n.$$

Then, $A(t)x \to -x$ for $x = (x_k) \in X$ as $t \to \infty$ and $s(A) = -1$. But $U(t,0)x = x$ if $0 \le t \le n$ and $x_k = 0$ for $k = 1, \cdots, n$, so that $\|U(n,0)\| = 1$.

Theorem 4.3 only gives an exponential dichotomy on an interval $[a, \infty)$ where a may be rather large. If $s(A) < 0$, then $U(\cdot,\cdot)$ is exponentially stable on $[a,\infty)$, and hence on $\mathbb{R}_+$. In the case of a non trivial exponential dichotomy, one cannot extend the exponential dichotomy to the left, in general, as the next example shows (in which (AT) is not quite fulfilled).

Example 4.7. On $X = L^1(\mathbb{R})$ we consider the semigroups $T_1(\cdot)$ generated by the second derivative and $T_2(t)f = e^t f|\mathbb{R}_+ + e^{-t} f|\mathbb{R}_-$, and define the evolution family

$$U(t,s) = \begin{cases} T_1(t-s), & s \le t \le 1, \\ T_2(t-1)T_1(1-s), & s \le 1 \le t, \\ T_2(t-s), & 1 \le s \le t. \end{cases}$$

Clearly, $U(\cdot,\cdot)$ has exponential dichotomy on $[1,\infty)$ with $Q(t)f = f|\mathbb{R}_+$ for $t \ge 1$. However, $U(1,0)f$ is a smooth function so that condition (b) of Definition 3.1 cannot be satisfied on $J = [0, \infty)$.

The next result gives sufficient conditions to extend exponential dichotomy to larger time intervals. It is essentially due to [36, §2].

Proposition 4.8. *Let $U(\cdot,\cdot)$ be an exponentially bounded evolution family on the time interval $[a,\infty)$ having an exponential dichotomy on $[b,\infty)$ for $b > a$ with projections $P(t)$ and constants $N, \delta > 0$. If $\dim Q(t)X < \infty$ for some/all $t \ge b$ and $U(b,a)^* x^* \ne 0$ for every $x^* \in Q(b)^* X^* \setminus \{0\}$, then $U(\cdot,\cdot)$ has an exponential dichotomy on $[a,\infty)$ with exponent δ and projections $\tilde{P}(t)$, $t \ge a$. Moreover, for $t \ge b$,*

$$\dim \ker \tilde{P}(t) = \dim \ker P(t), \qquad \tilde{P}(t)X = P(t)X, \tag{4.4}$$

$$\|\tilde{P}(t) - P(t)\| \le c\, e^{-2\delta(t-b)}. \tag{4.5}$$

If $U(\cdot,\cdot)$ is strongly continuous on $[a,\infty)$, then $\tilde{P}(t)$ and $U_{\tilde{Q}}(t,s)\tilde{Q}(s)$ are strongly continuous for $t, s \ge a$.

Proof. (1) We introduce the closed subspaces $X_-(t) := \{x \in X : U(b,t)x \in P(b)X\}$ for $t \in [a,b]$. We first want to verify

$$\operatorname{codim} X_-(a) = \operatorname{codim} P(b)X < \infty. \tag{4.6}$$

Observe that $\operatorname{codim} P(b)X = \dim Q(t)X$ for all $t \ge b$. We define $X_+^*(b) := Q(b)^* X^*$ and $X_+^*(a) := U(b,a)^* X_+^*(b)$. The assumptions imply that $X_+^*(a)$ and $X_+^*(b)$ are finite dimensional and that $U(b,a)^* : X_+^*(b) \to X_+^*(a)$ is an isomorphism. Using [30, Lem.III.1.40], we obtain

$$\begin{aligned} [P(b)X]^\perp &:= \{x^* \in X^* : \langle x, x^* \rangle = 0 \quad \forall\, x \in P(b)X\} \\ &= \{x^* \in X^* : \langle y, P(b)^* x^* \rangle = 0 \quad \forall\, y \in X\} = \ker P(b)^* = X_+^*(b), \end{aligned} \tag{4.7}$$

$$\operatorname{codim} P(b)X = \dim\, [P(b)X]^\perp = \dim X_+^*(b) = \dim X_+^*(a). \tag{4.8}$$

Let $x \in X_-(a)$ and $x^* \in X_+^*(b)$. By (4.7) we have $0 = \langle U(b,a)x, x^*\rangle = \langle x, U(b,a)^*x^*\rangle$, which gives $x \in {}^\perp[X_+^*(a)] := \{y \in X : \langle y, y^*\rangle \quad \forall\, y^* \in X_+^*(a)\}$. Inversing this argument, we deduce $X_-(a) = {}^\perp[X_+^*(a)]$. Since $X_+^*(a)$ is finite dimensional, we have $X_+^*(a) = [{}^\perp[X_+^*(a)]]^\perp$ by [53, Thm.4.7], so that

$$\dim X_+^*(a) = \dim [{}^\perp[X_+^*(a)]]^\perp = \dim X_-(a)^\perp = \operatorname{codim} X_-(a)$$

by [30, Lem.III.1.40] again. Combined with (4.8), this equality shows (4.6).

(2) Due to (4.6), there exists a finite dimensional complement $X_+(a)$ of $X_-(a)$ in X. We further define

$$X_+(t) := U(t,a)X_+(a) \quad \text{for} \quad t \geq a \qquad \text{and} \qquad X_-(t) := P(t)X \quad \text{for} \quad t \geq b.$$

Observe that $X_\pm(t)$, $t \geq a$, are closed subspaces. We want to show that $X_-(t) \oplus X_+(t) = X$ for $t \geq a$. Let $x \in X_-(t) \cap X_+(t)$. Then $x = U(t,a)y$ for some $y \in X_+(a)$. If $t \leq b$, we have $U(b,a)y = U(b,t)x \in P(b)X$, hence $y \in X_-(a)$. This yields $y = 0$ and $x = 0$. If $t > b$, we have $x \in P(t)X$. Thus $U(s,b)U(b,a)y = U(s,t)x \to 0$ as $s \to \infty$, and so $U(b,a)y \in P(b)X$ by (3.3). Again we obtain $y = x = 0$. This argument also shows that $U(t,a)y \neq 0$ for $y \in X_+(a) \setminus \{0\}$ and $t \geq a$. Therefore $U(t,s) : X_+(s) \to X_+(t)$ is an isomorphism for $t \geq s \geq a$. From this fact and (4.6) we deduce

$$\dim X_+(t) = \dim X_+(a) = \operatorname{codim} X_-(a) = \operatorname{codim} P(b)X = \operatorname{codim} P(t)X \tag{4.9}$$

for $t \geq b$. Consequently, $X_-(t) \oplus X_+(t) = X$ in this case. If $t \in [a,b)$ and $x \in X$, then $U(b,t)x = y_+ + y_-$ with $y_\pm \in X_\pm(b)$. There exists $x_+ \in X_+(t)$ with $U(b,t)x_+ = y_+$. Hence, $U(b,t)(x - x_+) = y_- \in X_-(b)$ which means that $x - x_+ \in X_-(t)$. So we arrive at $X_-(t) \oplus X_+(t) = X$ for all $t \geq a$.

(3) We denote by $\tilde{P}(t)$ the projection onto $X_-(t)$ with kernel $X_+(t)$ for $t \geq a$. Assertion (4.4) holds due to (4.9). It is clear that $U(t,s)\tilde{P}(s)X \subseteq \tilde{P}(t)X$ and $U(t,s) : \tilde{Q}(s)X \to \tilde{Q}(t)X$ is an isomorphism for $t \geq s \geq a$. As a consequence, $U(t,s)$ and $\tilde{P}(t)$ satisfy (a) and (b) of Definition 3.1 for $t \geq s \geq a$. For $a \leq t \leq s \leq b$ we have

$$\|U_{\tilde{Q}}(t,s)\tilde{Q}(s)\| \leq \|U(t,a)U_{\tilde{Q}}(a,b)\tilde{Q}(b)U(b,s)\| \leq M^2\, \|U_{\tilde{Q}}(a,b)\tilde{Q}(b)\|.$$

where M is a bound for $\|U(t,s)\|$ on $[a,b]$. So we obtain the boundedness of Green's function of $U(\cdot,\cdot)$ and $\tilde{P}(t)$ on $[a,b]$. Further, $\tilde{P}(t) = P(t)\tilde{P}(t)$ and $P(t) = \tilde{P}(t)P(t)$ for $t \geq b$ since $P(t)$ and $\tilde{P}(t)$ have the same range. These facts lead to

$$\begin{aligned}
\tilde{P}(t) &= \tilde{P}(t)P(t) + \tilde{P}(t)Q(t) = P(t) + \tilde{P}(t)U(t,b)U_Q(b,t)Q(t) \\
&= P(t) + U(t,b)P(b)\tilde{P}(b)U_Q(b,t)Q(t), \\
\|\tilde{P}(t) - P(t)\| &\leq N^2 \|\tilde{P}(b)\|\, e^{-2\delta(t-b)}
\end{aligned}$$

for $t \geq b$. Thus (4.5) holds and $\tilde{P}(t)$ is uniformly bounded on $[a,\infty)$. The first estimate in Definition 3.1(c) now follows from $U(t,s)\tilde{P}(s) = U(t,s)P(s)\tilde{P}(s)$; the second one from

$$\begin{aligned}
\tilde{Q}(t) &= \tilde{Q}(t)Q(t) + \tilde{Q}(t)P(t) = \tilde{Q}(t)Q(t) \\
&= \tilde{Q}(t)U(t,s)U_Q(s,t)Q(t) = U(t,s)\tilde{Q}(s)U_Q(s,t)Q(t), \\
U_{\tilde{Q}}(s,t)\tilde{Q}(t) &= \tilde{Q}(s)U_Q(s,t)Q(t)
\end{aligned}$$

for $t \geq s \geq b$. If $U(\cdot,\cdot)$ is strongly continuous, then $\tilde{P}(t)$ and $U_{\tilde{Q}}(t,s)\tilde{Q}(s)$ are strongly continuous for $t, s \geq a$ due to (3.1) and (3.2). $\square$

In the context of Theorem 4.3, the above assumptions are satisfied if, in addition, the resolvent of A is compact and the adjoint system possesses 'backward uniqueness' (at least) on the unstable subspaces, as we demonstrate in the next example.

Example 4.9. Let the operators $A(t)$ be given on $X = L^p(\Omega)$, $1 < p < \infty$, or $X = C(\overline{\Omega})$ as in Example 2.8. Assume there are functions $\tilde{a}_{kl} \in C^1(\overline{\Omega})$, $k, l = 1, \cdots, n$, and $\tilde{a}_0 \in C(\overline{\Omega})$ such that $(\tilde{a}_{kl})$ is real, symmetric, satisfies (2.38), and

$$a_{kl}(t,\cdot) \to \tilde{a}_{kl}(\cdot) \text{ in } L^n(\Omega) \quad \text{and} \quad a_0(t,\cdot) \to \tilde{a}_0(\cdot) \text{ in } L^{n/2}(\Omega)$$

as $t \to \infty$, where $n/2$ is replaced by 1 if $n = 1$. Define the operator A on X as in (2.39) for $\tilde{a}_{kl}, \tilde{a}_0$. Arguing as in (2.40) with $\nu = 0$, one sees that $R(w, A(t)) \to R(w, A)$ in $\mathcal{L}(X)$. Thus, due to Theorem 4.3, the evolution family generated by $A(\cdot)$ has exponential dichotomy on an interval $[a, \infty)$ provided that $\sigma(A) \cap i\mathbb{R} = \emptyset$.

We further want to show that here the exponential dichotomy of $U(\cdot,\cdot)$ can be extended to $\mathbb{R}_+$. To that purpose we restrict ourselves to $X = L^p(\Omega)$, $1 < p < \infty$, and assume additionally that $a_{kl}, a_0 \in C^1(\mathbb{R}_+, L^\infty(\Omega))$. Since A has compact resolvent, the range of $Q(t)$ is finite dimensional by Theorem 4.3. Observe that the adjoints $U(t,s)^*$ on X^* have properties analogous to those stated Theorem 2.2 since the operators $A(t)^*$, $t \geq 0$, also satisfy (AT). Thus $v(s) = U(a,s)^* y^*$ solves the adjoint backward problem

$$v'(s) = A(s)^* v(s), \quad s < a, \qquad v(a) = y^*, \tag{4.10}$$

on $X^* = L^{p'}(\Omega)$, $1/p + 1/p' = 1$. Due to [33, Thm.1.1], $v(0) = 0$ implies $y^* = 0$ if $y^* \in D(A_2(a))$.[6] Here we have $y^* = Q(a)^* x^*$ for some $x^* \in X^*$. By duality the formula $A(a)^* U(a+1,a)^* U_Q(a,a+1)^* Q(a)^* = A(a)^* Q(a)^*$ holds, so that $Q(a)^* X^* \subseteq D(A(a)^*) = D(A_{p'}(a))$. Thus the assumptions of Proposition 4.8 are satisfied if $p' \geq 2$, i.e., if $p \leq 2$. In the case $p > 2$, we use that for each $a' > a$ there is $z^* \in X^*$ such that $Q(a)^* x^* = U(a',a)^* Q(a')^* z^*$. For $a < a'' < a'$, we have $Q(a)^* x^* = U(a'',a)^* U(a',a'')^* Q(a')^* z^*$. Observe that $U(a',a'')^* Q(a')^* z^* \in D(A(a'')^*) \subseteq W^{2,p'}(\Omega) \hookrightarrow L^r(\Omega)$ with $r = np'/(n-2p')$ for $n > 2p'$ and r is arbitrarily large if $n \leq 2p'$, by Sobolev's embedding theorem. Moreover, the backward evolution families on different spaces $L^{p'}(\Omega)$ coincide on the intersection of these spaces by uniqueness of the problem (4.10). Hence, $Q(a)^* x^* \in D(A_r(a))$. This argument can be iterated until we reach an exponent $\tilde{r} \geq 2$. Therefore Proposition 4.8 can be applied for every $p \in (1,\infty)$ and $U(\cdot,\cdot)$ has an exponential dichotomy on $\mathbb{R}_+$.

4.3. **Slowly oscillating coefficients.** We have seen in Example 3.2 that the hyperbolicity of the semigroups $(e^{\tau A(t)})_{\tau \geq 0}$ (with common constants $N, \delta > 0$) does not imply the exponential dichotomy of the evolution family $U(\cdot,\cdot)$ generated by $A(\cdot)$. However, one can expect that $U(\cdot,\cdot)$ has an exponential dichotomy if in addition $A(\cdot)$ does not 'oscillate too much'. Such results were established by W.A. Coppel in [15, Prop.6.1] for matrices $A(t)$ and by A.G. Baskakov in [9] for bounded operators $A(t)$. Both authors required that the Lipschitz constant of $A(\cdot)$ is small compared to $N, \delta, \|A(\cdot)\|_\infty$. M.P.

[6] In fact, it suffices that $y^* \in W^{1,2}(\Omega)$, cf. [33].

Lizana used Coppel's result to prove an analogous theorem for ordinary functional differential equations, [38, Thm.4]. For parabolic problems one knows estimates of the type $\|U(t,s)\| \le c_1 e^{(w+c_2L^\kappa)(t-s)}$, where w and L are given by (AT) and $c_k, \kappa > 0$ are positive, see Theorems II.5.1.1 and IV.2.3.2 in [7], [23, Thm.2.3], or [27, Thm.7.4.2]. In particular, $U(\cdot,\cdot)$ is exponentially stable if $w < 0$ and the Hölder constant L is small. In our main Theorem 4.13 we show that $U(\cdot,\cdot)$ also inherits the exponential dichotomy (with the same ranks) of $A(t)$ if L is sufficiently small. Here we extend [55, Thm.3.7] and [56, Thm.5], where the case of (constant) dense domains $D(A(t))$ was treated. Our approach is based on Theorem 3.17, more precisely, we show that the 'generator' G of the evolution semigroup $T(\cdot)$ on E is invertible (recall that $T(\cdot)$ is only strongly continuous on $(0,\infty)$ if the domains are not dense). The rank of the dichotomy projections is computed by means of (3.7) and (3.8) and a continuity argument.

We start with several preparations. In order to simplify the proof of Theorem 4.13, we choose a slightly different Hölder condition in (AT) and require that

(AT') $A(t)$, $t \in J$, $J \in \{[a,\infty), \mathbb{R}\}$, satisfy (AT1), have uniformly bounded inverses, and

$$\|r^\nu A(t)R(r,A(t))\,(A(t)^{-1} - A(s)^{-1})\| \le \ell\,|t-s|^\mu$$

for $t,s \in J$, $r > w$, and constants $\ell \ge 0$ and $\mu,\nu \in (0,1]$ with $\mu+\nu > 1$.

If $J = [a,\infty)$, we set $A(t) := A(a)$ for $t \le a$. This definition preserves (AT'). It is straightforward to verify that (AT') is stronger than (AT) and that (AT) implies (AT') (possibly after replacing w by $w' > w$) if the operators $A(t)$ have uniformly bounded inverses.

We introduce the completion X_{-1}^t of $X_0^t = \overline{D(A(t))}$ with respect to the norm $\|x\|_{-1}^t := \|A^{-1}(t)x\|$, see e.g. [7, Chap.V], [21, §II.5].[7] Observe that $A(t)$ has a unique continuous extension $A_{-1}(t) : X_0^t \to X_{-1}^t$. Since one can also extend $R(\lambda, A(t))$ to an operator in $\mathcal{L}(X_{-1}^t, X_0^t)$, $A_{-1}(t)$ is sectorial in X_{-1}^t of the same type. Now we can proceed as in Section 2. We define

$$X_{\alpha-1,\infty}^t := (X_{-1}^t)_{\alpha,\infty}^{A_{-1}(t)} \quad \text{and} \quad X_{\alpha-1}^t := (X_{-1}^t)_{\alpha}^{A_{-1}(t)},$$
$$\text{with the norm} \quad \|x\|_{\alpha-1}^t := \sup_{r>w} \|r^\alpha R(r, A_{-1}(t))x\|$$

for $0 < \alpha < 1$ and $t \in \mathbb{R}$. The domain $D(A(t))$ is dense in $X_{\alpha-1}^t$ for all $\alpha \in [0,1]$ and the continuous embeddings

$$X_0^t \subseteq X \hookrightarrow X_{\beta-1}^t \hookrightarrow X_{\beta-1,\infty}^t \hookrightarrow X_{\alpha-1}^t \hookrightarrow X_{-1}^t \tag{4.11}$$

hold for $0 < \alpha < \beta < 1$. The norms of the embeddings depend only on the type of $A(t)$ and the norm of its inverse. The operator $A(t)$ can be extended to an isometric isomorphism $A_{\alpha-1}(t) : X_\alpha^t \to X_{\alpha-1}^t$ for $0 \le \alpha < 1$ which coincides with the part of $A_{-1}(t)$ in $X_{\alpha-1}^t$ (and analogously for $X_{\alpha,\infty}^t$). The semigroup $e^{\tau A(t)}$ extends to an analytic C_0–semigroup on $X_{\alpha-1}^t$ generated by $A_{\alpha-1}(t)$, $0 \le \alpha < 1$. In addition, (2.6) yields

$$\|A(t)^k\, e^{sA_{-1}(t)}x\|_X \le C_\alpha\, s^{\alpha-k-1}\, \|x\|_{\alpha-1}^t\,, \quad \text{where } C_\alpha \le C, \tag{4.12}$$

for $k = 0,1$, $0 < \alpha < 1$, $x \in X_{\alpha-1,\infty}^t$, $0 < s \le 1$, $t \in \mathbb{R}$, and constants C_α and C depending only on the type of $A(t)$ and $\|A^{-1}(t)\|$.

[7] One usually takes the norm $\|R(w,A(t))x\|$ which is equivalent to $\|x\|_{-1}^t$ (uniformly in $t \in \mathbb{R}$) by (AT').

Analogously we treat the multiplication operator $A(\cdot)$ on $E = C_0(\mathbb{R}, X)$ as defined in (2.12). There exist the corresponding inter- and extrapolation spaces $E_\alpha := E_\alpha^{A(\cdot)}$, $-1 \le \alpha \le 1$, and $E_{\alpha,\infty} := E_{\alpha,\infty}^{A(\cdot)}$, $-1 < \alpha < 1$, where $E_0 = \overline{D(A(\cdot))}$. The extrapolated operator $A(\cdot)_{-1} : E_0 \to E_{-1}$ is given by the multiplication operator $A_{-1}(\cdot)$ and $f \in E_{-1}$ can be identified with an element of $\prod_{t\in\mathbb{R}} X_{-1}^t$, cf. [25, Thm.4.7]. As above we see that $E_{\alpha-1}$ is the closure of $D(A(\cdot))$ in E_{-1} with respect to the norm

$$\|f\|_{\alpha-1} := \sup_{r>w} \sup_{s\in\mathbb{R}} \|r^\alpha R(r, A_{-1}(s))f(s)\|,$$

$A_{\alpha-1}(\cdot) : E_\alpha \to E_{\alpha-1}$ is an isometric isomorphism for $0 \le \alpha < 1$,

$$\begin{aligned} &D(A(\cdot)) \hookrightarrow E_\beta \subseteq E_{\beta,\infty} \hookrightarrow D((w - A(\cdot))^\alpha) \hookrightarrow E_\alpha \hookrightarrow E_0 \subseteq E \\ &E \hookrightarrow E_{\beta-1} \subseteq E_{\beta-1,\infty} \hookrightarrow E_{\alpha-1} \hookrightarrow E_{-1} \end{aligned} \tag{4.13}$$

for $0 < \alpha < \beta < 1$ (the domain of the fractional power is endowed with the norm $\|(w - A(\cdot))^\alpha f\|_\infty$).

Due to (2.15), (2.5), the reiteration theorem [42, Thm.1.2.15], and $\omega(U) < \infty$, we can extend $U(t,s)$ to an operator $\overline{U}(t,s) : X_{\alpha-1}^s \to X$ with norm

$$\|\overline{U}(t,s)\| \le c(\alpha)\,((t-s)^{\alpha-1} \vee 1)\, e^{\gamma(t-s)} \tag{4.14}$$

for $1 - \mu < \alpha < 1$ and constants $c(\alpha) \ge 0$ and $\gamma \in \mathbb{R}$ depending on the constants in (AT).

Let $T(\cdot)$ be the evolution semigroup on $E = C_0(\mathbb{R}, X)$ induced by $U(\cdot,\cdot)$ with 'generator' $G = \overline{A(\cdot) - d/dt}$, compare Proposition 3.16. Due to (4.14) and (3.24) we can extend $R(\lambda, G)$ to a bounded operator $\overline{\mathbb{K}}_\lambda : E_{\alpha-1} \to E$, $1 - \mu < \alpha < 1$, $\lambda > \gamma$, given by

$$\left(\overline{\mathbb{K}}_\lambda f\right)(t) = \int_{-\infty}^t e^{-\lambda(t-s)}\overline{U}(t,s) f(s)\, ds, \quad t \in \mathbb{R},\ f \in E_{\alpha-1}.$$

The following 'integration py parts' formula will be crucial for the proof of Theorem 4.13.

Lemma 4.10. *Let (AT) hold, $\lambda > \gamma$, and $f \in C_0^1(\mathbb{R}, X) \cap D((w - A(\cdot))^\alpha)$ for some $\alpha \in (1-\mu, \nu)$. Then $R(\lambda, G)f' = f - \lambda R(\lambda, G)f + \overline{\mathbb{K}}_\lambda A_{-1}(\cdot) f$.*

Proof. By rescaling we may assume that $\lambda = 0 > \gamma$ and $w = 0$. Let $f \in D(A(\cdot))$, $\alpha \in (1-\mu, \nu)$, $s \in \mathbb{R}$, and $0 < h \le 1$. We write

$$\begin{aligned} D_{h,f}(s) &:= (-A(s+h))^{\alpha-1}\, \tfrac{1}{h}\, [U(s+h,s) - I] f(s) + (-A(s))^\alpha f(s) \\ &= \frac{1}{h}\int_0^h \left[(-A(s+h))^{\alpha-1} - (-A(s+\tau))^{\alpha-1}\right] A(s+\tau) U(s+\tau,s) f(s)\, d\tau \\ &\quad - \frac{1}{h}\int_0^h \left[(-A(s+\tau))^\alpha\, U(s+\tau,s)(-A(s))^{-\alpha} - I\right] (-A(s))^\alpha f(s)\, d\tau. \end{aligned}$$

Due to (2.10), (2.13), and (2.5) the X–norm of the first integral can be estimated by

$$\frac{c}{h}\int_0^h (h-\tau)^\mu\, \tau^{\alpha-1}\, d\tau\ \|(-A(\cdot))^\alpha f\|_\infty = c\, h^{\alpha+\mu-1}\, \|(-A(\cdot))^\alpha f\|_\infty$$

for constants c independent of h and s. One verifies that the second integral also tends to 0 as $h \searrow 0$ uniformly in s using (2.19) and $(-A(\cdot))^\alpha f \in C_0(\mathbb{R}, X)$ for large s and the strong continuity of $(s,\tau) \mapsto (-A(s+\tau))^{\alpha} U(s+\tau),s)(-A(s))^{-\alpha}$ (proved in Theorem 2.2) for s in compact sets. As a result,

$$\lim_{h\to 0} \|D_{h,f}\|_\infty = 0 \tag{4.15}$$

for $f \in D(A(\cdot))$. In the same way one sees that $\|D_{h,f}\|_\infty \le c\,\|(-A(\cdot))^{\alpha} f\|_\infty$ so that (4.15) holds for $f \in D((-A(\cdot))^{\alpha})$ by approximation. For $f \in C_0^1(\mathbb{R},X) \cap D((-A(\cdot))^{\alpha})$, we further compute

$$\begin{aligned}
-\int_{-\infty}^{t} U(t,s)f'(s)\,ds &= \lim_{h\to 0}\int_{-\infty}^{t-h} U(t,s+h)\,\frac{1}{h}(f(s+h)-f(s))\,ds\\
&= \lim_{h\to 0}\left(\frac{1}{h}\int_{t-h}^{t} U(t,s)f(s)\,ds + \frac{1}{h}\int_{-\infty}^{t-h} U(t,s+h)[U(s+h,s)-I]f(s)\,ds\right)\\
&= f(t) + \lim_{h\to 0}\frac{1}{h}\int_{-\infty}^{t-h} U(t,s+h)[U(s+h,s)-I]f(s)\,ds,
\end{aligned} \tag{4.16}$$

where we used $f \in E_0$ in the last line. To determine the remaining limit, we note that the operator $U(t,s)(-A(s))^{1-\alpha}$, $t>s$, defined on $D((-A(s))^{1-\alpha})$ has a unique bounded extension $V(t,s) : X_0^s \to X_0^t$ satisfying

$$\|V(t,s)\| \le c(\alpha)\,((t-s)^{\alpha-1} \vee 1)\,e^{\gamma(t-s)} \tag{4.17}$$

for $t>s$ and a constant $c(\alpha)$ by (2.15) and $\omega(U)<\infty$. Approximating f in $D((-A(\cdot))^{\alpha})$ by $f_n \in D(A(\cdot))$, one sees that $\overline{U}(t,s)A_{-1}(s)f(s) = -V(t,s)(-A(s))^{\alpha}f(s)$. Hence,

$$\begin{aligned}
&\frac{1}{h}\int_{-\infty}^{t-h} U(t,s+h)[U(s+h,s)-I]f(s)\,ds - \int_{-\infty}^{t} \overline{U}(t,s)A_{-1}(s)f(s)\,ds\\
&= \int_{-\infty}^{t-h} V(t,s+h)\left\{(-A(s+h))^{\alpha-1}\tfrac{1}{h}[U(s+h,s)-I]f(s) + (-A(s))^{\alpha}f(s)\right\}ds\\
&\quad - \int_{-\infty}^{t-h} V(t,s+h)\,[(-A(s))^{\alpha}f(s) - (-A(s+h))^{\alpha}f(s+h)]\,ds.
\end{aligned} \tag{4.18}$$

As consequence of (4.17), (4.15), and the uniform continuity of $(-A(\cdot))^{\alpha}f(\cdot)$, the right hand side of (4.18) tends to 0 in X as $h \searrow 0$, and the assertion follows from (4.16) $\square$

Our second assumption reads as follows.

(ED) Each semigroup $(e^{\tau A(t)})_{\tau\ge 0}$, $t \in J$, has an exponential dichotomy with constants $N,\delta>0$ and projection P_t.

Notice that the extension $A(t) := A(a)$ for $t<a$ preserves (ED). The next observation says that (ED) could also be formulated in terms of resolvents. We use the variant stated in (ED) to simplify the presentation.

Lemma 4.11. *Let A be a sectorial operator of type (ϕ,K,w).*

(a) *If $[-\delta,\delta]+i\mathbb{R} \subseteq \rho(A)$ and $\|R(\lambda,A)\| \le r$ for $\lambda \in \pm\delta + i\mathbb{R}$ and constants $\delta, r > 0$, then $(e^{tA})_{t\ge 0}$ is hyperbolic with constants δ and $N = N(\phi,K,w,\delta,r)$. The projection on the unstable subspace is given by*

$$Q = \frac{1}{2\pi i}\int_{\Gamma} R(\lambda,A)\,d\lambda, \tag{4.19}$$

where Γ is a suitable path around the spectral set $\{\lambda\in\sigma(A) : \operatorname{Re}\lambda \ge \delta\}$.

(b) *If $(e^{tA})_{t\geq 0}$ is hyperbolic with constants $N, \delta > 0$, then we have $[-\delta', \delta'] + i\mathbb{R} \subseteq \rho(A)$ and $\|R(\lambda, A)\| \leq \frac{2N}{\delta - \delta'}$ for $\lambda \in \pm\delta + i\mathbb{R}$ and each $\delta' \in [0, \delta)$,*

Proof. (a) The exponential dichotomy of $(e^{tA})_{t\geq 0}$ and the formula (4.19) is shown e.g. in [42, Prop.2.3.3]. Clearly, the norms of Q and $P = I - Q$ can be estimated by a constant c_1 depending on ϕ, K, w, δ, r. Recall that there is a constant c_2 depending only on the type of A such that $\|e^{tA}\| \leq c_2$ for $0 \leq t \leq (w+\delta)^{-1}$, see e.g. [21, p.98], [42, p.36]. Further,

$$e^{t(A+\delta)}P = \frac{1}{2\pi i}\int_{\Gamma'} e^{\lambda t}\, R(\lambda, A+\delta)P\, d\lambda,$$

where the path Γ' consists of the straight line between $\pm i(w+\delta)\tan(\pi - \phi)$ and the rays of angle $\pm\phi$ starting at $\pm i(w+\delta)\tan(\pi-\phi)$. A straightforward computation then yields

$$\|e^{t(A+\delta)}P\| \leq c_3\, c_1 \qquad \text{for } \ t \geq (w+\delta)^{-1}$$

and a constant $c_3 = c_3(\phi, K, w, \delta, r)$. Thus, $\|e^{tA}P\| \leq N\, e^{-\delta t}$ for $t \geq 0$ and $N = N(\phi, K, w, \delta, r)$. In the same way one derives the exponential estimate for $e^{-tA_Q}Q$.

(b) By the spectral mapping theorem [42, Cor.2.3.7] and rescaling, the strip $[-\delta', \delta'] + i\mathbb{R}$ belongs to $\rho(A)$ for $0 \leq \delta' < \delta$. Moreover, for $|\operatorname{Re}\lambda| \leq \delta'$ we have

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t}\, e^{tA} P\, d\lambda - \int_0^\infty e^{\lambda t}\, e^{-tA_Q} Q\, d\lambda$$

which implies (b). ☐

Notice that Lemma 4.11 and (ED) yield that $\|A(t)^{-1}\| \leq \frac{2N}{\delta}$ for $t \in \mathbb{R}$. Hypothesis (ED) allows us to define

$$\Gamma_t(s) := \begin{cases} e^{sA_P(t)}P_t, & s \geq 0,\ t \in \mathbb{R}, \\ -e^{sA_Q(t)}Q_t, & s < 0,\ t \in \mathbb{R}. \end{cases}$$

Lemma 4.12. *If (AT') and (ED) hold, then the following assertions are true.*

(a) *$t \mapsto P_t \in \mathcal{L}(X)$ is globally Hölder continuous for $t \in \mathbb{R}$ and $(t,s) \mapsto \Gamma_t(s) \in \mathcal{L}(X)$ is locally Hölder continuous for $t \in \mathbb{R}$ and $s \in \mathbb{R} \setminus \{0\}$.*

(b) *P_t and $\Gamma_t(s)$ leave X^t_α and $X^t_{\alpha,\infty}$ invariant and have unique bounded extensions to $X^t_{\alpha-1}$ and $X^t_{\alpha-1,\infty}$ for $0 < \alpha \leq 1$ and $t, s \in \mathbb{R}$ (the extensions are denoted by the same symbol). Moreover,*

$$\|A(t)^k\, \Gamma_t(s)x\| \leq \begin{cases} NC_\alpha\, s^{\alpha-1-k}\, \|x\|^t_{\alpha-1}, & 0 < s \leq 1, \\ NC_\alpha\, e^{-\delta(s-1)}\, \|x\|^t_{\alpha-1}, & s \geq 1, \\ NC_\alpha\, e^{\delta(s-1)}\, \|x\|^t_{\alpha-1}, & s < 0, \end{cases}$$

for $k = 0, 1$, $\alpha \in (0,1)$, $t \in \mathbb{R}$, and $x \in X^t_{\alpha-1,\infty}$, where $C_\alpha \leq C$ are given by (4.12).

(c) *Set $\psi(s) := \sup_{t\in\mathbb{R}} \|A(t)\Gamma_t(s)\|_{\mathcal{L}(X^t_{\nu-1,\infty}, X)}$ for $s \neq 0$. Then the upper integral $q := \bar{\int}_{\mathbb{R}} \psi(s)|s|^\mu\, ds$ is finite; more precisely,*

$$q \leq NC_\nu\Big(\int_0^1 s^{\nu+\mu-2}\, ds + \int_0^\infty ((s+1)^\mu + e^{-\delta}\, s^\mu)\, e^{-\delta s}\, ds\Big).$$

Proof. (a) By (2.9) the map $t \mapsto e^{sA(t)}$ is Hölder continuous uniformly for $s \in [s_0, s_1] \subseteq (0,\infty)$ Moreover, $s \mapsto e^{sA(t)}$ is locally Lipschitz continuous for $s > 0$ uniformly in $t \in \mathbb{R}$ since the semigroup is analytic. Equation (2.8) shows that $\|R(\lambda, A(t)) - R(\lambda, A(\tau))\| \leq c\,|t-\tau|^\mu$ for a constant c, $t, \tau \in \mathbb{R}$, and λ contained in the path Γ used in (4.19). The global Hölder continuity of $\mathbb{R} \ni t \mapsto Q_t \in \mathcal{L}(X)$ and $\mathbb{R} \ni t \mapsto P_t \in \mathcal{L}(X)$ now follow from (4.19). Finally, we have

$$\begin{aligned} e^{-sA_Q(t)}Q_t - e^{-\sigma A_Q(\tau)}Q_\tau =& e^{-sA_Q(t)}Q_t\,(Q_t - Q_\tau) + e^{-sA_Q(t)}Q_t\,(e^{\sigma A(\tau)} - e^{sA(t)})e^{-\sigma A_Q(\tau)}Q_\tau \\ &+ (Q_t - Q_\tau)\,e^{-\sigma A_Q(\tau)}Q_\tau \end{aligned}$$

for $t, \tau \in \mathbb{R}$ and $s, \sigma > 0$. Hence, (a) is proved.
(b) The first sentence is an easy consequence of the definition of the inter- and extrapolation spaces and formula (4.19). To establish the asserted estimates, we write

$$A(t)^k\,\Gamma_t(s) = \begin{cases} P_t\,A(t)^k\,e^{sA(t)}, & 0 < s \leq 1, \\ e^{(s-1)A(t)}P_t\,A(t)^k\,e^{A(t)}, & s \geq 1, \\ -e^{(s-1)A_Q(t)}Q_t\,A(t)^k\,e^{A(t)}, & s < 0, \end{cases}$$

for $t \in \mathbb{R}$ and use (4.12). Assertion (c) follows from (b). □

We now come to the main result in this paragraph.

Theorem 4.13. *Assume that (AT') and (ED) hold and let q be given by Lemma 4.12. If $q\ell < 1$, then the evolution family $U(\cdot,\cdot)$ generated by $A(\cdot)$ has an exponential dichotomy with exponent $\delta' \in (0, (1-q\ell)/\kappa)$ and constant N'. Here κ depends on N, δ, and the constants in (AT') and N' depends on N, δ, q, δ', and the constants in (AT'). Moreover, the Hölder continuous dichotomy projections $P(\cdot)$ of $U(\cdot,\cdot)$ satisfy*

$$\dim P(t)X = \dim P_sX \quad \textit{and} \quad \dim\ker P(t) = \dim\ker P_s\,, \qquad t, s \in J. \tag{4.20}$$

Proof. If $A(\cdot)$ is given on $J = [a, \infty)$, we extend it to $\mathbb{R}$ by setting $A(t) := A(a)$ for $t < a$. Then the extension satisfies (AT') and (ED) with the same constants (but observe that q was defined in Lemma 4.12 for the extended family). By restriction the assertions then follow from the proof for $J = \mathbb{R}$ given below.

(1) Assume that $Gf = 0$ for some $f \in D(G)$. Then there are $f_n \in D(A(\cdot)) \cap C_0^1(\mathbb{R}, X)$ such that $f_n \to f$ and $Gf_n = A(\cdot)f_n - f_n' \to 0$ in E as $n \to \infty$ (cf. Proposition 3.16). By [1, Prop.3.2,5.1] we have

$$f_n(t) = U(t,s)f_n(s) - \int_s^t U(t,\tau)(Gf_n)(\tau)\,d\tau, \qquad t \geq s.$$

Hence, $f(t) = U(t,s)f(s)$ for $t \geq s$ so that $f'(t) = A(t)f(t)$ exists by Theorem 2.2. Taking $s = t-1$ we see that $f \in D(A(\cdot)) \cap C_0^1(\mathbb{R}, X)$. For $g \in E$, the function

$$(Lg)(t) = \int_{-\infty}^{\infty} \Gamma_t(t-s)g(s)\,ds, \quad t \in \mathbb{R},$$

belongs to E by Lemma 4.12. Integrating by parts and using [42, Thm.4.3.1], we obtain

$$\begin{aligned} (Lf')(t) &= \lim_{\varepsilon\to 0}\int_{-\infty}^{t-\varepsilon} e^{(t-s)A(t)}\,P_t\,f'(s)\,ds - \int_t^\infty e^{(t-s)A_Q(t)}\,Q_t\,f'(s)\,ds \\ &= A(t)\,(Lf)(t) + Q_tf(t) + \lim_{\varepsilon\to 0}\Big(A(t)\int_{t-\varepsilon}^t e^{(t-s)A(t)}\,P_t\,f(s)\,ds + e^{\varepsilon A(t)}P_tf(t-\varepsilon)\Big) \\ &= f(t) + A(t)\,(Lf)(t) \end{aligned}$$

for $t \in \mathbb{R}$. (Note that $A_Q(t) \in \mathcal{L}(Q_t X)$.) We also define

$$(Vf)(t) := \int_{\mathbb{R}} \Gamma_t(t-s)(A(s) - A_{-1}(t))f(s)\,ds$$
$$= \int_{\mathbb{R}} A(t)\Gamma_t(t-s)(A(t)^{-1} - A(s)^{-1})A(s)f(s)\,ds$$

for $t \in \mathbb{R}$, where the integral in the first line is understood in the topology of X^t_{-1}. Combining these formulas, we derive

$$0 = LGf = -Lf' + LA(\cdot)f = Vf - f, \quad \text{i.e.,} \;\; Vf = f.$$

Using (AT'), Lemma 4.12(c), and $\|A(t)^2\Gamma_t(\tau)\|_{\mathcal{L}(X^t_{\nu,\infty},X)} = \|A(t)\Gamma_t(\tau)\|_{\mathcal{L}(X^t_{\nu-1,\infty},X)}$, we further deduce that

$$\|A(t)f(t)\| = \|A(t)\,(Vf)(t)\| \le \int_{\mathbb{R}} \|A(t)^2\Gamma_t(\tau)\|_{\mathcal{L}(X^t_{\nu,\infty},X)}\,\ell\,|\tau|^\mu\,d\tau\;\|A(\cdot)f\|_\infty$$
$$\le q\ell\,\|A(\cdot)f\|_\infty < \|A(\cdot)f\|_\infty\,.$$

Since each $A(t)$, $t \in \mathbb{R}$, is invertible, this estimate shows that $f = 0$. Hence G is injective.

(2) For $f \in E$ and $t \in \mathbb{R}$ we define

$$(Rf)(t) = \int_{-\infty}^{\infty} \Gamma_s(t-s)f(s)\,ds.$$

By Lemma 4.12, R is a bounded operator on E which can be extended to an operator $\tilde{R} : E_{\alpha-1} \to E$ for $\alpha \in (0,1)$ with norm

$$\|\tilde{R}\|_{\mathcal{L}(E_{\alpha-1},E)} \le NC_\alpha \left(\frac{1}{\alpha} + \frac{1+e^{-\delta}}{\delta}\right) =: \rho_\alpha. \tag{4.21}$$

For $f \in D(A(\cdot))$, one easily sees that $Rf \in C^1_0(\mathbb{R},X)$ and $\frac{d}{dt}Rf = f + RA(\cdot)f$. Let $\alpha \in (1-\mu,\nu]$. Then (AT') and Lemma 4.12 imply

$$\sup_{r>w}\left\| r^\alpha A(t)R(r,A(t)) \int_{\mathbb{R}} (A(s)^{-1} - A(t)^{-1})\Gamma_s(t-s)A(s)f(s)\,ds \right\|$$
$$\le c(\alpha)\int_{\mathbb{R}} \ell\,|t-s|^\mu\,\|A(s)\Gamma_s(t-s)f(s)\|\,ds \le c'(\alpha)\,\|f\|_{\alpha-1}$$

for $f \in D(A(\cdot))$ and constants $c(\alpha)$ and $c'(\alpha)$ with $c'(\nu) = q\ell$. Therefore $Rf \in E_{\nu,\infty}$ and the function

$$(Sf)(t) := A_{-1}(t)\int_{\mathbb{R}} (A(s)^{-1} - A(t)^{-1})A(s)\Gamma_s(t-s)f(s)\,ds, \quad t \in \mathbb{R},$$

belongs to $E_{\nu-1,\infty}$. Moreover, S can be extended to a bounded operator from $E_{\alpha-1}$ to $E_{\alpha-1,\infty}$ having the same representation and then restricted to an operator $\tilde{S} : E_{\nu-1,\infty} \to E_{\nu-1,\infty}$ with norm less than $q\ell$. We set $\tilde{G}h := -h' + A_{-1}(\cdot)h \in E_{-1}$ for $h \in C^1_0(\mathbb{R},X) \cap E_0$. For $f \in D(A(\cdot))$ the above observations yield

$$\tilde{G}Rf = -f - RA(\cdot)f + A_{-1}(\cdot)Rf = (S-I)f. \tag{4.22}$$

For a given $g \in E$, there exists $f := (\tilde{S}-I)^{-1}g \in E_{\nu-1,\infty}$. Fix $\alpha \in (1-\mu,\nu)$. There are $f_n \in D(A(\cdot))$ converging to f in $\|\cdot\|_{\alpha-1}$. Equation (4.22) now gives

$$\tilde{G}Rf_n = (S-I)f_n \;\longrightarrow\; g \qquad \text{in } \|\cdot\|_{\alpha-1} \text{ as } n \to \infty.$$

Since $Rf_n \in E_{\nu,\infty} \hookrightarrow D((w - A(\cdot))^\alpha)$, Lemma 4.10 shows that

$$\overline{\mathbb{K}}_\lambda \tilde{G} R f_n = \overline{\mathbb{K}}_\lambda A_{-1}(\cdot) R f_n - R(\lambda, G) \tfrac{d}{dt} R f_n = \lambda R(\lambda, G) R f_n - R f_n.$$

Letting $n \to \infty$ and using the continuity of $\overline{\mathbb{K}}_\lambda, \tilde{R} : E_{\alpha-1} \to E$, we deduce

$$R(\lambda, G) g = \overline{\mathbb{K}}_\lambda g = \lambda R(\lambda, G) \tilde{R} f - \tilde{R} f.$$

Therefore, $\tilde{R} f \in D(G)$ and $G \tilde{R} f = g$. This means that G is surjective. Together with step (1) we have proved the invertibility of G.

The above argument also shows that $G^{-1} g = \tilde{R}(\tilde{S} - I)^{-1} g$ for $g \in E$, and thus $\|G^{-1}\| \le \frac{\sigma\rho}{1 - q\ell}$ where $\rho := \rho_\nu$ is given by (4.21) and $\sigma := \sup_{r > w, t \in \mathbb{R}} \|r^\nu R(r, A(t))\|$. Hence Theorem 3.17 and Proposition 3.18 establish the theorem except for (4.20).

(3) Recall that the dimensions of the kernel and the range of two projections P and $\hat{P}$ coincide whenever $\|P - \hat{P}\| < 1$, see [32, p.298]. So Lemma 4.12(a) shows that the dimensions of $\ker P_t$ and $P_t X$ do not depend on t.

We set $A_\varepsilon(t) := A(\varepsilon t)$ for $0 \le \varepsilon \le 1$ and $t \in \mathbb{R}$. Observe that $A_\varepsilon(\cdot)$ satisfies (AT') and (ED) with the same constants and the same q. Hence the corresponding evolution families $U_\varepsilon(\cdot,\cdot)$ fulfill the same exponential estimate and are hyperbolic with common constants $\delta', N' > 0$ and projections $P_\varepsilon(t)$ due to steps (1) and (2). Note that $P_1(t) = P(t)$ and $P_0(t) = P_0$ for $t \in \mathbb{R}$. Applying [32, p.298] once more, it remains to show that $\varepsilon \mapsto P_\varepsilon(t) \in \mathcal{L}(X)$ is continuous for each $t \in \mathbb{R}$. Since (AT) holds, we have

$$\|R(w, A_\varepsilon(t)) - R(w, A_\eta(t))\| \le c\, |t|^\mu\, |\varepsilon - \eta|^\mu$$

for $t \in \mathbb{R}$ and $\varepsilon, \eta \in [0,1]$. Thus Proposition 2.6 shows that

$$\|U_\varepsilon(s, s-1) - U_\eta(s, s-1)\| \le \tilde{c}\,(1 + r)^{\mu\beta}\, |\varepsilon - \eta|^{\mu\beta} \tag{4.23}$$

for $|s| \le r$ and some $\beta > 0$ provided that $(1 + r)\,|\varepsilon - \eta|$ is sufficiently small, where the constant $\tilde{c}$ does not depend on r, ε, η.

Let $T_\varepsilon(\cdot) = T_{U_\varepsilon}(\cdot)$. Formula (3.8) yields $\|R(\lambda, T_\varepsilon(1))\| \le 2N'\,(1 - e^{-\delta'}) =: d$ for each $|\lambda| = 1$ and $0 \le \varepsilon \le 1$. Moreover, $M := \sup_{s,\varepsilon} \|U_\varepsilon(s, s-1)\| < \infty$. Given $t \in \mathbb{R}$ and $x \in X$, we set $f = \varphi(\cdot) x$ for a function $\varphi \in C(\mathbb{R})$ with support in $(t - \frac{1}{2}, t + \frac{1}{2})$, $\varphi(t) = 1$, and $0 \le \varphi \le 1$. Let $0 \le \varepsilon, \eta \le 1$. Then (3.7), (3.8), and (4.23) imply

$$\begin{aligned}
\|P_\eta(t)x - P_\varepsilon(t)x\| &\le \|P_\eta(\cdot) f - P_\varepsilon(\cdot) f\|_\infty \\
&\le d \sup_{|\lambda|=1} \|(T_\eta(1) - T_\varepsilon(1)) R(\lambda, T_\varepsilon(1)) f\|_\infty \\
&\le d \max\{ \sup_{|s| \le r, |\lambda|=1} \|U_\varepsilon(s, s-1) - U_\eta(s, s-1)\|\, \|R(\lambda, T_\varepsilon(1)) f\|_\infty, \\
&\qquad \sup_{|s| \ge r, |\lambda|=1} 2M \|R(\lambda, T_\varepsilon(1)) f(s-1)\| \} \\
&\le \max\{ d^2 \tilde{c}\,(1+r)^{\mu\beta}\, |\varepsilon - \eta|^{\mu\beta}, 2dMN' \sup_{|s| \ge r} \sum_{n \in \mathbb{Z}} e^{-\delta' |n|} \varphi(s - 1 - n) \}\, \|x\| \\
&\le \max\{ d^2\, \tilde{c}\,(1+r)^{\mu\beta}\, |\varepsilon - \eta|^{\mu\beta},\ 2dMN'\, e^{-\delta' N(r)} \}\, \|x\|
\end{aligned}$$

for each $r > \max\{t + \frac{1}{2}, -t - \frac{3}{2}\}$ and sufficiently small $|\varepsilon - \eta|$ (depending on r), where $N(r)$ is the minimum of the integer parts of $r - t - \frac{1}{2}$ and $r + t + \frac{3}{2}$. Therefore $P_\eta(t) \to P_\varepsilon(t)$ in $\mathcal{L}(X)$ as $\eta \to \varepsilon$. As observed above, (4.20) follows from this fact. $\square$

In the above proof we have used the operators $A(\varepsilon t)$ and noted that they fulfill (AT') and (ED) with the same constants and the same q. More precisely, in (AT') the Hölder constant ℓ can be replaced by $\varepsilon^{\mu}\ell$. This observation leads to the next corollary.

Corollary 4.14. *Assume that the operators $A(t)$ satisfy (AT') and (ED). Let $0 < \varepsilon < \varepsilon_0 := (q\ell)^{-1/\mu}$. Then the evolution family $U_\varepsilon(\cdot,\cdot)$ generated by the operators $A(\varepsilon t)$ is hyperbolic. Moreover, its dichotomy projections $P_\varepsilon(t)$ satisfy*

$$\dim P_\varepsilon(t)X = \dim P_s X \quad \textit{and} \quad \dim\ker P_\varepsilon(t) = \dim\ker P_s, \quad t,s \in J, \tag{4.24}$$

and the dichotomy constants can be chosen independent of $\varepsilon \in (0,\varepsilon_1]$ if $\varepsilon_1 < \varepsilon_0$.

The operators $\frac{1}{\varepsilon}A(t)$, $t \in J$, generate the evolution family $\hat{U}_\varepsilon(t,s) = U_\varepsilon(\frac{t}{\varepsilon},\frac{s}{\varepsilon})$, $t \geq s$, where $U_\varepsilon(\cdot,\cdot)$ is generated by $A(\varepsilon t)$. So we obtain the following result on singular perturbation, cf. [32, §IV.1]. Similar theorems are contained in [35, §10.7] for bounded $A(t)$, in [38, Thm.6] for delay equations, and in [27, p.215] for a special class of exponentially stable parabolic equations. Notice that the dichotomy exponent blows up as $\varepsilon \searrow 0$.

Corollary 4.15. *Assume that $A(\cdot)$ satisfies (AT') and (ED). Let $0 < \varepsilon < \varepsilon_0 := (q\ell)^{-1/\mu}$. Then the solutions of*

$$\varepsilon\, u'(t) = A(t)u(t), \quad t \geq s, \qquad u(s) = x,$$

have exponential dichotomy with constants δ'/ε and N', where δ' and N' do not depend on $\varepsilon \in (0,\varepsilon_1]$ for $\varepsilon_1 < \varepsilon_0$. Moreover, (4.24) holds.

Example 4.16. We consider the situation of Example 2.8 assuming additionally that Ω is connected, $p = 2$, $a_0 = 0$, but requring instead of (2.37) only that $a_{kl} \in C_b^{\mu}(\mathbb{R}_+, L^\infty(\Omega))$ for some $\mu > \frac{1}{2}$. Our aim is to calculate the constants q and ℓ in Theorem 4.13 explicitely (which is hardly possible in the framework of Example 2.8 and 2.9). We combine the approach of [64] with Hilbert space techniques.

On $X = L^2(\Omega)$ we define the operator $A(t)$ by the closed symmetric quadratic form

$$a_t(f,g) = -\sum_{k,l=1}^{n} \int_\Omega a_{kl}(t,x)\, D_k f(x)\, \overline{D_l g(x)}\, dx$$

for $f, g \in D(a_t) = W^{1,2}(\Omega)$, cf. [30, Chap.6], [62, §2.2, 3.6]. If we can verify (AT) for $A(\cdot)$, then we obtain functions $u \in C^1((s,\infty), L^2(\Omega)) \cap C([s,\infty), L^2(\Omega))$ satisfying (2.36) in a weak sense w.r.t. the space variables (see Example 2.8 concerning better regularity).

Since the domain is connected, 0 is a simple eigenvalue of the Neumann Laplacian Δ_N on Ω. By the Rayleigh–Ritz formula, see e.g. [18, §4.5], 0 is also a simple eigenvalue of $A(t)$ and the first non–zero eigenvalue of $A(t)$ is smaller than $\eta\lambda_1$, where λ_1 is the first nonzero eigenvalue of Δ_N and η is the ellipticity constant from (2.38). Set

$$\delta := -\tfrac{1}{2}\,\eta\,\lambda_1. \tag{4.25}$$

Then $\tilde{A}(t) := A(t) + \delta$ has an exponential dichotomy with constants δ and $N = 1$ and orthogonal projection P_t, where $\dim Q_t X = 1$. Moreover $\tilde{A}(t)$ is invertible and sectorial of type (ϕ, K, w) for all $\phi \in (\frac{\pi}{2}, \pi)$, $w > \delta$, and a constant $K = K(w,\phi)$. We fix some $w > \delta$. Using the orthogonality of P_t and the functional calculus, see e.g. [18, Thm.2.5.3], one sees that

$$\|\tilde{A}(t)e^{\tau\tilde{A}(t)}\| \leq \frac{1}{e\tau} \quad \text{and} \quad \|\tilde{A}(t)^2 e^{\tau\tilde{A}(t)}\| = \|\tilde{A}(t)e^{\tau\tilde{A}(t)}\|_{\mathcal{L}(X_{-1}^t, X)} \leq \frac{4}{e^2\tau^2}$$

for $t \geq 0$ and $0 < \tau \leq \frac{2}{\delta}$, where X^t_{-1} is the extrapolation space for $\tilde{A}(t)$. By real interpolation, see e.g. [42, Prop.1.2.6], this implies

$$\|\tilde{A}(t)e^{\tau\tilde{A}(t)}f\| \leq 2e^{-3/2}\,\tau^{-3/2}\ |||f|||^t_{-\frac{1}{2}}$$

for $0 < \tau \leq \frac{2}{\delta}$ and $f \in X^t_{-\frac{1}{2},\infty}$, where

$$|||f|||^t_{-\frac{1}{2}} := \sup_{s>0} s^{-\frac{1}{2}}\,K(s,f), \quad K(s,f) := \inf\{\|u\|^t_{-1} + s\,\|v\| : f = u+v,\ u \in X^t_{-1},\ v \in X\}.$$

For $s < \frac{1}{w}$, set $u := -\tilde{A}_{-1}(t)R(\frac{1}{s},\tilde{A}_{-1}(t))f$ and $v := \frac{1}{s}R(\frac{1}{s},\tilde{A}_{-1}(t))f$. Then

$$\sup_{0<s<1/w} s^{-\frac{1}{2}}\,K(s,f) \leq \sup_{0<s<1/w} 2s^{-\frac{1}{2}}\,\|R(\tfrac{1}{s},\tilde{A}_{-1}(t))f\| = 2\,\|f\|^t_{-\frac{1}{2}}.$$

For $s \geq \frac{1}{w}$ we choose $u := f$ and $v := 0$ and obtain

$$\begin{aligned}\sup_{s\geq 1/w} s^{-\frac{1}{2}}\,K(s,f) &\leq \sup_{s\geq 1/w} s^{-\frac{1}{2}}\,\|\tilde{A}_{-1}(t)^{-1}f\| = w^{\frac{1}{2}}\,\|\tilde{A}_{-1}(t)^{-1}f\| \\ &= \|(w-\tilde{A}(t))\tilde{A}(t)^{-1}\|\,w^{\frac{1}{2}}\,\|R(w,\tilde{A}_{-1}(t))f\| \leq (1+\tfrac{w}{\delta})\,\|f\|^t_{-\frac{1}{2}}\end{aligned}$$

since $\|\tilde{A}(t)^{-1}\| = \frac{1}{\delta}$. As a result,

$$\|\tilde{A}(t)e^{\tau\tilde{A}(t)}\|_{\mathcal{L}(X^t_{-\frac{1}{2},\infty},X)} \leq 2e^{-3/2}\,(1+\tfrac{w}{\delta})\,\tau^{-3/2} =: c_1(w)\tau^{-3/2} \tag{4.26}$$

for $0 < \tau \leq \frac{2}{\delta}$. This allows to estimate the quantity q from Lemma 4.12 for $\tilde{A}(\cdot)$ by

$$\begin{aligned} q &\leq c_1\int_0^{2/\delta} s^{\mu-3/2}\,ds + c_1\Big(\frac{\delta}{2}\Big)^{\frac{3}{2}}\int_{2/\delta}^{\infty} e^{-\delta(s-2/\delta)}s^{\mu}\,ds + c_1\Big(\frac{\delta}{2}\Big)^{\frac{3}{2}}\int_0^{\infty} e^{-\delta(s+2/\delta)}s^{\mu}\,ds \\ &= c_1\left[\frac{1}{\mu-1/2}\Big(\frac{2}{\delta}\Big)^{\mu-\frac{1}{2}} + \Big(\frac{\delta}{2}\Big)^{\frac{3}{2}}\int_0^{\infty} e^{-\delta s}\Big((s+\tfrac{2}{\delta})^{\mu}+e^{-2}s^{\mu}\Big)ds\right] =: c_2(w). \end{aligned} \tag{4.27}$$

For instance, $c_2(w) = c_1(w)\,[2^{\frac{3}{2}} + 2^{-\frac{1}{2}} + (1+e^{-2})\,2^{-\frac{3}{2}}]\,\delta^{-\frac{1}{2}}$ if $\mu = 1$.

Let $[a]_\mu$ be the maximum of the Hölder constants of a_{kl}, $k,l = 1,\cdots,n$, $\varepsilon := w-\delta > 0$, $f \in L^2(\Omega)$, and $g \in D((\varepsilon - A(t))^{\frac{1}{2}})$. Then (2.40) implies

$$\begin{aligned}&|((\varepsilon - A(t))^{-\frac{1}{2}}\,[R(\varepsilon,A(t)) - R(\varepsilon,A(s))]f,g)| \\ &\qquad\qquad \leq n\,[a]_\mu\,|t-s|^{\mu}\,\|\nabla R(\varepsilon,A(s))f\|_2\,\|\nabla(\varepsilon-A(t))^{-\frac{1}{2}}g\|_2\,.\end{aligned}$$

Further, (2.38) yields

$$\begin{aligned}\|\nabla R(\varepsilon,A(s))f\|_2^2 &= \sum_{k=1}^n \int_\Omega |D_kR(\varepsilon,A(s))f(x)|^2\,dx \\ &\leq \frac{1}{\eta}\sum_{k,l=1}^n \int_\Omega a_{kl}(s,x)\,D_kR(\varepsilon,A(s))f(x)\,\overline{D_lR(\varepsilon,A(s))f(x)}\,dx \\ &= \tfrac{1}{\eta}\,(-A(s)R(\varepsilon,A(s))f, R(\varepsilon,A(s))f) \\ &\leq \tfrac{1}{\eta}\,\|f\|^2\,(1+\varepsilon\|R(\varepsilon,A(s))\|)\,\|R(\varepsilon,A(s))\| = \tfrac{2}{(w-\delta)\eta}\,\|f\|^2.\end{aligned}$$

In the same way we obtain

$$\begin{aligned}\|\nabla(\varepsilon - A(t))^{-\frac{1}{2}}g\|_2^2 &\le \tfrac{1}{\eta}\,(-A(t)(\varepsilon - A(t))^{-\frac{1}{2}}g, (\varepsilon - A(t))^{-\frac{1}{2}}g)\\ &= \tfrac{1}{\eta}\,(-A(t)R(\varepsilon, A(t))g, g) \le \tfrac{2}{\eta}\,\|g\|^2.\end{aligned}$$

So we have shown that

$$\|(\varepsilon - A(t))^{\frac{1}{2}}\,[R(\varepsilon, A(t)) - R(\varepsilon, A(s))]\|_{\mathcal{L}(X)} \le [a]_\mu\,\frac{2n}{\eta(w-\delta)^{\frac{1}{2}}}\,|t-s|^\mu\,. \tag{4.28}$$

It remains to relate (4.28) with the Hölder estimate for $\tilde{A}(t)$ in (AT'). To do this, we first write for $r > w$

$$\begin{aligned} r^{\frac{1}{2}}\,\tilde{A}(t)R(r, \tilde{A}(t))\,(\varepsilon - A(t))^{-\frac{1}{2}} &= \frac{1}{\pi}\,r^{\frac{1}{2}}\,\tilde{A}(t)R(r, \tilde{A}(t))\int_0^\infty (rs)^{-\frac{1}{2}}\,R(rs+\varepsilon, A(t))\,r\,ds\\ &= \frac{1}{\pi}\bigg(\int_0^1 s^{-\frac{1}{2}}rR(r-\delta, A(t))\,\tilde{A}(t)R(rs+\varepsilon+\delta, \tilde{A}(t))\,ds\\ &\qquad + \int_1^\infty s^{-\frac{3}{2}}\,rsR(rs+\varepsilon, A(t))\,\tilde{A}(t)R(r, \tilde{A}(t))\,ds\bigg)\end{aligned}$$

using a standard formula for fractional powers, see e.g. [47, (2.6.4)]. So we can estimate

$$\begin{aligned} &\|r^{\frac{1}{2}}\,\tilde{A}(t)R(r, \tilde{A}(t))\,(\varepsilon - A(t))^{-\frac{1}{2}}\|\\ &\quad\le \frac{1}{\pi}\bigg(\int_0^1 s^{-\frac{1}{2}}\,\frac{r}{r-\delta}\Big(\frac{rs+\varepsilon+\delta}{rs+\varepsilon}+1\Big)ds + \int_1^\infty s^{-\frac{3}{2}}\,\frac{rs}{rs+\varepsilon}\,\Big(\frac{r}{r-\delta}+1\Big)ds\bigg)\\ &\quad\le \frac{2}{\pi}\Big(1+\frac{w}{w-\delta}\Big)^2 =: c_3(w). \end{aligned} \tag{4.29}$$

Finally, we compute

$$\begin{aligned}\tilde{A}(s)^{-1} - \tilde{A}(t)^{-1} &= (\varepsilon - A(t))(\delta + A(t))^{-1}[R(\varepsilon, A(t)) - R(\varepsilon, A(s))](\varepsilon - A(s))(\delta + A(s))^{-1}\\ &= (w(A(t)+\delta)^{-1} - I)\,[R(\varepsilon, A(t)) - R(\varepsilon, A(s))]\,(w(A(s)+\delta)^{-1} - I).\end{aligned}$$

Putting all this together, we conclude that

$$\|\tilde{A}(t)^{-1} - \tilde{A}(s)^{-1}\|_{\mathcal{L}(X, X^t_{\frac{1}{2},\infty})} \le c_3\,\frac{2n(1+\frac{w}{\delta})^2}{\eta(w-\delta)^{1/2}}\,[a]_\mu\,|t-s|^\mu =: c_4(w)\,[a]_\mu\,|t-s|^\mu\,. \tag{4.30}$$

Thus (AT') holds for $\tilde{A}(\cdot)$ with $\ell := c_4\,[a]_\mu$. As a consequence of Theorem 4.13, the operators $\tilde{A}(t)$ generate an evolution family having exponential dichotomy with exponent $\delta' > 0$ and $\dim Q(t) = 1$ provided that

$$[a]_\mu < (c_2\,c_4)^{-1}, \tag{4.31}$$

where c_2 and c_4 depend on $\mu, w, \eta, \lambda_1, n$ as described in (4.25), (4.26), (4.27), (4.29), and (4.30). Consequently, there are subspaces $X_1(s)$ of codimension 1 such that the solutions of (2.36) starting in $X_1(s)$ tend to 0 exponentially fast. On the other hand, (2.36) has the constant solution $u(t,x) = c$ since $a_0 = 0$. Estimate (4.31) can always be achieved if we replace $a_{kl}(t,x)$ by $a_{kl}(\varepsilon t, x)$ for a small $\varepsilon > 0$.

5. Inhomogeneous problems

In this section we investigate the inhomogeneous problems

$$u'(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R}, \tag{5.1}$$

$$u'(t) = A(t)u(t) + f(t), \quad t > a, \qquad u(a) = x, \tag{5.2}$$

assuming that $f \in C_b(J, X)$, $x \in X$, and $A(t)$, $t \in J$, $J \in \{\mathbb{R}, [a, \infty)\}$, satisfy (AT) and that the evolution family $U(\cdot,\cdot)$ generated by $A(\cdot)$ has an exponential dichotomy with projections $P(t)$ and Green's function $\Gamma(\cdot,\cdot)$. We want to show that u inherits the convergence and almost periodicity of f. We have seen in Section 2 that the classical solution of (5.2) with $x \in X_0^a$ is given by the mild solution

$$u(t) = U(t,a)x + \int_a^t U(t,\tau)f(\tau)\,d\tau, \qquad t \geq a. \tag{5.3}$$

Conversely, the mild solution is a classical or strict one if the data are sufficiently regular due to Theorem 2.7. Writing $f(\tau) = P(\tau)f(\tau) + Q(\tau)f(\tau)$, we deduce from (5.3) that

$$u(t) = U(t,a)\Big(x + \int_a^\infty U_Q(a,\tau)Q(\tau)f(\tau)\,d\tau\Big) + \int_a^\infty \Gamma(t,\tau)f(\tau)d\tau, \quad t \geq a. \tag{5.4}$$

Since $U(\cdot,\cdot)$ is exponentially dichotomic, the function u is bounded if and only if the term in brackets belongs to $P(a)X$ if and only if

$$Q(a)x = -\int_a^\infty U_Q(a,\tau)Q(\tau)f(\tau)\,d\tau. \tag{5.5}$$

This condition automatically holds if $U(\cdot,\cdot)$ is exponentially stable. In the general case it says that the unstable part of the initial value of a bounded solution is determined by the inhomogeneity (and by U_Q). If (5.5) is valid, then the mild solution is given by

$$u(t) = U(t,a)P(a)x + \int_a^\infty \Gamma(t,\tau)f(\tau)\,d\tau =: v_1(t) + v_2(t), \qquad t \geq a. \tag{5.6}$$

In the case that $J = \mathbb{R}$ we can apply the above arguments for $x = u(s)$ and all $s = a \in \mathbb{R}$. So we call a function $u \in C(\mathbb{R}, X)$ satisfying

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)\,d\tau \quad \text{ for all } \; t \geq s \tag{5.7}$$

the *mild solution* of (5.1). Such a function is continuously differentiable, belongs to $D(A(t))$, and solves (5.1) if f is locally Hölder continuous or $\|f(t)\|_\alpha^t$ is locally bounded by virtue of Theorem 2.7. It is easy to verify that

$$u(t) = \int_{\mathbb{R}} \Gamma(t,\tau)f(\tau)\,d\tau, \qquad t \in \mathbb{R}, \tag{5.8}$$

is a bounded mild solution. Conversely, letting $a \to -\infty$ in (5.6) with $x = u(a)$, we see each bounded mild solution is given by (5.8).

5.1. **Asymptotically autonomous problems.** We first study (5.2) in the framework of Paragraph 4.2 where we require that $f \in C([a,\infty), X)$ converges to f_∞ in X as $t \to \infty$. Supposing the assumptions of Theorem 4.3, the evolution family $U(\cdot,\cdot)$ has an exponential dichotomy on an interval $[a, \infty)$.[8] Our next result generalizes a theorem due to H.Tanabe

[8] We recall that Proposition 4.8 provides conditions allowing to take $a = 0$.

from 1961, see [62, Thm.5.6.1], where the situation of Remark 4.5 with $w = 0$ was studied. In [26, Thm.1.3] Tanabe's result was extended to the case of Kato–Tanabe conditions for f converging in a weaker sense, but still for the case of exponential stability. In [57, Thm.4.1] we proved our theorem for dense domains $D(A(t))$. It was partly extended to partial functional differential equations in [59]. Recall that the condition (5.5) is equivalent to the boundedness of the mild solution of (5.2).

Theorem 5.1. *Assume that the operators $A(t)$, $t \geq 0$, satisfy (AT) from Section 2, the operator A satisfies (AT1), $R(w, A(t)) \to R(w, A)$ in $\mathcal{L}(X)$ as $t \to \infty$, and $\sigma(A) \cap i\mathbb{R} = \emptyset$. Fix the number $a \geq 0$ as obtained in Theorem 4.3. Suppose that $f \in C([a, \infty), X)$ tends to f_∞ in X as $t \to \infty$ and that $x \in X$ satisfies* (5.5). *Then the mild solution u of* (5.2) *is given by* (5.6) *and converges to $u_\infty := -A^{-1} f_\infty$ as $t \to \infty$. If, in addition, either*

(a) *$f \in C_b^\alpha([a, \infty), X)$ for some $\alpha > 0$ or*

(b) *$\sup_{t \geq a} \|f(t)\|_\beta^t < \infty$ for some $\beta > 0$,*

then $u \in C^1((a, \infty), X)$, $u(t) \in D(A(t))$ for $t > a$, and (5.2) *holds. Moreover, $u'(t) \to 0$ and $A(t)u(t) \to Au_\infty$ as $t \to \infty$. Finally, u is a classical solution if $x \in X_0^a$.*

Proof. (1) Let u be the mild solution given by (5.3) or (5.6). Theorems 2.2 and 2.7 show that $u \in C^1((a, \infty), X)$, $u(t) \in D(A(t))$ for $t > a$, and (5.2) holds if (a) or (b) are true. Moreover, $u(t) \to x$ as $t \to a$ if $x \in X_0^s$.

We use the functions v_k defined in (5.6), where we extend f by 0 to $\mathbb{R}$. Thus v_2 is given by (5.8) for $t \geq a$. Clearly, $v_1(t) \to 0$ and

$$\|v_1'(t)\| = \|A(t)v_1(t)\| \leq \|A(t)U(t, t-1)\| \, \|U(t-1, a)P(a)x\| \longrightarrow 0$$

as $t \to \infty$ due to (2.13). In Theorem 4.3 we proved that $U(s+t, s) \to e^{tA}$ and $P(s) \to P$ strongly as $s \to \infty$, where P is the dichotomy projection of e^{tA}. We further have

$$v_2(t) = \int_0^\infty U(t, t-\tau)P(t-\tau)f(t-\tau)\, d\tau - \int_0^\infty U_Q(t, t+\tau)Q(t+\tau)f(t+\tau)\, d\tau$$

for $t \geq a$. The first integrand converges to $e^{\tau A} P f_\infty$ as $t \to \infty$. Concerning the second integrand, we observe that

$$\begin{aligned} &U_Q(t, t+\tau)Q(t+\tau)f(t+\tau) - e^{-\tau A_Q} Q f_\infty \\ &\quad = U_Q(t, t+\tau)Q(t+\tau)\,[Q(t+\tau)f(t+\tau) - Qf_\infty] + U_Q(t, t+\tau)Q(t+\tau) \\ &\qquad \cdot [e^{\tau A} - U(t+\tau, t)]\, e^{-\tau A_Q} Q f_\infty + [Q(t+\tau) - Q]\, e^{-\tau A_Q} Q f_\infty \end{aligned}$$

tends to 0 as $t \to \infty$. Hence,

$$\begin{aligned} \lim_{t\to\infty} u(t) = \lim_{t\to\infty} v_2(t) &= \int_0^\infty e^{\tau A_P} P f_\infty \, d\tau - \int_0^\infty e^{-\tau A_Q} Q f_\infty \, d\tau \\ &= -A_P^{-1} f_\infty + (-A_Q^{-1}) f_\infty = -A^{-1} f_\infty \end{aligned} \tag{5.9}$$

by the theorem of dominated convergence and standard semigroup theory. The remaining assertions follow from

$$\lim_{t\to\infty} v_2'(t) = 0. \tag{5.10}$$

To this establish this fact, we will use the formula

$$v_2(t) = U(t, r)v_2(r) + \int_r^t U(t, \tau)f(\tau)\, d\tau, \quad t \geq r \geq a, \tag{5.11}$$

which is a consequence of (5.3).

(2) Assume that (a) holds. We first want to show that v_2' is uniformly Hölder continuous on $[a+1,\infty)$. In view of (5.11) and Theorems 2.2 and 2.7, v_2' is given by

$$v_2'(t) = A(t)U(t,r)v_2(r) + A(t)\int_r^t U(t,\tau)f(\tau)\,d\tau + f(t) \tag{5.12}$$

for $t > r \geq a$. For $t \geq s \geq r + \frac{1}{2}$, Proposition 2.4 shows that

$$\|A(t)U(t,r)v_2(r) - A(s)U(s,r)v_2(r)\| \leq c\,(t-s)^\alpha\,(s-r)^{-1-\alpha}\,\|v_2\|_\infty \leq c\,(t-s)^\alpha\,\|f\|_\infty$$

for constants c independent of t,s,r satisfying $r+1 \geq t \geq s \geq r+\frac{1}{2}$. Further, (2.32) with $\varepsilon = 1/2$ yields

$$\left\|A(t)\int_r^t U(t,\tau)f(\tau)d\tau - A(s)\int_r^s U(s,\tau)f(\tau)d\tau\right\| \leq c\,(t-s)^\alpha\,\|f\|_{C^\alpha}$$

if $r+1 \geq t \geq s \geq r+\frac{1}{2}$, where c does not depend on t,s,r. As a result,

$$\|v_2'(t) - v_2'(s)\| \leq c\,(t-s)^\alpha\,\|f\|_{C^\alpha}$$

where c is independent of $t \geq s$ provided we take $t,s \in [r+\frac{1}{2}, r+1]$. This can be achieved for $0 \leq t-s \leq 1/2$ by choosing $r = t-1 \geq a$. Since $v_2'(t)$ is bounded for $t \geq a+1$ due to (5.12) (for $r = t-1$), (2.13), and (2.32), we obtain that $v_2' \in C_b^\alpha([a+1,\infty),X)$.

Set $\tilde{v}_2(t) := v_2(t) - u_\infty$. The interpolation result [42, Prop.1.2.19] and [42, Prop.0.2.2] then imply

$$\begin{aligned}\sup_{n\leq t\leq n+1}\|v_2'(t)\| &\leq \|\tilde{v}_2\|_{C^\gamma([n,n+1],X)} \leq c\,\|\tilde{v}_2\|^{1-\theta}_{C([n,n+1],X)}\,\|\tilde{v}_2\|^\theta_{C^{1+\alpha}([n,n+1],X)}\\ &\leq c\sup_{n\leq t\leq n+1}\|v_2(t)-u_\infty\|^{1-\theta}\end{aligned} \tag{5.13}$$

for some $\gamma \in (1, 1+\alpha)$, $\theta := \frac{\gamma}{1+\alpha}$, and constants independent of n. Thus (5.10) is true.

(3) Now assume that (b) holds. To verify again (5.10), we want to employ

$$\sup_{t\geq a+1}\|v_2'(t)\|_\beta^t < \infty. \tag{5.14}$$

This fact is an immediate consequence of (5.12) (for $r = t-1$), Proposition 2.4 and (2.34) with $\varepsilon = 1/2$. On the other hand, (5.2) implies

$$R(w,A(t))v_2'(t) = -v_2(t) + wR(w,A(t))v_2(t) + R(w,A(t))f(t).$$

The right hand side converges to 0 as $t \to \infty$ because of (5.9) and $R(w,A(t)) \to R(w,A)$. This means that

$$\lim_{t\to\infty} R(w,A(t))v_2'(t) = 0. \tag{5.15}$$

Using (2.7), (2.5), and (5.14), we derive

$$\|v_2'(t)\| \leq c\,\|(w-A(t))^{-1}v_2'(t)\|^{1-\theta}\,\|(w-A(t))^\gamma v_2'(t)\|^\theta \leq c\,\|R(w,A(t))v_2'(t)\|^{1-\theta}$$

for $\gamma \in (0,\beta)$, $\theta := \frac{1}{1+\gamma}$, and constants c independent of t. Now (5.15) yields (5.10). □

Example 5.2. One can directly apply the above theorem to the situation of Example 4.9.

5.2. **Almost periodic equations.** We assume in this paragraph that the resolvents $R(w, A(\cdot))$ and the inhomogeneity f are almost periodic (in $\mathcal{L}(X)$ and X, respectively) in the sense of the following definition due to H. Bohr, compare [8], [35], [50].

Definition 5.3. *Let Y be a Banach space. A continuous function $g : \mathbb{R} \to Y$ is* almost periodic *if for every $\epsilon > 0$ there exist a set $P(\varepsilon) \subseteq \mathbb{R}$ and a number $\ell(\varepsilon) > 0$ such that each interval $(a, a + \ell(\varepsilon))$, $a \in \mathbb{R}$, contains an* almost period *$\tau = \tau_\varepsilon \in P(\varepsilon)$ and the estimate $\|g(t + \tau) - g(t)\| \le \varepsilon$ holds for all $t \in \mathbb{R}$ and $\tau \in P(\varepsilon)$. The space of almost periodic functions is denoted by $AP(\mathbb{R}, Y)$.*

We recall that $AP(\mathbb{R}, Y)$ is a closed subspace of the space of bounded and uniformly continuous functions, see [35, Chap.1]. In addition to (AT) we make the following assumptions.

(AP) $R(w, A(\cdot)) \in AP(\mathbb{R}, \mathcal{L}(X))$ with pseudo periods $\tau = \tau_\varepsilon$ belonging to sets $P(\varepsilon, A)$.
(H) The evolution family $U(\cdot, \cdot)$ generated by $A(\cdot)$ is hyperbolic with projections $P(t)$ and constants $N, \delta > 0$

The almost periodicity of inhomogeneous problems has been studied by many authors in the autonomous and the periodic case, see [8], [35], [50], and the references given there and in [43], [58]. Equations with almost periodic $A(\cdot)$ are treated in, e.g., [15] and [22] for $X = \mathbb{C}^n$ and in [27] for a certain class of parabolic problems, see also [11], [35]. For general evolution families $U(\cdot, \cdot)$ (but subject to an extra condition not assumed here), it is shown in [28] that $U(\cdot, \cdot)$ has an exponential dichotomy *with* an almost periodic Green's function if and only if there is a unique almost periodic mild solution u of (5.1) for each almost periodic f, see also [15, Prop.8.3]. Our main Theorem 5.9 extends [15, Prop.8.4], [22, Thm.7.7], and [27, p.240], and complements [28, Thm.5.4] in the case of parabolic evolution equations. Theorem 5.9 is taken from [43], where we also treat problems on the time interval $\mathbb{R}_+$.

Below we first show the almost periodicity of Green's function $\Gamma(\cdot, \cdot)$ of $U(\cdot, \cdot)$ and then deduce from (5.8) that u is almost periodic. Our strategy is similar to Henry's approach in [27, §7.6] who derived the almost periodicity of $\Gamma(\cdot, \cdot)$ from a formula for $\Gamma(t + \tau, s + \tau) - \Gamma(t, s)$ (compare (5.16)). In the context of [27], this equation allows to verify almost periodicity by straightforward estimates in operator norm if τ is a pseudo period of $A(\cdot)$. However, in the present more general situation one obtains such a formula only on a subspace of X (see [43, Cor.4.3]) so that we cannot proceed in this way.

The Yosida approximations $A_n(t) = nA(t)R(n, A(t))$, $n \ge w$, allow us to overcome this difficulty. By Lemma 2.3 there is a number $\overline{n}$ such that $A_n(\cdot)$ satisfies (AT) with constants K', L', ϕ', and $w' := w + 1$ for $n \ge \overline{n}$, and thus generates an evolution family $U_n(\cdot, \cdot)$. Proposition 2.5, Theorem 3.15, and (3.22) imply that $A_n(\cdot)$ also fulfills (H) for large n and that the projections converge.

Corollary 5.4. *Let (AT) and (H) hold. Then there is a number $n_1 \ge n_0(1) \ge \overline{n}$ such that $U_n(\cdot, \cdot)$ has an exponential dichotomy for $n \ge n_1$ with constants $\delta' \in (0, \delta)$ and $N' = N'(\delta')$ independent of n. Moreover, the dichotomy projections $P_n(t)$ of $U_n(\cdot, \cdot)$ satisfy $\|P_n(t) - P(t)\| \le c\, n^{-\theta}$ for $t \in \mathbb{R}$, where $\theta \in (0, 1)$ and $n_0(1)$ are given by Proposition 2.5.*

The almost periodicity of the resolvents is inherited by the Yosida approximations, too.

Lemma 5.5. *If (AT) and (AP) hold, then $R(w', A_n(\cdot)) \in AP(\mathbb{R}, \mathcal{L}(X))$ for $n \geq \overline{n}$, with pseudo periods belonging to $P(\epsilon/\kappa, A)$, where $\kappa := (1 + (1 + w)K)^2$.*

Proof. Let $\tau > 0$ and $t \in \mathbb{R}$. Equations (2.20) and (2.8) yield

$$R(w', A_n(t+\tau)) - R(w', A_n(t)) = \tfrac{n^2}{(w'+n)^2}\,[R(\tfrac{w'n}{w'+n}, A(t+\tau)) - R(\tfrac{w'n}{w'+n}, A(t)],$$
$$\|R(w', A_n(t+\tau)) - R(w', A_n(t))\| \leq (1 + (1+w)K)^2 \|R(w, A(t+\tau)) - R(w, A(t))\|.$$

The assertion thus follows from (AP). □

We next establish the almost periodicity of Green's function Γ_n of $U_n(\cdot,\cdot)$.

Lemma 5.6. *Assume that (AT), (AP), and (H) hold. Let $n \geq n_1$, $\eta > 0$, and $\tau \in P(\eta/\kappa, A)$, where n_1 and κ were given in Corollary 5.4 and Lemma 5.5. Then*

$$\|\Gamma_n(t+\tau, s+\tau) - \Gamma_n(t,s)\| \leq c\,\eta\, n^2\, e^{-\frac{\delta}{2}|t-s|} \quad \textit{for} \;\; t, s \in \mathbb{R}.$$

Proof. The operators $\Gamma_n(t,s)$ exist by Corollary 5.4. It is easy to see that

$$\begin{aligned} g_n(\sigma) &:= \frac{d}{d\sigma}\Big(\Gamma_n(t,\sigma)\Gamma_n(\sigma+\tau, s+\tau)\Big) \\ &= \Gamma_n(t,\sigma)(A_n(\sigma+\tau) - A_n(\sigma))\Gamma_n(\sigma+\tau, s+\tau) \\ &= \Gamma_n(t,\sigma)(A_n(\sigma) - w')((A_n(\sigma) - w')^{-1} - (A_n(\sigma+\tau) - w')^{-1})\cdot \\ &\qquad \cdot (A_n(\sigma+\tau) - w')\Gamma_n(\sigma+\tau, s+\tau), \end{aligned}$$

for $\sigma \neq t, s$ and $n \geq n_1$. Hence,

$$\int_{\mathbb{R}} g_n(\sigma)\,d\sigma = \begin{cases} \int_{-\infty}^s g_n(\sigma)d\sigma + \int_s^t g_n(\sigma)d\sigma + \int_t^\infty g_n(\sigma)d\sigma, & t \geq s, \\ \int_{-\infty}^t g_n(\sigma)d\sigma + \int_t^s g_n(\sigma)d\sigma + \int_s^\infty g_n(\sigma)d\sigma, & t < s, \end{cases}$$
$$= \begin{cases} -\Gamma_n(t,s)Q_n(s+\tau) + P_n(t)\Gamma_n(t+\tau, s+\tau) - \Gamma_n(t,s)P_n(s+\tau) + Q_n(t)\Gamma_n(t+\tau, s+\tau) \\ P_n(t)\Gamma_n(t+\tau, s+\tau) - \Gamma_n(t,s)Q_n(s+\tau) + Q_n(t)\Gamma_n(t+\tau, s+\tau) - \Gamma_n(t,s)P_n(s+\tau) \end{cases}$$
$$= \Gamma_n(t+\tau, s+\tau) - \Gamma_n(t,s).$$

We have shown that

$$\begin{aligned} \Gamma_n(t+\tau, s+\tau) - \Gamma_n(t,s) = \int_{\mathbb{R}} &\Gamma_n(t,\sigma)(A_n(\sigma) - w')\,[R(w', A_n(\sigma+\tau)) - R(w', A_n(\sigma))]\cdot \\ &\cdot (A_n(\sigma+\tau) - w')\Gamma_n(\sigma+\tau, s+\tau)\,d\sigma \end{aligned} \tag{5.16}$$

for $s, t \in \mathbb{R}$ and $n \geq n_1$. Lemma 5.5 and Corollary 5.4 now yield

$$\|\Gamma_n(t+\tau, s+\tau) - \Gamma_n(t,s)\| \leq c\eta n^2 \int_{\mathbb{R}} e^{-\frac{3\delta}{4}|t-\sigma|}\, e^{-\frac{3\delta}{4}|\sigma-s|}\,d\sigma$$

for $\tau \in P(\eta/\kappa, A)$, which implies the asserted estimate. □

Lemma 5.7. *Assume that (AT) and (H) hold. Fix $0 < t_0 < t_1$ and let $\theta > 0$, $n_0(t_0)$, and n_1 be given by Proposition 2.5 and Corollary 5.4. Then $\|\Gamma(t,s) - \Gamma_n(t,s)\| \leq c(t_1)\, n^{-\theta}$ holds for $t_0 \leq |t-s| \leq t_1$ and $n \geq \max\{n_0(t_0), n_1\}$.*

Proof. If $t_0 \le t-s \le t_1$, we write

$$\Gamma_n(t,s) - \Gamma(t,s) = (U_n(t,s) - U(t,s))P_n(s) + U(t,s)(P_n(s) - P(s)).$$

For $-t_1 \le t-s \le -t_0$, we have

$$\begin{aligned}\Gamma(t,s) - \Gamma_n(t,s) &= U_{n,Q}(t,s)Q_n(s) - U_Q(t,s)Q(s) \\ &= U_Q(t,s)Q(s)(Q_n(s) - Q(s)) + (Q_n(t) - Q(t))U_{n,Q}(t,s)Q_n(s) \\ &\quad - U_Q(t,s)Q(s)(U_n(s,t) - U(s,t))U_{n,Q}(t,s)Q_n(s).\end{aligned}$$

In both cases the claim is a consequence of Proposition 2.5 and Corollary 5.4. ☐

Proposition 5.8. *Assume that (AT), (AP), and (H) hold. Let $\varepsilon > 0$, $h > 0$, and $|t-s| \ge h$. Then*

$$\|\Gamma(t+\tau, s+\tau) - \Gamma(t,s)\| \le \varepsilon\, e^{-\frac{\delta}{2}|t-s|}$$

holds for $\tau \in P(\eta/\kappa, A)$, where $\kappa = (1+(1+w)K)^2$ and $\eta = \eta(\varepsilon, h) \to 0$ as $\varepsilon \to 0$ and h is fixed.

Proof. Let $\varepsilon > 0$ and $h > 0$ be fixed. Then there is a $t_\varepsilon > h$ such that

$$\|\Gamma(t+\tau, s+\tau) - \Gamma(t,s)\| \le \epsilon\, e^{-\frac{\delta}{2}|t-s|}$$

for $|t-s| \ge t_\varepsilon$. Let $h \le |t-s| \le t_\epsilon$, $\eta > 0$, and $\tau \in P(\eta/\kappa, A)$. Lemma 5.6 and 5.7 yield

$$\|\Gamma(t+\tau, s+\tau) - \Gamma(t,s)\| \le (c(t_\epsilon)e^{\frac{\delta}{2}t_\varepsilon}\, n^{-\theta} + c\eta n^2)\, e^{-\frac{\delta}{2}|t-s|}$$

for $n \ge \max\{n_0(h), n_1\}$. We now choose first a large n and then a small $\eta > 0$ (depending on ε and h) in order to obtain the assertion. ☐

Theorem 5.9. *Assume that (AT), (AP), and (H) hold. Then $r \mapsto \Gamma(t+r, s+r)$ belongs to $AP(\mathbb{R}, \mathcal{L}(X))$ for $t, s \in \mathbb{R}$, where we may take the same pseudo periods for t and s with $|t-s| \ge h > 0$. If $f \in AP(\mathbb{R}, X)$, then the unique bounded mild solution $u = \int_{\mathbb{R}} \Gamma(\cdot, s) f(s)\, ds$ of (5.1) is almost periodic.*

Proof. In Lemma 5.6 we have seen that $P_n(\cdot) \in AP(\mathbb{R}, \mathcal{L}(X))$. Corollary 5.4 then shows that $P(\cdot) \in AP(\mathbb{R}, \mathcal{L}(X))$. Thus the first assertion follows from Proposition 5.8. Further, for $\tau, h > 0$, $t \in \mathbb{R}$, we can write

$$\begin{aligned}u(t+\tau) - u(t) &= \int_{\mathbb{R}} \Gamma(t+\tau, s+\tau) f(s+\tau)\, ds - \int_{\mathbb{R}} \Gamma(t,s) f(s)\, ds \\ &= \int_{\mathbb{R}} \Gamma(t+\tau, s+\tau)(f(s+\tau) - f(s))\, ds + \int_{|t-s|\ge h} (\Gamma(t+\tau, s+\tau) - \Gamma(t,s)) f(s)\, ds \\ &\quad + \int_{|t-s|\le h} (\Gamma(t+\tau, s+\tau) - \Gamma(t,s)) f(s)\, ds.\end{aligned}$$

For $\overline{\varepsilon} > 0$ let $\eta = \eta(\overline{\varepsilon}, h)$ be given by Proposition 5.8. Let $P(\epsilon, A, f)$ be the set of pseudo periods for the almost periodic function $t \mapsto (f(t), R(\omega, A(t)))$, cf. [35, p.6]. Taking $\tau \in P(\eta/\kappa, A, f)$, we deduce from Proposition 5.8 and (AP) that

$$\|u(t+\tau) - u(t)\| \le \tfrac{2N}{\delta\kappa}\, \eta(\overline{\varepsilon}, h) + (\tfrac{4}{\delta}\, \overline{\varepsilon} + 4Nh)\|f\|_\infty.$$

for $t \in \mathbb{R}$. Given an $\varepsilon > 0$, we can take first a small $h > 0$ and then a small $\overline{\varepsilon} > 0$ such that $\|u(t+\tau) - u(t)\| \le \varepsilon$ for $t \in \mathbb{R}$ and $\tau \in P(\eta/\kappa, A, f) =: P(\varepsilon)$. ☐

Remark 5.10. For $g \in AP(\mathbb{R}, Y)$ and $\lambda \in \mathbb{R}$ the means

$$\lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} e^{-i\lambda s} g(s)\, ds$$

exist and they are different from zero for at most countable many λ, which are called the *frequencies* of g, see e.g. [8, §4.5], [35, §2.3]. The *module* of g is the smallest additive subgroup of $\mathbb{R}$ containing all frequencies of g. By [35, p.44] (see also [22, Thm.4.5]) and the proof of Theorem 5.9 the module of the solution u to (5.1) is contained in the joint module of f and $R(w, A(\cdot))$ (which is the smallest additive subgroup of $\mathbb{R}$ containing the frequencies of the function $t \mapsto (f(t), R(\omega, A(t))))$. Similarly, the modules of $\Gamma(t+\cdot, s+\cdot)$, $t \neq s$, are contained in those of $R(w, A(\cdot))$.

Example 5.11. In the situation of Example 2.8 we replace the time interval $\mathbb{R}_+$ by $\mathbb{R}$ and assume additionally that $p \in (1, \infty)$, $a_{kl} \in AP(\mathbb{R}, L^n(\Omega))$ and $a_0 \in AP(\mathbb{R}, L^{n/2}(\Omega))$ for $k, l = 1, \cdots, n$. (We replace $n/2$ by 1 if $n = 1$.) Using (2.40) with $\nu = 0$, one can then verify that $R(w, A(\cdot))$ is almost periodic. As a result, the mild solution of (5.1) is almost periodic if $f \in AP(\mathbb{R}, X)$ and the evolution family generated by $A(t)$ has an exponential dichotomy.

6. Convergent solutions for a quasilinear equation

In this section we want to illustrate how the linear theory developed so far can be used to investigate the long term behaviour of nonlinear equations. Here we concentrate on the quasilinear autonomous problem

$$u'(t) = A(u(t))u(t) + f(u(t)), \quad t > a, \qquad u(a) = x. \tag{6.1}$$

One way to deal with such problems is to choose a suitable function space F, fix some $u \in F$ in the arguments of the nonlinear operators, and solve the corresponding inhomogeneous, non–autonomous, linear problem

$$v'(t) = A(u(t))v(t) + f(u(t)), \quad t > a, \qquad v(a) = x. \tag{6.2}$$

This procedure should yield a mapping $u \mapsto v =: \Phi(u)$ whose fixed points $\Phi(u) = u$ correspond to solutions of (6.1). Indeed, in many cases one can employ this approach to obtain local (in time) solutions of the quasilinear problem. T. Kato developed and applied this method in particular for hyperbolic problems, see e.g. [29], [31]. Local and global solvability of parabolic equations was studied by several authors within this framework in differing situations, see e.g. [4], [5], [6], [39], [40], [61], [65]. In the same context principles of linearized stability of equlibria were established in [20], [23], [24], [41], [49], and center manifolds were constructed in [60]. We also refer to the monographs [27] for semilinear and [42] for fully nonlinear problems.

We want to find convergent solutions of (6.1) assuming in particular that $\sigma(A(y)) \cap i\mathbb{R} = \emptyset$. Our arguments are based on Theorems 4.3 and 5.1. For technical reasons we have to restrict ourselves to the case of constant domains $D(A(y))$, cf. Remark 6.4. Thus we are led to the following hypotheses, where $X_1 \hookrightarrow X$ are Banach spaces, $X_\beta := (X, X_1)_{\beta,\infty}$ is the real interpolation space with exponent $\beta \in (0, 1)$ and coefficient ∞ (see e.g. [42, Chap.1]), $\|\cdot\|$ is the norm on X and $\|\cdot\|_\beta$ is the norm on X_β for $0 < \beta \leq 1$.

(A) There are $R>0$ and $\alpha\in(0,1)$ such that for $y\in X_\alpha$ with $\|y\|_\alpha\le R$ the operators $A(y)$ are sectorial of type (ϕ,K,w), $D(A(y))=X_1$ with uniformly equivalent graph norms, and $\|A(y)-A(z)\|_{\mathcal{L}(X_1,X)}\le L^A\,\|y-z\|_\alpha$ for $\|y\|_\alpha,\|z\|_\alpha\le R$.

(S) $i\mathbb{R}\subseteq\rho(A(y))$ and $\|R(i\tau,A(y))\|\le\tilde K$ for $\|y\|_\alpha\le R$, $\tau\in\mathbb{R}$, and a constant $\tilde K>0$.

(f) $f(0)=0$ and $\|f(y)-f(z)\|\le L(r)\,\|y-z\|_\alpha$ for $\|y\|_\alpha,\|z\|_\alpha\le r\le R$, where $L(r)\to0$ as $r\to0$.

Observe that (f) implies that $\|f(y)\|\le rL(r)$ for $\|y\|_\alpha\le r\le R$. We fix $\beta\in(\alpha,1)$, denote by κ the norm of the embedding $X_\beta\hookrightarrow X_\alpha$ and set $R':=R/\kappa$. We introduce the space

$$\begin{aligned}F=\{u:\mathbb{R}_+\to X \text{ such that } &\|u(t)\|_\beta\le R',\ \|u(t)-u_\infty\|_\alpha\le(1+t)^{-1},\\ &\|u(t)-u(s)\|\le M_2\,|t-s|^\beta \text{ for } t,s\ge0 \text{ and some } u_\infty\in X_\alpha\}\end{aligned}$$

for a fixed number $M_2>0$. It is clear that $A_u(t):=A(u(t))$ are sectorial of uniform type for $u\in F$ and $t\ge0$. Assumption (A) and the reiteration theorem [42, Thm.1.2.15] yield

$$\|A_u(t)-A_u(s)\|_{\mathcal{L}(X_1,X)}\le L^A\|u(t)-u(s)\|_\alpha\le L^AM_2\,(R')^{\alpha/\beta}\,|t-s|^{\beta-\alpha}$$

for $t,s\ge0$. Thus $A_u(t)$, $t\ge0$, satisfy (AT) with $\nu=1$, $\mu=\beta-\alpha$, and $L=cL^AM_2(R')^{\alpha/\beta}$. We denote by $U_u(\cdot,\cdot)$ the evolution family generated by $A_u(\cdot)$. Moreover,

$$\|R(w,A_u(t))-R(w,A(u_\infty))\|\le c\,\|A(u(t))-A(u_\infty)\|_{\mathcal{L}(X_1,X)}\le cL^A(1+t)^{-1}$$

for $t\ge0$. By (S) and Lemma 4.11, the semigroups generated by $A(y)$, $\|y\|_\alpha\le R$, have an exponential dichotomy with common constants $\tilde N,\tilde\delta>0$. Theorem 4.3 now shows that the evolution famlies $U_u(\cdot,\cdot)$, $u\in F$, have exponential dichotomies with uniform constants $N\ge1$ and $\delta\in(0,\tilde\delta)$ on an interval $[a,\infty)$ for some $a\ge0$. We fix this a and some $\delta'\in(0,\delta)$, choose $M_1>0$ such that $M_1(1+t)\le e^{\delta' t}$ for $t\ge0$, and define

$$\begin{aligned}F_r=\{u:[a,\infty)\to X \text{ such that } &\|u(t)\|_\beta\le r,\ \|u(t)-u_\infty\|_\alpha\le M_1e^{-\delta' t},\\ &\|u(t)-u(s)\|\le M_2\,|t-s|^\beta \text{ for } t,s\ge a \text{ and some } u_\infty\in X_\alpha\}\end{aligned}$$

for $r\le R'$. Functions in F_r are extended to $\mathbb{R}$ by setting $u(t)=u(a)$ for $t\le a$ and thus considered as elements of F. The norm

$$|||f|||_\beta=\sup_{t\ge a}\|u(t)\|_\beta$$

induces a complete metric on F_r. For a given $x\in X_\beta$ we define

$$(\Phi_xu)(t)=U_u(t,a)P_u(a)x+\int_a^\infty\Gamma_u(t,\tau)f(u(\tau))\,d\tau$$

for $t\ge a$ and $u\in F_{R'}$. Here $P_u(t)$ and $\Gamma_u(\cdot,\cdot)$ denote the projections and Green's function of $U_u(\cdot,\cdot)$. The following two lemmas provide the estimates establishing that Φ is a strict contraction on F_r for small $r>0$ and $\|x\|_\beta$.

Lemma 6.1. *Let (A) and (S) hold. Fix β, a, and δ' as above. Let $u,v\in F_{R'}$ and $t,s\ge a$. Then there is a constant c independent of u,v,t,s such that*

$$\|\Gamma_u(t,s)-\Gamma_v(t,s)\|_{\mathcal{L}(X_\beta)}\le c\,e^{-\delta'|t-s|}\sup_{\tau\ge a}\|u(\tau)-v(\tau)\|_\alpha\,.$$

Proof. Recall that we have extended $u(t)$ constantly to $\mathbb{R}$. As seen in the proof of Theorem 4.3, the operators $A_u(t)$, $t \in \mathbb{R}$, generate an evolution family $U_u(\cdot,\cdot)$ having an exponential dichotomy on $J = \mathbb{R}$, with Green's function $\Gamma_u(t,s)$. Moreover, the corresponding projections $P_u(t)$ are Hölder continuous in t. Take $u, v \in F_{R'}$. Employing [42, Prop.6.2.6], the formula $U_{Q,u}(t,\tau)Q_u(\tau) = U_{Q,u}(t,t')Q_u(t')U_u(t',\tau)$ for $t < \tau < t'$, and Proposition 3.18, we conclude that

$$\tfrac{\partial}{\partial \tau}\,\Gamma_u(t,\tau)\Gamma_v(\tau,s)x = \Gamma_u(t,\tau)\,[A_v(\tau) - A_u(\tau)]\,\Gamma_v(\tau,s)x$$

for $\tau \neq t, s$. Because of (2.13) and Proposition 3.18, the right hand side of this equality is integrable in $\tau \in \mathbb{R}$ if $x \in X_\beta$. Moreover, $\tau \mapsto \Gamma_u(t,\tau)\Gamma_v(\tau,s)x$ has one-sided limits as $\tau \to t, s$ due to (3.2), $x \in X_\beta \subseteq \overline{X_1}$, and the continuity of $P_u(\cdot)$. Thus we obtain as in Lemma 5.7 that

$$\Gamma_v(t,s)x - \Gamma_u(t,s)x = \int_{\mathbb{R}} \Gamma_u(t,\tau)\,[A_v(\tau) - A_u(\tau)]\,\Gamma_v(\tau,s)x\,d\tau \tag{6.3}$$

for $t, s \in \mathbb{R}$ and $x \in X_\beta$. Using Proposition 3.18, (2.13), and (A), we estimate

$$\begin{aligned}\|\Gamma_u(t,s)x - \Gamma_v(t,s)x\|_\beta &\le c\int_{\mathbb{R}} \lceil t-\tau\rceil^{-\beta} e^{-\delta|t-\tau|}\,\|u(\tau)-v(\tau)\|_\alpha\,\lceil \tau - s\rceil^{\beta-1} e^{-\delta|\tau-s|}\|x\|_\beta\,d\tau\\ &\le c\,e^{-\delta'|t-s|}\|x\|_\beta \sup_{\tau\ge a}\|u(\tau)-v(\tau)\|_\alpha\end{aligned}$$

where $\lceil t\rceil := t$ if $0 < t \le 1$ and $\lceil t\rceil := 1$ if $t \le 0$ or $t \ge 1$. $\square$

Lemma 6.2. *Let (A), (S), and (f) hold. Fix β, a, and δ' as above. Take $t \ge a$ and $u, v \in F_r$ with $r \le R'$. Then there are constants c independent of u, v, t, r such that*

$$\Big\|\int_a^\infty (\Gamma_u(t,\tau)f(u(\tau)) - \Gamma_v(t,\tau)f(v(\tau)))\,d\tau\Big\|_\beta \le cL(r)\sup_{\tau\in\mathbb{R}}\|u(\tau)-v(\tau)\|_\alpha$$

$$\Big\|e^{\delta' t}\int_a^\infty (\Gamma_u(t,\tau)f(u(\tau)) - \Gamma_v(t,\tau)f(v(\tau)))\,d\tau\Big\|_\beta \le cL(r)(r + \sup_{\tau\ge a} e^{\delta'\tau}\|u(\tau)-v(\tau)\|_\alpha).$$

Proof. We only show the second assertion since the first one can be treated similarly. Let $t \ge a$ and $u, v \in F_r$. We write

$$\begin{aligned}&\int_a^\infty e^{\delta' t}\,(\Gamma_u(t,\tau)f(u(\tau)) - \Gamma_v(t,\tau)f(v(\tau)))\,d\tau\\ &= \int_a^\infty e^{\delta'(t-\tau)}\Gamma_u(t,\tau)\,e^{\delta'\tau}(f(u(\tau)) - f(v(\tau)))\,d\tau + \int_a^\infty e^{\delta' t}\,(\Gamma_u(t,\tau) - \Gamma_v(t,\tau))f(v(\tau))\,d\tau\\ &=: I_1 + I_2.\end{aligned}$$

Using Proposition 3.18, (2.13), and (f), we estimate $\|I_1\|_\beta \le cL(r)\sup_{\tau\ge a} e^{\delta'\tau}\|u(\tau) - v(\tau)\|_\alpha$. In order to deal with I_2, we employ as in Lemma 5.6 the Yosida approximations $A_n(u(t))$ of $A(u(t))$ for $t \in \mathbb{R}$ and $n \ge \overline{n}$. As in Corollary 5.4 and Lemma 5.7, we see that the evolution families $U_{u,n}(\cdot,\cdot)$ generated by $A_n(u(t))$ have exponential dichotomies with uniform constants for sufficiently large n and that the corresponding Green's functions $\Gamma_{u,n}(t,s)$ converge in operator norm to $\Gamma_u(t,s)$ as $n \to \infty$, for $t, s \in \mathbb{R}$. Arguing as in

(5.16), we deduce

$$
\begin{aligned}
I_2 &= \lim_{n\to\infty} \int_a^\infty e^{\delta' t}\,(\Gamma_{u,n}(t,\tau) - \Gamma_{v,n}(t,\tau)) f(v(\tau))\,d\tau \\
&= -\lim_{n\to\infty} \Big(\int_a^\infty \int_a^\infty e^{\delta'(t-\sigma)} \Gamma_{u,n}(t,\sigma)\, e^{\delta'\sigma} [A_n(v(\sigma)) - A_n(u(\sigma))]\, \Gamma_{v,n}(\sigma,\tau) f(v(\tau))\,d\sigma\,d\tau \\
&\qquad + \int_a^\infty e^{\delta' t}\,\Gamma_{u,n}(t,a)\Gamma_{v,n}(a,\tau) f(v(\tau))\,d\tau \Big) =: -\lim_{n\to\infty}(I_{21}^n + I_{22}^n).
\end{aligned}
$$

It is easy to see that $\|I_{22}^n\|_\beta \le crL(r)$ due to Proposition 3.18, (2.13), and (f). Interchanging the order of integration in I_{21}^n yields

$$
\begin{aligned}
I_{21}^n &= \int_a^\infty e^{\delta'(t-\sigma)} \Gamma_{u,n}(t,\sigma)\, e^{\delta'\sigma}\, [A_n(v(\sigma)) - A_n(u(\sigma))] \int_a^\infty \Gamma_{v,n}(\sigma,\tau) f(v(\tau))\,d\tau\,d\sigma, \\
&= \int_a^\infty e^{\delta'(t-\sigma)} \Gamma_{u,n}(t,\sigma)\, nR(n, A(u(\sigma)))\, e^{\delta'\sigma}\, [A(v(\sigma)) - A(u(\sigma))] R(w, A(v(\sigma))) \\
&\qquad \cdot n(w - A(v(\sigma))) R(n, A(v(\sigma))) \int_a^\infty \Gamma_{v,n}(\sigma,\tau) f(v(\tau))\,d\tau\,d\sigma,
\end{aligned}
$$

$$
\begin{aligned}
\|I_{21}^n\|_\beta \le c \sup_{\sigma\ge a} \Big\{ e^{\delta'\sigma}\, \|u(\sigma) - v(\sigma)\|_\alpha \Big(\Big\| A_n(v(\sigma)) \int_a^\infty \Gamma_{v,n}(\sigma,\tau) f(v(\tau))\,d\tau \Big\| \qquad (6.4) \\
+ \Big\| \int_a^\infty \Gamma_{v,n}(\sigma,\tau) f(v(\tau))\,d\tau \Big\| \Big) \Big\}
\end{aligned}
$$

by Proposition 3.18 and (A). The integral in (6.4) is equal to

$$
\begin{aligned}
P_{v,n}(\sigma) \int_{(\sigma-1)\vee a}^{\sigma} U_{v,n}(\sigma,\tau) f(v(\tau))\,d\tau + U_{v,n}(\sigma,\sigma-1) \int_a^{(\sigma-1)\vee a} U_{v,n}(\sigma-1,\tau) P_{v,n}(\tau) f(v(\tau))\,d\tau \\
- U_{v,n}(\sigma,\sigma-1) \int_\sigma^\infty U_{v,n,Q}(\sigma-1,\tau) Q_{v,n}(\tau) f(v(\tau))\,d\tau.
\end{aligned}
$$

Using Proposition 3.18, [42, Lem.6.2.1] or [65, Thm.2.4], and (2.13), we conclude that

$$
\|I_{21}^n\|_\beta \le c\, \|f(v)\|_{C_b^{\beta-\alpha}([a,\infty),X)} \sup_{\tau\ge a} e^{\delta'\tau}\, \|u(\tau) - v(\tau)\|_\alpha\,.
$$

Finally, assumption (f) and the reiteration theorem, [42, Thm.1.2.15], give

$$
\begin{aligned}
\|f(v(\tau)) - f(v(\tau'))\| &\le L(r)\, \|v(\tau) - v(\tau')\|_\alpha \le L(r)\, \|v(\tau) - v(\tau')\|_\beta^{\alpha/\beta}\, \|v(\tau) - v(\tau')\|^{1-\alpha/\beta} \\
&\le c\, r^{\alpha/\beta} L(r)\, |\tau - \tau'|^{\beta-\alpha} \qquad (6.5)
\end{aligned}
$$

for $\tau, \tau' \ge a$. The second assertion is verified combining the above estimates. $\square$

In the following theorem we establish the existence and uniqueness of convergent solutions to (6.1). Recall from Paragraph 5.1 that we can only prescribe the stable part of the initial value of a bounded mild solution to (6.2).

Theorem 6.3. *Assume that (A), (S), and (f) hold. Fix $\beta \in (\alpha, 1)$ and the initial time $a > 0$ as obtained in Theorem 4.3. Then there are numbers $r_0, \rho_0 > 0$ such that for $x \in X_\beta$ with $\|x\|_\beta \le \rho_0$ there is a unique $u \in F_{r_0} \cap C^1((a,\infty), X) \cap C((a,\infty), X_1)$ solving*

$$
u'(t) = A(u(t))u(t) + f(u(t)), \qquad t > a, \qquad (6.6)
$$

such that $P_u(a)u(a) = P_u(a)x$. Moreover,

$$u(a) = P_u(a)x - \int_a^\infty U_{u,Q}(0,\tau)Q_u(\tau)f(u(\tau))\,d\tau, \tag{6.7}$$

$\|u(t) - u_\infty\|_\alpha \le M_1 e^{-\delta' t}$ for $t \ge a$, the above chosen $\delta', M_1 > 0$, and $u_\infty \in X_1$ satisfying $A(u_\infty)u_\infty = -f(u_\infty)$, and $u'(t) \to 0$ and $A(u(t))u(t) \to A(u_\infty)u_\infty$ in X as $t \to \infty$.

Proof. To construct a solution, we employ the spaces F_r and the operator Φ_x introduced above. Let $u \in F_r$ and $\|x\|_\beta \le \rho$. Then

$$\|(\Phi_x u)(t)\|_\beta \le c\|x\|_\beta + c\int_a^\infty \lceil t-\tau\rceil^{-\beta} e^{-\delta|t-\tau|}\,\|f(u(\tau))\|\,d\tau \le c\,(\rho + rL(r)) \tag{6.8}$$

for $t \ge a$ by Proposition 3.18, (2.13), and (f). For $t' \ge t \ge a$ we have

$$(\Phi_x u)(t') - (\Phi_x u)(t) = (U_u(t',t) - I)(\Phi u)(t) + \int_t^{t'} U_u(t,\tau)f(u(\tau))\,d\tau.$$

Thus (2.16) and (6.8) yield

$$\|(\Phi_x u)(t') - (\Phi_x u)(t)\| \le c\,(\rho + rL(r))\,|t'-t|^\beta. \tag{6.9}$$

For the function $v \equiv u_\infty$, we calculate

$$\begin{aligned} e^{\delta' t}\Phi_x u(t) &- e^{\delta' t}R(0, A(u_\infty))f(u_\infty) \\ &= e^{\delta' t}U_u(t,a)P_u(a)x + e^{\delta' t}\int_a^\infty (\Gamma_u(t,\tau)f(u(\tau)) - \Gamma_{u_\infty}(t-\tau)f(u_\infty))d\tau \\ &\quad - e^{\delta' t}e^{(t-a)A(u_\infty)}P_{u_\infty}\int_{-\infty}^a e^{(a-\tau)A(u_\infty)}P_{u_\infty}f(u_\infty)\,d\tau. \end{aligned}$$

Due to Proposition 3.18, (2.13), Lemma 6.2, and (f) this identity leads to the estimate

$$\begin{aligned} e^{\delta' t}\,\|\Phi_x u(t) - R(0,A(u_\infty))f(u_\infty)\|_\alpha &\le c\,\|x\|_\alpha + cL(r)(r + \sup_{\tau\ge a} e^{\delta'\tau}\|u(\tau) - u_\infty\|_\alpha) + crL(r) \\ &\le c\,(\rho + rL(r) + L(r)M_1). \end{aligned} \tag{6.10}$$

The above constants do not depend on u, v, t, t', r, ρ. In view of (6.8), (6.9), and (6.10), we may choose $r_1 > 0$ and $\rho_1 > 0$ such that Φ_x maps F_r into F_r if $r \le r_1$ and $\|x\|_\beta \le \rho_1$. Using Lemmas 6.1 and 6.2 and (2.5), we find $r_0 \in (0, r_1]$ and $\rho_0 \in (0, \rho_1]$ such that $\Phi_x : F_r \to F_r$ is a strict contraction for $|||\cdot|||_\beta$ if $r \le r_0$ and $\|x\|_\beta \le \rho_0$. The resulting unique fix point $\Phi_x u = u \in F_r$ has the asserted properties due to Theorem 5.1 and (6.5). □

Remark 6.4. We have not strived for optimal regularity results in the above theorem. In principle, our approach can be extended to time varying domains in the framework of the Acquistapace–Terreni conditions, cf. [23], [24], [65]. However, since the operator Φ changes the value of the function at $t = a$, some difficulties with compatibility conditions arise which lead to unsatisfactory restrictions on x. For this reason and to obtain a simpler exposition, we have chosen the present setting.

Example 6.5. We consider the quasilinear initial–boundary value problem

$$\begin{aligned} D_t\,u(t,x) &= A(x,u(t,x),\nabla u(t,x),D)u(t,x) + g(x,u(t,x),\nabla u(t,x)), \quad t > a,\ x \in \Omega, \\ u(t,x) &= 0, \quad t > a,\ x \in \partial\Omega, \\ u(a,\cdot) &= u_0, \end{aligned} \tag{6.11}$$

on a bounded domain $\Omega \subseteq \mathbb{R}^n$ with boundary $\partial\Omega$ of class C^2, where

$$A(x,\sigma,\tau,D) = \sum\nolimits_{kl} a_{kl}(x,\sigma,\tau)\, D_k D_l + \sum\nolimits_{k} a_k(x,\sigma,\tau)\, D_k + a_0(x,\sigma,\tau).$$

It is assumed that the coefficients $a_{kl} = a_{lk}, a_k, a_0, g$ belong to $C(\overline{\Omega} \times [\tilde{R}, \tilde{R}] \times \overline{B}(0,\tilde{R}))$, are real–valued, $g(x,0,0) = 0$, and

$$\sum_{k,l=1}^{n} a_{kl}(x,\sigma,\tau)\, v_k\, v_l \geq \eta\, |v|^2$$

for $k,l = 1,\cdots,n$, constants $\tilde{R}, \eta > 0$, and $x \in \overline{\Omega}$, $\sigma \in [\tilde{R}, \tilde{R}]$, $\tau \in \overline{B}(0,\tilde{R})$, $v \in \mathbb{R}^n$. Here $\overline{B}(0,\tilde{R})$ is the closed ball in $\mathbb{R}^n$ with radius $\tilde{R}$ and center 0. We further require that

$$|a_\gamma(x,\sigma,\tau) - a_\gamma(x,\sigma',\tau')| \leq \tilde{L}\,(|\sigma - \sigma'| + |\tau - \tau'|), \quad x \in \overline{\Omega},\ |\sigma|, |\sigma'|, |\tau|, |\tau'| \leq \tilde{R},$$
$$|g(x,\sigma,\tau) - g(x,\sigma',\tau')| \leq \tilde{L}(r)\,(|\sigma - \sigma'| + |\tau - \tau'|), \quad x \in \overline{\Omega},\ |\sigma|, |\sigma'|, |\tau|, |\tau'| \leq r \leq \tilde{R},$$

where $\tilde{L}(r) \to 0$ as $r \to 0$ and $a_\gamma \in \{a_{kl}, a_k, a_0 : k,l = 1,\cdots,n\}$. Fix $p > n$ and $\alpha \in (1/2 + n/(2p), 1)$. We set $X = L^p(\Omega)$ and $X_1 = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$. Then $X_\alpha \hookrightarrow C^{1+\varepsilon}(\overline{\Omega})$ for some $\varepsilon \in (0,1)$ due to e.g. [27, Thm.1.6.1] and (2.5), where we denote the norm of this embedding by $d > 0$. Thus we can define

$$A(\psi)\varphi = A(\cdot,\psi(\cdot),\nabla\psi(\cdot))\varphi \quad \text{with} \quad D(A(\psi)) = X_1 \quad \text{and} \quad f(\psi) = g(\cdot,\psi(\cdot),\nabla\psi(\cdot))$$

for $\psi \in X_\alpha$ with $\|\psi\|_\alpha \leq R := d\tilde{R}$. It is then straightforward to check that (A) and (f) hold using standard elliptic regularity, cf. [42], [62], [63]. If also (S) is valid, we can fix an initial time a in (6.11) as obtained in Theorem 4.3. Theorem 6.3 then shows the existence of converging solutions to (6.11) for suitable initial values u_0.

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