Equivalents to Ekeland's Variational Principle in $\mathcal{F}$-type Topological Spaces

Andreas H. Hamel*

Abstract

We shall show that a recent version of Ekeland's principle in $\mathcal{F}$-type topological spaces due to Fang from 1996 is implied by the Brezis-Browder principle on ordered sets.

We give a series of equivalent formulations of Ekeland's principle in $\mathcal{F}$-type topological spaces, i.e. Penot's flower petal theorem, Takahashi's minimization principle and two theorems due to Oettli and Théra and show the equivalence of these to the version of Caristi's fixed point theorem given by Fang.

Keywords. Ekeland's variational principle, Kirk-Caristi fixed point theorem, petal theorem, $\mathcal{F}$-type topological spaces

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1 Introduction

The fundamental idea of the celebrated variational principle due to Ekeland [4] is to assign to a minimization problem a slightly perturbed problem having a solution which is at the same time an approximate solution of the original problem. The distance function serves as perturbation, i.e. the function which generates the topology of the complete metric space the principle is based on.

The godfather of all Ekeland-type theorems seems to be the lemma 1 of [14] due to Phelps from 1963 formulated in Haussdorff locally convex topological vector spaces.

*Department of Mathematics and Computer Science, Martin-Luther-University Halle-Wittenberg, 06099 Halle (Saale), Germany, E-mail: hamel@mathematik.uni-halle.de

The perturbation function appearing in this formulation is the Minkowski functional of a closed convex starshaped and bounded set and has at the first glance nothing to do with the topology of the space.

Both the theorems are easy consequences of the Brézis-Browder principle on ordered sets [1]. This principle does without any topology.

Recently, Fang [5] proved a version of Ekeland's principle in $\mathcal{F}$-type topological spaces which include complete metric spaces as well as Hausdorff topological vector spaces as special cases. The topology of $\mathcal{F}$-type spaces can be generated by means of a family of quasi-metrics as described below. At the same time, elements of this family serve as perturbation functions. In this sense, the variational principle due to Fang is more in the spirit of Ekeland's original version than the Phelps lemma.

The aim of this paper is first to show that Fang's theorem is a consequence of the Brézis-Browder principle and second to give and prove a series of equivalent formulations in $\mathcal{F}$-type spaces, namely Penot's flower petal theorem, Takahashi's minimization principle, the Kirk-Caristi fixed point theorem and two formulations of Ekeland's principle due to Oettli and Théra [11]. We close the paper with some comments.

2 The variational principle in $\mathcal{F}$-type spaces

We consider a nonempty set X with a partial ordering relation "$\preceq$". A sequence $\{x_n\}$ is said to be increasing provided $x_n \preceq x_{n+1}$ for all n. The following proposition from which we will derive Fang's result is an implication of the Brézis-Browder principle on ordered sets [1].

Proposition 1 *(corollary 3 of [1])*
Let X be a Hausdorff topological space with an ordering structure. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function, bounded below. Assume
(a) $x \preceq y$ and $x \neq y$ imply $f(y) < f(x)$
(b) $\{y \in X : x \preceq y\}$ is sequentially closed for each $x \in X$,
(c) any nondecreasing sequence is relatively compact.
Then for each $x_0 \in X$ there exists some $x^* \in X$ such that $x_0 \preceq x^*$ and x^* is maximal.

For a proof we refer to [1]. Note that it is enough to assume the sets $\{y \in X : x \preceq y\}$ to be sequentially closed.

Using an equivalent characterization proved by Fang [5] we introduce the concept of $\mathcal{F}$-type topological spaces by means of quasi-metrics.

Definition 1 *Let X be a topological space, $(D, \prec)$ a directed set. X is said to be $\mathcal{F}$-type iff there exists a family $\{d_\lambda\}_{\lambda \in D}$ of quasi-metrics $d_\lambda : X \times X \to [0, +\infty)$ with the following properties:*

(F1) *$d_\lambda(x, y) = 0$ for all $\lambda \in D$ iff $x = y$ for all $x, y \in X$,*

(F2) *$d_\lambda(x, y) = d_\lambda(y, x)$ for all $x, y \in X$,*

(F3) *$d_\lambda(x, y) \leq d_\mu(x, y)$ if $\lambda \prec \mu$ for all $x, y \in X$,*

(F4) *for every $\lambda \in D$ there exists $\mu \in D$ with $\lambda \prec \mu$ such that*

$$d_\lambda(x, y) \leq d_\mu(x, z) + d_\mu(z, y) \ \forall x, y, z \in X.$$

Note that there is a unique Hausdorff topology on X generated by the family of quasi-metrics. The sets

$$\mathcal{B}_x := \{B_x(\lambda, r) : \lambda \in D, r > 0\}$$

where

$$B_x(\lambda, r) := \{y \in X : d_\lambda(x, y) < r\}$$

define a base of neighborhoods of x for this topology. Every Hausdorff topological vector space and clearly every metric space is an $\mathcal{F}$-type topological space.

We give a slightly altered version of Fang's theorem 3.2 in [5].

Theorem 1 *Altered Ekeland's principle in $\mathcal{F}$-type spaces*
Let X be a sequentially complete $\mathcal{F}$-type topological space and $\{d_\lambda\}_{\lambda \in D}$ a family of quasi-metrics generating the topology. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a sequentially lower semi-continuous function, $\not\equiv +\infty$, bounded from below and $h : D \to (0, +\infty)$ a nondecreasing function. Then for every $x_0 \in X$, $\gamma > 0$ there exists some $x^* \in X$ such that

$$f(x^*) + \gamma h(\lambda) d_\lambda(x^*, x_0) \leq f(x_0) \ \forall \lambda \in D, \tag{1}$$

$$\forall x \neq x^* \; \exists \eta \in D : \; f(x^*) < f(x) + \gamma h(\eta) \, d_\eta(x, x^*) . \tag{2}$$

The proof of Fang [5] follows the lines of Ekeland's original proof in [4]. We deduce theorem 1 from proposition 1.

Proof: We define a partial ordering on X by

$$x \preceq y \quad \Longleftrightarrow \quad f(y) + \gamma h(\lambda) \, d_\lambda(y, x) \leq f(x) \;\; \forall \lambda \in D.$$

Then f is strictly decreasing with respect to " $\preceq$ " because of (F1) and the sets $\{y \in X : \; x \preceq y\}$ are sequentially closed (see [5], proof of theorem 3.2), i.e. (a) and (b) of proposition 1 are satisfied.

Let $\{x_n\}$ be a nondecreasing sequence. Then $\{f(x_n)\}$ is a nonincreasing sequence, bounded from below, hence convergent. For all $\lambda \in D$, we have for $n < m$

$$d_\lambda(x_n, x_m) \leq d_{\mu_1}(x_n, x_{n+1}) + \ldots + d_{\mu_{m-n}}(x_{m-1}, x_m)$$

$$\leq \frac{1}{\gamma h(\lambda)} \left[f(x_n) - f(x_m) \right]$$

for some $\mu_1, \mu_2, \ldots, \mu_{m-n} \in D$, $\lambda \prec \mu_1 \prec \mu_2 \prec \ldots \prec \mu_{m-n}$. Since $\{f(x_n)\}$ is Cauchy, so $\{x_n\}$ is which implies (c) of proposition 1. The conclusions of this proposition imply (1) and (2) of theorem 1. ■

The proof shows that Fang's result is indeed implied in a simple manner by the Brézis-Browder principle on ordered sets [1].

3 Equivalences in $\mathcal{F}$-type spaces

In this section we state generalizations to $\mathcal{F}$-type topological spaces of several theorems well-known in complete metric spaces. We prove these theorems in proving their equivalence to theorem 1.

Theorem 2 *Caristi's fixed point theorem in $\mathcal{F}$-type spaces (theorem 3.1 in [5])*
Let X be a sequentially complete $\mathcal{F}$-type topological space and $\{d_\lambda\}_{\lambda \in D}$ a family of quasi-metrics generating the topology. Let $f : X \to \mathbb{R}_+$ be a sequentially lower semicontinuous

function and $k : D \to (0, +\infty)$ a nonincreasing function. Suppose that the multi-valued mapping $T : X \to X$ satisfies the condition

$$d_\lambda (x, y) \leq k (\lambda) [f (x) - f (y)] \;\forall \lambda \in D \;\forall y \in T (x) . \tag{3}$$

Then T has a stationary point in X, i. e. there exists $x^* \in X$ such that $\{x^*\} = T (x^*)$.

We define the flower petal connected with $a, b \in X$, $\mu \in D$ and $\gamma > 0$ to be the set

$$P_\gamma^\mu (a, b) := \{x \in X : \; d_\mu (x, b) + \gamma h (\lambda) d_\lambda (x, a) \leq d_\mu (a, b) \;\; \forall \lambda \in D\} .$$

Theorem 3 *Penot's flower petal theorem in $\mathcal{F}$-type spaces*
Let X be a $\mathcal{F}$-type topological space and $\{d_\lambda\}_{\lambda \in D}$ a family of quasi-metrics generating the topology. Let $M \subset X$ be a sequentially complete subset of X, $a \in M$ and $b \in X \backslash M$. Then for each $\gamma > 0$, $\mu \in D$ there exists $x^* \in X$ such that

$$x^* \in M \cap P_\gamma^\mu (a, b) \quad \text{and} \quad \{x^*\} = M \cap P_\gamma^\mu (x^*, b) .$$

Theorem 4 *Takahashi's minimization principle in $\mathcal{F}$-type spaces*
Let X be a sequentially complete $\mathcal{F}$-type topological space and $(d_\lambda)_{\lambda \in D}$ a family of quasi-metrics generating the topology. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a sequentially lower semi-continuous function, $\not\equiv +\infty$, bounded from below and $h : D \to (0, +\infty)$ a nondecreasing function. Suppose that for each $x \in X$ with $\inf_{u \in X} f (u) < f (x)$ there exists $y \in X$ such that

$$y \neq x \quad \text{and} \quad f (y) + h (\lambda) d_\lambda (y, x) \leq f (x) \;\; \forall \lambda \in D. \tag{4}$$

Then there exists an $x^* \in X$ such that $f (x^*) = \inf_{u \in X} f (u)$.

In the following, let $F : X \times X \to \mathbb{R} \cup \{+\infty\}$ be a function such that
(1) For all $x, y, z \in X$,

$$F (x, z) \leq F (x, y) + F (y, z) , \tag{5}$$

(2) $F (x, \cdot) : X \to \mathbb{R} \cup \{+\infty\}$ is sequentially lower semicontinuous for all $x \in X$,
(3) there exists $x_0 \in X$ such that $\inf_{x \in X} F (x_0, x) > -\infty$.

We define the sequentially closed sets

$$S (x) := \{y \in X : \; F (x, y) + \gamma h (\lambda) d_\lambda (x, y) \leq 0 \;\forall \lambda \in D\} .$$

Theorem 5 *Equilibrium version of Ekeland's principle in $\mathcal{F}$-type spaces*
Let X be a sequentially complete $\mathcal{F}$-type topological space and $\{d_\lambda\}_{\lambda\in D}$ a family of quasi-metrics generating the topology. Let $F : X \times X \to \mathbb{R} \cup \{+\infty\}$ be a function satisfying the conditions (1)-(3) together with some $x_0 \in X$. Then, there exists $x^* \in S(x_0)$ such that

$$\forall x \neq x^* \; \exists \eta \in D : \; F(x^*, x) + \gamma h(\eta)\, d_\eta(x, x^*) > 0 \tag{6}$$

Theorem 6 *Oettli-Théra theorem in $\mathcal{F}$-type spaces*
Let X be a sequentially complete $\mathcal{F}$-type topological space and $\{d_\lambda\}_{\lambda\in D}$ a family of quasi-metrics generating the topology. Let $A \subset X$ have the property that for each $x \in S(x_0) \setminus A$ there exists $y \in S(x) \setminus \{x\}$. Then there exists an $x^* \in S(x_0) \cap A$.

Theorem 7 *Theorems 1 through 6 are equivalent.*

4 Proof of the equivalence theorem

(A) We shall show that theorem 1 is equivalent theorem 2 generalizing arguments of [1].

First, it follows from theorem 1 that there exists some $x^* \in X$ satisfying (2) of theorem 1 with $h(\lambda) = \frac{1}{k(\lambda)}$. We have $\{x^*\} = T(x^*)$; otherwise $\{x^*\} \neq T(x^*)$ would imply $f(x^*) < f(x) + h(\lambda_0)\, d_\lambda(x, x^*)$ for some $x \in T(x^*)$, $x \neq x^*$ which contradicts the assumption.

Conversely, suppose that the conclusion of theorem 1 does not hold. Then for each $x \in X$ there exists some $y \in X$, $y \neq x$ such that $f(y) + \frac{1}{k(\lambda)} d_\lambda(y, x) \leq f(x)$ for all $\lambda \in D$. Hence we could build a mapping $T : X \to X$ satisfying (3) and having no fixed point. ∎

(B) We show that theorem 1 is equivalent theorem 3.

It is easy to see that theorem 1 implies theorem 3 using the function $f(x) := d_\mu(x, b)$. The opposite direction is more complicated.

Without loss of generality we may assume $\inf_{x\in X} f(x) = 0$. Let $x_0 \in \mathrm{dom} f$. Choosing a number $k > 0$ such that

$$\left(1 + \frac{1}{\gamma h(\mu)}\right) f(x_0) \leq k \tag{7}$$

we define

$$M = \mathrm{epi} f, \; a = (x_0, f(x_0)), \; b = (x_0, f(x_0) - k) \in (X \times \mathbb{R}) \setminus M.$$

Note that $r - f(x_0) + k \geq 0$ for all $r \geq \inf f = 0$.

We introduce a family of quasi-metrics on $X \times \mathbb{R}$ by

$$\rho_\lambda(w_1, w_2) := \max\{d_\lambda(x_1, x_2), |r_1 - r_2|\} \ \lambda \in D$$

where $w_1 = (x_1, r_1) \in X \times \mathbb{R}$, $w_2 = (x_2, r_2) \in X \times \mathbb{R}$. Theorem 3 with X replaced by $X \times \mathbb{R}$ provides a point $w^* = (x^*, r^*) \in M = \mathrm{epi} f$ such that

$$\rho_\mu(w^*, b) + \gamma h(\lambda) \rho_\lambda(w^*, a) \leq \rho_\mu(a, b) \ \forall \lambda \in D \tag{8}$$

and for all $w = (x, r) \in M$, $w \neq w^*$ there exists $\eta \in D$ such that

$$\rho_\mu(w^*, b) < \rho_\mu(w, b) + \gamma h(\eta) \rho_\eta(w, w^*). \tag{9}$$

From (8) we may deduce

$$f(x^*) + \gamma h(\lambda) d_\lambda(x^*, x_0) \leq r^* + \gamma h(\lambda) d_\lambda(x^*, x_0) \leq f(x_0) \ \forall \lambda \in D \tag{10}$$

which is (1) of theorem 1.

Relation (9) implies the inequality

$$\begin{aligned} & r^* - f(x_0) + k < \\ & \qquad \max\{d_\mu(x, x_0), |r - f(x_0) + k|\} + \gamma h(\eta) \max\{d_\eta(x, x^*), |r - r^*|\}. \end{aligned}$$

We shall consider different cases.

Case 1.1: $d_\mu(x, x_0) \leq r - f(x_0) + k$ and $|r - r^*| \leq d_\eta(x, x^*)$ implies immediately (b) of theorem 1.

Case 1.2: If $d_\mu(x, x_0) \leq r - f(x_0) + k$ and $|r - r^*| > d_\eta(x, x^*)$ we obtain

$$r^* < r + \gamma h(\eta) |r - r^*|.$$

If we assume that $r < r^*$ this leads to

$$r^* < r + \gamma h(\eta)(r^* - r)$$

which implies

$$(1 - \gamma h(\eta)) r^* < (1 - \gamma h(\eta)) r,$$

a contradiction because without loss of generality we may assume that $0 < h(\eta) < 1$ and $0 < \gamma \leq 1$, see the lemma at the end of this section.

Case 2: If $d_\mu(x, x_0) > r - f(x_0) + k$ we may conclude from (7)

$$d_\mu(x, x_0) > r - f(x_0) + \left[1 + \frac{1}{\gamma h(\mu)}\right] f(x_0) = r + \frac{1}{\gamma h(\mu)} f(x_0).$$

Together with (10) this implies

$$r^* + \gamma h(\lambda) d_\lambda(x^*, x_0) < \gamma h(\mu)\left[d_\mu(x, x_0) - r\right] \quad \forall \lambda \in D.$$

Assume that (2) of theorem 1 is not valid, i. e.

$$r^* \geq r + \gamma h(\lambda) d_\lambda(x, x^*) \quad \forall \lambda \in D. \tag{11}$$

The last two inequalities imply

$$r + \gamma h(\lambda)\left[d_\lambda(x, x^*) + d_\lambda(x^*, x_0)\right] < \gamma h(\mu)\left[d_\mu(x, x_0) - r\right] \tag{12}$$

for all $\lambda \in D$. Axiom (F4) of definition 1 implies the existence of $\mu' \in D$, $\mu \prec \mu'$ such that

$$d_\mu(x, x_0) \leq d_{\mu'}(x, x^*) + d_{\mu'}(x^*, x_0).$$

Choosing $\lambda = \mu'$ in (12) we obtain

$$r + \gamma h(\mu) d_\mu(x, x_0) < \gamma h(\mu)\left[d_\mu(x, x_0) - r\right]$$

which implies

$$r < -\gamma h(\mu) r,$$

a contradiction to $r \geq 0$. Hence (11) can not be true, i. e. there exists some $\eta \in D$ such that

$$r^* < r + \gamma h(\eta) d_\eta(x, x^*).$$

in case 2. The last relation valid in any case gives (2) of theorem 1 if one observes that $f(x^*) \leq r^*$ if $(x^*, r^*) \in epi\, f$ and (x, r) can be replaced by $(x, f(x))$. ■

(C) The equivalence of theorem 1 and theorem 4 can be deduced generalizing arguments of [16] and [7], respectively into the context of $\mathcal{F}$-type topological spaces.

First, we derive theorem 4 from theorem 1. By means of theorem 1, there exists $x^* \in X$ such that

$$\forall x \neq x^* \; \exists \eta \in D: \; f(x^*) < f(x) + h(\eta) \, d_\eta (x, x^*) \, .$$

If x^* is not a minimizer of f condition (4) implies the existence of $\hat{x} \neq x^*$ such that

$$f(\hat{x}) + h(\eta) \, d_\eta (\hat{x}, x^*) \leq f(x^*) \, ,$$

hence

$$f(\hat{x}) + h(\eta) \, d_\eta (\hat{x}, x^*) \leq f(x^*) < f(\hat{x}) + h(\eta) \, d_\eta (\hat{x}, x^*) \, ,$$

a contradiction.

For the inverse implication let us consider, for some $x_0 \in X$, the set

$$X_0 := \{ x \in X : \; f(x) + \gamma h(\lambda) \, d_\lambda (x, x_0) \leq f(x_0) \;\; \forall \lambda \in D \} \, .$$

Then X_0 is nonempty and sequentially closed. Suppose

$$\forall x \in X_0 \; \exists \hat{x} \in X : \; \hat{x} \neq x, \; f(\hat{x}) + \gamma h(\lambda) \, d_\lambda (\hat{x}, x) \leq f(x) \;\; \forall \lambda \in D.$$

Then

$$\gamma h(\lambda) \, d_\lambda (\hat{x}, x) \leq f(x) - f(\hat{x}) \leq f(x_0) - f(\hat{x})$$

which implies $\hat{x} \in X_0$. Theorem 4 tells us that there exists $x^* \in X_0$ such that $f(x^*) = \inf_{u \in X} f(u)$. This is a contradiction because, by assumption there exists $y^* \in X$ such that $f(y^*) < f(x^*)$. ■

(D) Next, let us prove the equivalence of theorem 1 and 5.

Suppose theorem 1 is true. We apply it to the function

$$f(x) := F(x_0, x)$$

which is proper, sequentially lower semicontinuous and bounded from below. We get some $x^* \in S(x_0)$ such that

$$\forall x \neq x^* \; \exists \eta \in D: \; F(x_0, x^*) < F(x_0, x) + \gamma h(\eta) \, d_\eta (x, x^*) \, .$$

The last inequality and (5) imply the result.

The opposite direction can easily be proved by considering the function

$$F(x, y) := f(y) - f(x)$$

as it was done in the complete metric case in [11]. ■

(E) Finally, we prove the equivalence of theorem 5 and 6.

First, let theorem 5 be true, that is, there exists an $x^* \in S(x_0)$ such that (6) holds. This implies $x^* \in A$ because $x^* \in S(x_0)$ and $x^* \notin S(x_0) \setminus A$. Hence $x^* \in S(x_0) \cap A$.

Conversely, let theorem 6 be true. Define for $x \in X$

$$A(x) := \{y \in X : \ y \neq x, \ F(x, y) + \gamma h(\lambda) \, d_\lambda(x, y) \leq 0 \ \forall \lambda \in D\} = S(x) \setminus \{x\}$$

and choose $A = \{x \in X : \ A(x) = \emptyset\}$. If $x \in S(x_0) \setminus A$ then there exists $z \in A(x_0)$, i.e. the assumption of theorem 6 is satisfied. Hence there exists $x^* \in S(x_0) \cap A$ which implies $A(x^*) = \emptyset$, i.e. (6) is satisfied. ■

Finally, we give the lemma necessary for part (B) of the proof above.

Lemma 1 *If theorem 1 is true for a nondecreasing function $h : D \to (0, \infty)$ with $0 < h(\lambda) < 1$ for all $\lambda \in D$ then it is true for an arbitrary nondecreasing function $h : D \to (0, \infty)$.*

Proof: Let $h : D \to (0, \infty)$ be an arbitrary nondecreasing function. Define for $\lambda \in D$

$$\tilde{h}(\lambda) := \frac{h(\lambda)}{1 + h(\lambda)}.$$

Then $0 < \tilde{h}(\lambda) < 1$ for all $\lambda \in D$. Consider the set

$$X_0 := \{x \in X : \ f(x) + \gamma h(\lambda) \, d_\lambda(x, x_0) \leq f(x_0) \ \ \forall \lambda \in D\}$$

which is nonempty and sequentially closed. We define the function

$$\tilde{f}(x) := \begin{cases} f(x) & : \ \text{if } x \in X_0, \\ +\infty & : \ \text{otherwise} \end{cases}$$

and apply theorem 1 with $\tilde{f}$, $\tilde{h}$. We get a point $x^* \in X_0$ such that for all $x \neq x^*$ there is a $\eta \in D$ such that

$$\tilde{f}(x^*) < \tilde{f}(x) + \gamma \tilde{h}(\eta) \, d_\eta(x, x^*) \,. \tag{13}$$

$x^* \in X_0$ implies (a) of theorem 1 with h. Relation (13) implies

$$[1 + h(\eta)] f(x^*) < [1 + h(\eta)] f(x) + \gamma h(\eta) d_\eta(x, x^*).$$

Supposing that

$$f(x^*) \geq f(x) + \gamma h(\eta) d_\eta(x, x^*) \tag{14}$$

we obtain

$$[1 + h(\eta)] [f(x) + \gamma h(\eta) d_\eta(x, x^*)] < [1 + h(\eta)] f(x) + \gamma h(\eta) d_\eta(x, x^*).$$

This yields $d_\eta(x, x^*) < 0$, a contradiction. Hence (14) can not be true which proves the lemma. ∎

5 Comments

Remark 1 *To the knowlegde of the author there is no direct derivation of Ekeland's theorem from Penot's flower petal theorem appearing in the literature. So most of the proof of (B) in section 4 seems to be new. Compare [6], [13] for proofs using the drop theorem and the Phelps lemma as intermediate steps in the context of complete metric spaces.*

Remark 2 *Theorem 5 for complete metric spaces was proved by Oettli and Théra [11] and is in fact equivalent to the usual "primitive" (non-equilibrium) formulation of Ekeland's principle as well as the equilibrium versions of Takahashi's minimization principle. This seems to be a new observation even in the metric case.*

Remark 3 *It is an easy task to derive a version of theorem 1 for locally convex Hausdorff topological vector spaces substituting the quasi-metrics by a family of semi-norms which generates the topology, compare for example [9]. Then it turns out that this version is equivalent to the very original Phelps lemma (lemma 1 of [14]), the maximal point version of Phelp's lemma (lemma 1.2 in [15]) and the drop theorem, each in locally convex spaces. This has been proved by the author elsewhere [8].*

Remark 4 *In a recent paper [10] Li and Fang states the equivalence of Ekeland's principle and the Kirk-Caristi fixed point theorem in $\mathcal{F}$-type topological spaces. This paper is announced in Zentralblatt für Mathematik but it was not available for the author via the usual electronic order services. A letter to Prof. Fang remained without an answer until now.*

References

[1] H. Brezis, F. E. Browder, A General Principle on Ordered Sets in Nonlinear Functional Analysis, *Advances in Mathematics* 21, 355-364 (1976)

[2] A. Brønstedt, On a Lemma of Bishop and Phelps, *Pacific Journal of Mathematics* 55, No. 2, 335-341 (1974)

[3] J. Daněs, Equivalence of some Geometric and Related Results of Nonlinear Functional Analysis, *Commentationes Mathematicae Universitas Carolinae* 26, No. 3, 443-454 (1985)

[4] I. Ekeland, On the Variational Principle, *Journal of Mathematical Analysis and Applications* 47, 324-353 (1974)

[5] J.-X. Fang, The Variational Principle and Fixed Point Theoremsin Certain Topological Spaces, *Journal of Mathematical Analysis and Applications* 202, 398-412 (1996)

[6] P. G. Georgiev, The Strong Ekeland Variational Principle, the Strong Drop Theorem and Applications, *Journal of Mathematical Analysis and Applications* 131, 1-21 (1988)

[7] A. Hamel, Remarks to an Equivalent Formulation fo Ekeland's Variational Principle, *Optimization* 31, 233-238 (1994)

[8] A. Hamel, The Phelps Lemma, Daněs' Drop Theorem and Ekeland's Principle in Locally Convex Topological Vector Spaces, manuscript (2001)

[9] G. Isac, Ekeland's Principle and Nuclear Cones: A GeometricAspect, *Math. Comput. Modelling* 26 (11), 111-116 (1997)

[10] J. Li, J.-X. Fang, On the Variational Principle and Fixed Point Theorem in F-type topological spaces, *J. Nanjing Univ., Math. Biq.* 15, No. 2, 290-294 (1998)

[11] W. Oettli, M. Théra, Equivalents of Ekeland's Principle, *Bull. Austral. Math. Soc.* 48, 385-392 (1993)

[12] S. Park, On generalizations of the Ekeland-type variational principles, *Nonlinear Analysis. Theory, Methods & Applications* 39, 881-889 (2000)

[13] J.-P. Penot, The Drop Theorem, the Petal Theorem and Ekeland's Variational Principle, *Nonlinear Analysis. Theory, Methods & Applications* 10, No. 9, 813-822 (1986)

[14] R. R. Phelps, Support Cones and their Generalization, in V. L. Klee, "Convexity", Proceedings of Symposia in Pure Mathematics Vol. VII, AMS, Providence 1963, pp. 393-401

[15] R. R. Phelps, Support Cones in Banach Spaces and Their Generalization, *Advances in Mathematics* 13, 1-19 (1974)

[16] W. Takahashi, Existence Theorems Generalizing Fixed Point Theorems for Multivalued Mappings, in J.-B. Baillon/M. Thera, " Fixed Point Theory and Applications", Longman S&T 1991, 397-406