TIME–DEPENDENT PARABOLIC PROBLEMS ON NON–CYLINDRICAL DOMAINS WITH INHOMOGENEOUS BOUNDARY CONDITIONS

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To the memory of Ralph Phillips, inspiring mathematician and human being, collaborator and friend

Abstract. We study the relationship between the Dirichlet problem and the Cauchy problem with inhomogeneous boundary conditions for local operators. Our results are applied to non–autonomous parabolic problems on non–cylindrical domains.

1. Introduction

In this paper we continue the investigation of non–autonomous parabolic problems on non–cylindrical domains begun in [14]. In that work we have characterized the wellposedness of the problem with homogeneous Dirichlet boundary conditions in terms of a barrier condition using the concepts of semigroups and of local operators, cf. [10], [14]. Our main tool from semigroup theory was the Lumer–Phillips theorem, [12]. We now want to treat inhomogeneous boundary conditions in the same framework.

In Section 2 we first work in the following more general setting. Let V be a bounded open subset of a locally compact metric space Ω and A be a linear operator on $C(V)$. Following [11], we let A_0 be the part of A in $C_0(V)$ and define on $C(\overline{V})$ the operator A_1 with Ventcel boundary conditions, cf. (2.1). Assuming closedness and dissipativity of A, we prove that the Dirichlet problem

$$Au = 0 \quad \text{on} \quad V, \qquad u = \varphi \quad \text{on} \quad \partial V, \tag{1.1}$$

has a unique solution $u \in D(A) \cap C(\overline{V})$ for each $\varphi \in C(\partial V)$ provided that A_0 is invertible on $C_0(V)$ and A_1 is densely defined on $C(\overline{V})$, see Corollary 2.7. In fact, these properties characterize the solvability of (1.1) (under a mild extra assumption) by Theorem 2.13. Then A_1 generates a mean ergodic semigroup and the corresponding ergodic projection yields the solution operator of (1.1). We point out that there are degenerate problems such that A_0 is a generator but A_1 is not densely defined and (1.1) has no solution for certain φ, see Example 2.8. In the regular case this does not happen as shown in [4, Thm.4.1] (together with [4, Thm.4.10]) for uniformly elliptic operators A and in Theorem 3.5 for parabolic problems on non–cylindrical domains.

Combining these results with the theory developed in [14], we investigate in Section 3 the parabolic problem

$$Lu = F \quad \text{on} \quad \underline{V}_0, \qquad u = f \quad \text{on} \quad \Gamma \cup V(0), \tag{1.2}$$

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on a non–cylindrical domain $\underline{V}_0 \subseteq (0,T] \times \Omega$ with lateral boundary Γ and bottom $V(0)$, where L is a parabolic differential operator with continuous coefficients and F, f are continuous functions. This setting covers diffusion processes on open subsets of $(0,T]\times\mathbb{R}^N$ as well as on networks, see [14, §6]. In particular, we show in the non–degenerate case that (1.2) is solvable if and only if the corresponding homogeneous problem is well–posed, Theorem 3.5. Moreover, a problem of singular multiplicative perturbation is treated. We also study regularity properties of the variable space propagator solving the homogeneous problem in Theorem 3.6.

Different semigroup approaches to the Dirichlet problem and inhomogeneous initial boundary value problem are developed in e.g. [3], [4], [8]. These works, however, do not cover non–autonomous problems. We refer to [14] for further literature concerning local operator methods and parabolic problems on non–cylindrical domains, see also [1].

2. Dirichlet and Cauchy problems

Let Ω be a locally compact Hausdorff space having a countable base, Ω^* be its one–point compactification, and $V \neq \emptyset$ be a fixed open subset of Ω. We denote the closure and boundary of V in Ω^* by $\overline{V^*}$ and ∂V^*, respectively. Let $C(V)$ be endowed with the topology of uniform convergence on compact subsets of V. We also use the Banach space $C(\overline{V^*})$ equipped with the sup–norm $\|\cdot\|$ and its closed subspace $C_0(V)$ of functions vanishing on ∂V^*. We identify $C(\overline{V^*})$ with a subspace of $C(V)$ by restriction. Let $A : D(A) \subseteq C(V) \to C(V)$ be a linear operator. Our basic assumption reads as follows.

(H1) A is closed in $C(V)$. If $x \in V$ and $f \in D(A)$ with $|f(x)| = \|f\|$, then $\mathrm{Re}[(Af)(x)\,\overline{f(x)}] \le 0$.

This hypothesis will lead to closedness and disspativity of the operators

$$\begin{aligned} D(A_0) &= \{f \in D(A) \cap C_0(V) : Af \in C_0(V)\}, \quad A_0 f = Af, \\ D(A_1) &= \{f \in D(A) \cap C(\overline{V^*}) : Af \in C_0(V)\}, \quad A_1 f = Af. \end{aligned} \tag{2.1}$$

Throughout A_0 and A_1 are considered as operators on $C_0(V)$ and $C(\overline{V^*})$, respectively.

Lemma 2.1. *Let (H1) hold. Then A_0 and A_1 are closed and dissipative in $C_0(V)$ and $C(\overline{V^*})$, respectively.*

Proof. We only consider A_1, the operator A_0 can be treated in the same way. Let $D(A_1) \ni f_n \to f$ and $A_1 f_n \to g$ in $C(\overline{V^*})$. Since A is closed in $C(V)$, f belongs to $D(A)$ and $Af = g \in C_0(V)$ so that A_1 is closed in $C(\overline{V^*})$. Given $f \in D(A_1)$, there is $x \in \overline{V^*}$ such that $\|f\| = |f(x)|$. Set $\phi = \overline{f(x)}\,\delta_x \in C(\overline{V^*})'$. Then $\|\phi\|^2 = \|f\|^2 = \phi(f)$ and the quantity

$$\mathrm{Re}\langle A_1 f, \phi\rangle = \begin{cases} 0 & \text{if } x \in \partial V^*, \\ \mathrm{Re}[Af(x)\,\overline{f(x)}] & \text{if } x \in V, \end{cases}$$

is negative due to (H1). Thus A_1 is dissipative. □

The resolvent set of a linear operator B is denoted by $\rho(B)$ and its resolvent by $R(\lambda, B)$.

Lemma 2.2. *Let (H1) hold and $0 < \lambda \in \rho(A_1)$. Then $\lambda \in \rho(A_0)$ and $R(\lambda, A_1)f = R(\lambda, A_0)f$ for $f \in C_0(V)$.*

Proof. Given $g \in C_0(V)$, set $f = R(\lambda, A_1)g \in D(A_1)$. Then $\lambda f = A_1 f + g \in C_0(V)$ so that $f \in D(A_0)$ and $(\lambda - A_0)f = g$. Lemma 2.1 now yields the assertion. $\square$

The above results allow to relate the generation properties of A_0 and A_1. Example 2.8 below shows that the density of $D(A_1)$ cannot be omitted in assertions (a) and (c) of the next theorem, in general.

Theorem 2.3. *If (H1) holds, the following assertions are equivalent.*

(a) $(\lambda - A_0)D(A_0)$ *is dense in* $C_0(V)$ *for some* $\lambda > 0$ *and* $D(A_1)$ *is dense in* $C(\overline{V^*})$.
(b) A_1 *generates a* C_0*–semigroup* $T_1(\cdot)$ *on* $C(\overline{V^*})$.
(c) A_0 *generates a* C_0*–semigroup* $T_0(\cdot)$ *on* $C_0(V)$ *and* $D(A_1)$ *is dense in* $C(\overline{V^*})$.

If this is the case, $T_0(t)$ *and* $T_1(t)$ *are contractions,* $T_1(t)f = T_0(t)f$ *for* $f \in C_0(V)$ *and* $(T_1(t)f)(x) = f(x)$ *for* $x \in \partial V^*$, $f \in C(\overline{V^*})$, *and* $t \geq 0$.

Proof. (a)$\Rightarrow$(b): If A_1 were not a generator, then there would exist a finite Borel measure $\mu \neq 0$ on $\overline{V^*}$ with

$$\int_{\overline{V^*}} (\lambda f - A_1 f)\, d\mu = 0 \qquad \text{for all} \quad f \in D(A_1)$$

because of Lemma 2.1 and the Lumer–Phillips theorem. Using (a), we see that μ must vanish on V. Thus

$$0 = \int_{\partial V^*} (\lambda f - A_1 f)\, d\mu = \int_{\partial V^*} \lambda f \, d\mu$$

for all $f \in D(A_1)$. The second condition in (a) now implies $\mu = 0$ so that (b) is verified.

(b)$\Rightarrow$(c): In view of Lemma 2.1 and 2.2 and the Lumer–Phillips theorem, one only has to show that $D(A_0)$ is dense in $C_0(V)$. This fact follows from

$$\lambda R(\lambda, A_0)f = \lambda R(\lambda, A_1)f \to f$$

as $\lambda \to \infty$ for $f \in C_0(V)$.

The implication '(c)$\Rightarrow$(a)' is trivial. By Lemma 2.1, $T_0(t)$ and $T_1(t)$ are contractions, and by Lemma 2.2 both coincide on $C_0(V)$. The identity

$$T_1(t)f - f = A_1 \int_0^t T_1(s)f\, ds \ \in \ C_0(V), \qquad f \in C(\overline{V^*}),\ t \geq 0,$$

establishes the last assertion. $\square$

As an immediate application we can solve the Cauchy problem for A with inhomogeneous boundary conditions. In [3, Thm.6.5] the case of uniformly elliptic partial differential operators A in divergence form was treated allowing for time depending boundary values. Non–autonomous extensions of the next result are given in Theorem 3.3 and 3.5.

Corollary 2.4. *Let (H1) hold and* $f \in D(A_1)$. *If (a)–(c) of Theorem 2.3 hold, then there is a unique* $u \in C^1(\mathbb{R}_+, C(\overline{V^*}))$ *such that* $u(t) \in D(A_1)$ *for* $t \geq 0$ *and*

$$\begin{aligned} \tfrac{d}{dt} u(t,\cdot) &= Au(t,\cdot), && t \geq 0, \quad \text{on } V, \\ u(t,x) &= f(x), && t \geq 0,\ x \in \partial V^*, \\ u(0,x) &= f(x), && x \in V. \end{aligned} \tag{2.2}$$

Proof. The function $u(t,x) = (T_1(t)f)(x)$ on $\mathbb{R}_+ \times \overline{V^*}$ solves (2.2) due to Theorem 2.3. The solution is unique since A_0 is a generator. $\square$

We strengthen (H1) in order to obtain positive solutions:

(H2) A is closed in $C(V)$ and real (i.e., if $f \in D(A)$, then $\bar{f} \in D(A)$ and $A\bar{f} = \overline{Af}$). If $x \in V$ and $f \in D(A)$ is real–valued with $0 < f(x) = \sup_V f$, then $(Af)(x) \leq 0$.

As in [14, Cor.2.10] one proves that (H2) implies (H1). The converse can be shown if A is given by a real local operator that satisfies (H1) on open subsets of V, see [14, Cor.2.10] and the definitions given in the next section. The following result can easily be deduced from R.S. Phillips' characterization of generators of positive contraction semigroups [17, Thm.2.1] (compare the proof of Lemma 2.1).

Proposition 2.5. *If (H2) holds and A_1 is a generator, then $T_1(\cdot)$ is positive. Thus the solution of* (2.2) *is positive if $f \geq 0$.*

The results established so far are essentially stated in [11] without proof. In this work as well as in [3] further results were derived assuming that the Dirichlet problem (1.1) is solvable. Here we are looking for conditions on A_0 and A_1 characterizing the solvability of (1.1). Our main tool is the mean ergodic theorem which we apply to $T_1(\cdot)$.

Theorem 2.6. *Assume that (H1) holds, A_0 is invertible on $C_0(V)$, and A_1 is densely defined on $C(\overline{V^*})$. Then $T_1(\cdot)$ is mean ergodic with projection P onto $\ker A_1 = \operatorname{fix} T_1(\cdot)$. Moreover, $\|P\| \leq 1$ and $Pf(x) = f(x)$ for $x \in \partial V^*$ and $f \in C(\overline{V^*})$. If in addition (H2) holds, then P is positive and $\|T_1(t) - P\| \leq Me^{-\varepsilon t}$ for $t \geq 0$ and constants $M, \varepsilon > 0$.*

Proof. By Theorem 2.3 the operators A_k generate the contraction semigroups $T_k(\cdot)$, $k = 0,1$. For $\lambda > 0$ and $f \in D(A_1)$, we obtain

$$\begin{aligned} \lambda R(\lambda, A_1)f &= R(\lambda, A_1)A_1 f + f = R(\lambda, A_0)A_1 f + f \\ &\to -A_0^{-1}A_1 f + f \end{aligned} \tag{2.3}$$

as $\lambda \to 0$ due to Lemma 2.2. The mean ergodicity of $T_1(\cdot)$ follows from (2.3) and [5, Thm.5.1]. The other assertions except for the convergence of $T_1(\cdot)$ are straightforward consequences of Theorem 2.3 and Proposition 2.5. Let (H2) hold. Note that $T_0(\cdot)$ is positive and thus uniformly exponentially stable by [15, Thm.B-IV.1.4]. We now deduce

$$\|T_1(t)f - Pf\| = \|T_0(t)(f - Pf)\| \leq 2Ne^{-\varepsilon t}\,\|f\|$$

from $T_1(t)Pf = Pf$, for $t \geq 0$, $f \in C(\overline{V^*})$, and some constants $N, \varepsilon > 0$. $\square$

Observe that $Pf = f - A_0^{-1}A_1 f$ for $f \in D(A_1)$ due to (2.3).

Corollary 2.7. *Assume that (H1) holds, A_0 is invertible, and A_1 is densely defined. Then for each $\varphi \in C(\partial V^*)$ there exists a unique solution $u \in C(\overline{V^*}) \cap D(A)$ of*

$$Au = 0 \quad \text{on } V, \qquad u|\partial V^* = \varphi. \tag{2.4}$$

If (H2) holds and $\varphi \geq 0$, then $u \geq 0$.

Proof. The function $u = Pf$ solves (2.4), where P is given by Theorem 2.6 and f is any extension of φ to $C(\overline{V^*})$. If u satisfies (2.4) for $\varphi = 0$, then $u \in \ker A_0$ so that $u = 0$. $\square$

We refer to [7] for a detailed account on the Dirichlet problem for elliptic operators. The following two examples show that Theorem 2.6 and Corollary 2.7 are not valid if one only assumes that A_0 is an invertible generator or that A_1 is a generator and A_0 is injective.

Example 2.8. For $V = (0,1)^2$ and $\Omega = \mathbb{R}^2$ define

$$Af(x,y) = \partial_{xx} f(x,y) + y^2\, \partial_{yy} f(x,y) - f(x,y)$$

with $D(A) = \{f \in C(V) \cap W^2_{p,loc}(V) : Af \in C(V)\}$ for one (and hence all) $p > 2$. It can be checked that (H2) holds, cf. [16]. Clearly, A_0 is densely defined. Further, the function

$$h : V \to \mathbb{R}_+; \quad h(x,y) = x(1-x)(1-y)\sqrt{y}$$

is a Cauchy barrier in the sense of (CB) below. Therefore A_0 generates a contraction semigroup on $C_0(V)$ by [16, Thm.1], see also [10, Thm.5.4] or [14, Thm.3.27]. By the same reasons $A_0 + 1$ is also a dissipative generator so that A_0 is invertible. However, following the arguments of [7, §6.6, p.116] one shows that the Dirichlet problem corresponding to A cannot be solved for $\varphi \in C(\partial V)$ vanishing for $x = 0, 1$ but not for $y = 0$. Therefore the operator A_1 is not densely defined on $C(\overline{V})$ due to the above corollary.

Example 2.9. Let $\Omega = (0,\infty)$, $V = (1,\infty)$, and $Af = f''$ with $D(A) = C^2(V)$. Then (H2) holds and A_0 is an injective generator on $C_0(V)$. Further, $D(A_1)$ is dense in $C([1,\infty])$. Hence, A_1 is a generator by Theorem 2.3. However, (2.4) has no solution for φ with $\varphi(0) \neq \varphi(\infty)$, and so $T_1(\cdot)$ is not mean ergodic in view of the proof of Corollary 2.7.

Concluding this section we want to characterize the Dirichlet regularity of V in terms of A_0 and A_1. Here we use the following notion, cf. [10].

Definition 2.10. *The set V is called* weakly (A–)Dirichlet regular *if* (2.4) *has a solution for each $\varphi \in C(\partial V^*)$ and* (A–)Dirichlet regular *if, in addition, the solution is unique.*

Lemma 2.11. *If V is weakly Dirichlet regular and $D(A_0)$ is dense in $C_0(V)$, then $D(A_1)$ is dense in $C(\overline{V^*})$.*

Proof. For $f \in C(\overline{V^*})$ there is $v \in C(\overline{V^*}) \cap D(A)$ such that $Av = 0$ on V and $v = f$ on ∂V^*. By assumption, there exist $u_n \in D(A_0)$ converging to $f - v$ in $C_0(V)$. Hence, $u_n + v \in D(A_1)$ tends to f in $C(\overline{V^*})$. □

Lemma 2.12. *If V is weakly Dirichlet regular and $C_0(V) \subseteq A[D(A) \cap C(\overline{V^*})]$, then A_0 is surjective.*

Proof. For $g \in C_0(V)$ there is $u \in D(A) \cap C(\overline{V^*})$ with $Au = g$. By assumption, there exists $v \in D(A) \cap C(\overline{V^*})$ such that $Av = 0$ and $u = v$ on ∂V^*. Then $f = u - v$ belongs to $D(A_0)$ and $Af = Au = g$. □

The range condition used above holds if A is a local operator on Ω such that A is invertible on $C_0(\Omega)$, cf. [14, Prop.3.4] and the next section.

Theorem 2.13. *Assume that (H1) holds, $D(A_0)$ is dense in $C_0(V)$, and $C_0(V) \subseteq A[D(A) \cap C(\overline{V^*})]$. Then the following assertions are equivalent.*

(a) *V is A–Dirichlet regular.*
(b) *A_0 is invertible on $C_0(V)$ and $D(A_1)$ is dense in $C(\overline{V^*})$.*

(c) A_1 *generates a mean ergodic semigroup on* $C(\overline{V^*})$.

Proof. Assertion (b) follows from (a) in view of Lemma 2.1, 2.11, 2.12 and the uniqueness of (2.4). The implication '(b)$\Rightarrow$(c)' was shown in Theorem 2.6. If (c) holds, then V is weakly Dirichlet regular due to the proof of Corollary 2.7. The operator A_0 is thus surjective by Lemma 2.12. This implies the injectivity of A_0 since $T_0(\cdot)$ is also mean ergodic, cf. [5, Thm.5.1]. Therefore solutions of (2.4) are unique. □

3. Parabolic problems on non–cylindrical domains

In this section we make use of the theory of local operators, see [10], [14], and the references therein. Unexplained notation and concepts can be found in [14], but for the convenience of the reader we recall several basic definitions. A local operator A is a collection of linear operators $A^V : D(A,V) \subseteq C(V) \to C(V)$ which respect restriction to subsets, where V belongs to the set $\mathcal{O}(\Omega)$ of non–empty open subsets of Ω. Usually we omit the superscript V. We introduce further notions, where $V \in \mathcal{O}(\Omega)$.

(a) A is *locally closed u.c.* if, for $f, g \in C(V)$, the existence of $\mathcal{O}(\Omega) \ni V_n \uparrow V$ and $f_n \in D(A, V_n)$ such that $f_n \to f$ and $Af_n \to g$ uniformly on compact subsets of V implies that $f \in D(A,V)$ and $A^V f = g$.
(b) A is *locally dissipative* if, for all compact $K \subset V$ and $f \in D(A,V)$ with $\sup_{V\setminus K} |f| < \sup_V |f|$, there is $x_0 \in V$ such that $|f(x_0)| = \sup_V |f|$ and $\mathrm{Re}[(Af)(x_0)\,\overline{f(x_0)}] \le 0$.
(c) A is *real* if $f \in D(A,V)$ implies that $\overline{f} \in D(A,V)$ and $A\overline{f} = \overline{Af}$ on V.

If we assume that (a)–(c) hold and that

(S) for V contained in a base of $\mathcal{O}(\Omega)$ the part A_V of A^V in $C_0(V)$ is densely defined,

then Lemma 2.9 and 2.10 of [14] show that $A = A^V$ satisfies (H2) (and (H1) if A is not supposed to be real). In view of Section 2 we define the operator

$$\tilde{A}_V f = \begin{cases} Af & \text{on } V, \\ 0 & \text{on } \partial V^*, \end{cases} \qquad D(\tilde{A}_V) = \{f \in C(\overline{V^*}) \cap D(A,V) : Af \in C_0(V_0)\},$$

on $C(\overline{V^*})$. We set $A_0 = A_V$ and $A_1 = \tilde{A}_V$ if there is no danger of confusion.

The results in the previous section allow to treat a problem of degenerate multiplicative perturbation. For a local operator A and $p \in C(\Omega)$ we define the local operator $(pA)f = pAf$ with domain $D(pA, V) = D(A,V)$, cf. [10, §6].

Proposition 3.1. *Assume that A is a locally dissipative, locally closed u.c., local operator satisfying (S) on $\mathcal{O}(\Omega)$. Let $V \in \mathcal{O}(\Omega)$ and $p \in C(\Omega)$ be bounded and positive such that $W = V \setminus \{x \in V : p(x) = 0\}$ is dense in V. Further suppose that $\tilde{A}_V$ is densely defined on $C(\overline{V^*})$ and $(pA)_W$ is a generator on $C_0(W)$. Then $D(\widetilde{(pA)}_W)$ is dense in $C(\overline{W^*})$ and thus there exist unique solutions of* (2.2) *for pA on W.*

Proof. Note that $(pA)^W$ satisfies (H1) on W. For $f \in C(\overline{W^*}) = C(\overline{V^*})$ take $f_n \in D(\tilde{A}_V)$ converging to f in sup–norm. Since $pAf_n \in C_0(W)$, the functions $f_n|W$ belong to $D(\widetilde{(pA)}_W)$. Corollary 2.4 now implies the second assertion. □

The above result can be applied to the following situation. (One can deal with more general examples in view of [10, Thm.6.4], see also [13].)

Example 3.2. Let $V = B_1$ be the open unit ball in $\mathbb{R}^N$, $W = B_1 \setminus \{0\}$, $A = \Delta$ with maximal domain, and $p(x) = |x|^2$. Then $(pA)_W$ is a generator on $C_0(W)$ by e.g. the remarks after [10, Thm.6.10]. $D(\tilde{A}_V)$ is dense in $C(\overline{B_1})$ because of standard existence results for the Dirichlet problem for the Laplacian and Lemma 2.11. Thus we can solve (2.2) for $r^2\Delta$ on $B_1 \setminus \{0\}$.

Turning to the parabolic case, we let $\underline{\Omega} = [S,T] \times \Omega$ and, for $\underline{V} \in \mathcal{O}(\underline{\Omega})$ and $t \in [S,T]$,

$$V(t) = \{x \in \Omega : (t,x) \in \underline{V}\}, \ \ \underline{V}(t) = \{t\} \times V(t), \ \ \underline{V}_t = \{(s,x) \in \underline{V} : s > t\},$$
$$\underline{V}_0 = \underline{V}_S, \ \ \underline{V}'_t = \{(s,x) \in \underline{V} : s \geq t\}, \ \ I_{\underline{V}} = \{t \in [S,T] : V(t) \neq \emptyset\}.$$

Further, $\overline{\underline{V}^*}$ and $\partial\underline{V}^*$ designate the closure and boundary of $\underline{V}$ in $\underline{\Omega}^* = [S,T] \times \Omega^*$, respectively. Observe that $\partial\underline{V}_0^* = \partial\underline{V}^* \cup \underline{V}(S)$, [14, Prop.4.1]. We write $\underline{x} = (t,x)$ for a generic point of $\underline{\Omega}$ or $\underline{\Omega}^*$ and consider $C(\overline{\underline{V}^*})$ and $C(\underline{V}_0)$ as subspaces of $C(\underline{V})$.

A local operator L defined on $\mathcal{O}(\underline{\Omega})$ is called *parabolic* if, for $F \in D(L,\underline{V})$, $\underline{V} \in \mathcal{O}(\underline{\Omega})$, and $\varphi \in C^1(I_{\underline{V}})$, we have $\varphi F \in D(L,\underline{V})$ and $L(\varphi F) = \varphi\, LF - \varphi' F$.

Let $\underline{V} \in \mathcal{O}(\underline{\Omega})$ satisfy $V(t) \neq \emptyset$ for $t \in [S,T]$. Take $F \in C(\overline{\underline{V}^*})$, $f \in C(\overline{V(S)^*})$, and $g \in C(\partial\underline{V}^*)$ with $g(S,x) = f(x)$ for $x \in \partial V(S)^*$. We are looking for solutions $u \in C(\overline{\underline{V}^*}) \cap D(L,\underline{V}_0)$ of the problem

$$\begin{aligned} Lu &= F \qquad \text{on } \underline{V}_0, \\ u &= g \qquad \text{on } \partial\underline{V}^*, \\ u(S) &= f. \end{aligned} \tag{3.1}$$

The above setting covers the parabolic differential operator

$$L(\underline{x},D) = \sum_{k,l=1}^{N} a_{kl}(t,x)\partial_{kl} + \sum_{k=1}^{N} b_k(t,x)\partial_k + c(t,x) - \partial_t \tag{3.2}$$

on an open subset Ω of $\mathbb{R}^N$, where $\partial_t = \frac{\partial}{\partial t}$ and $\partial_k = \frac{\partial}{\partial x_k}$. The coefficients a_{kl}, b_k, and $c \leq 0$ are supposed to be real-valued and continuous and to satisfy

$$\sum_{k,l=1}^{N} a_{kl}(t,x)\, y_k\, y_l > 0 \tag{3.3}$$

for all $y \in \mathbb{R}^N$ and $(t,x) \in \underline{\Omega}$. The coefficients may be singular or degenerate at the boundary. (See e.g. [6] for the case of uniformly elliptic and Hölder continuous coefficients on regular domains $\underline{V}$.) Set $\underline{\Omega}_{00} = (S,T) \times \Omega$ and $\underline{W}_{00} = \underline{W} \cap \underline{\Omega}_{00}$ for $\underline{W} \in \mathcal{O}(\underline{\Omega})$. As in [14, §6.1], we define

$$\begin{aligned} D(L,\underline{W}_0) &= \{F \in C(\underline{W}_0) \cap W^{1,2}_{p,loc}(\underline{W}_0) : L(\underline{x},D)u = G \text{ a.e. on } \underline{W}_{00} \text{ for } G \in C(\underline{W}_0)\}, \\ LF &= G \qquad \text{on } \underline{W}_0. \end{aligned} \tag{3.4}$$

Here, $\underline{W} \in \mathcal{O}(\underline{\Omega})$, $p > N+2$ is fixed, and $W^{1,2}_{p,loc}(\underline{W}_0)$ ($W^{1,2}_{p,loc}(\underline{W})$) denotes the space of functions F such that $F|\underline{U}_{00}$ belongs to the Sobolev space

$$W^{1,2}_p(\underline{U}_{00}) = \{F \in L^p(\underline{U}_{00}) : \partial_k F, \ \partial_{kl} F, \ \partial_t F \in L^p(\underline{U}_{00}), \ k,l = 1,\dots,N\}$$

for all relatively compact open subsets $\underline{U}$ of $\underline{W}_0$ (of $\underline{W}$). In [14, §6.1] it is shown that $D(L, \underline{W}_0)$ does not depend on the choice of $p > N+2$ and that L satisfies the assumptions listed above. Analogous operators on networks were studied in [14, §6.2].

We come back to the general situation of a real, locally dissipative, locally closed u.c., parabolic, local operator L satisfying (S). Then (3.1) with $g = 0$, $F = 0$ on $\partial \underline{V}^*$, and S replaced by $s \in (S, T)$ can be solved if the part $L_{\underline{V}_0}$ of L in $C_0(\underline{V}_0)$ is a generator, [14, Thm.4.14]. Under some mild additional assumptions $L_{\underline{V}_0}$ is a generator if and only $D(L_{\underline{V}_0})$ is dense in $C_0(\underline{V}_0)$ and $\underline{V}_0$ has a *Cauchy barrier* in the sense that

(CB) there exists a compact subset $\underline{K}$ of $\underline{V}_0$ and a function $h \in D(L, \underline{V}_0 \setminus \underline{K})$ such that $h > 0$ and $(L - \lambda)h \leq 0$ on $\underline{V}_0 \setminus \underline{K}$ for some $\lambda \geq 0$, and for all $\varepsilon > 0$ there is a compact set $\underline{K}_\varepsilon$ with $\underline{K} \subseteq \underline{K}_\varepsilon \subseteq \underline{V}_0$ and $0 \leq h \leq \varepsilon$ on $\underline{V}_0 \setminus \underline{K}_\varepsilon$,

see [14, Thm.3.25, 3.27] and [10, Thm.5.4]. Cauchy barriers are discussed in [14, §3.1] where further literature can be found. Examples for Cauchy barriers are given in [14, §6.1]. The next result is a straightforward consequence of the theory developed in the previous section.

Theorem 3.3. *Let L be a locally dissipative, locally closed u.c., parabolic, local operator satisfying (S) and defined on $\mathcal{O}(\underline{\Omega})$, let $\underline{V} \in \mathcal{O}(\underline{\Omega})$ satisfy $V(t) \neq \emptyset$ for $t \in [S,T]$, and let $F \in C(\overline{\underline{V}^*})$, $f \in C(\overline{V(S)^*})$, and $g \in C(\partial \underline{V}^*)$ with $g(S,x) = f(x)$ for $x \in \partial V(S)^*$. Assume that $L_{\underline{V}_0}$ is a generator on $C_0(\underline{V}_0)$ and that $D(\tilde{L}_{\underline{V}_0})$ is dense in $C(\overline{\underline{V}^*})$. Then there is a unique solution $u \in C(\overline{\underline{V}^*}) \cap D(L, \underline{V}_0)$ of* (3.1). *If L is real and $-F, g, f \geq 0$, then $u \geq 0$.*

Proof. Uniqueness and positivity of solutions follow from the parabolic maximum principle [14, Thm.2.29, 2.30]. By Theorem 2.3, L_1 generates a contraction semigroup on $C(\overline{\underline{V}^*})$. Since $e^{tL_0} = 0$ for $t \geq T - S$, see [14, Thm.4.9], Theorem 2.6 provides us with a projection $P = P_{\underline{V}}$ mapping $C(\overline{\underline{V}^*})$ onto $\ker L_1$ such that $Pw(\underline{x}) = w(\underline{x})$ for $\underline{x} \in \partial \underline{V}_0^*$ and $w \in C(\overline{\underline{V}^*})$. Set $\hat{F}(t,x) = e^{-t}\, F(t,x)$, $\hat{v} = (L_1 - 1)^{-1} \hat{F}$, and $v(t,x) = e^t\, \hat{v}(t,x)$ on $\overline{\underline{V}^*}$. Then $v \in D(L, \underline{V}_0) \cap C(\overline{\underline{V}^*})$ and

$$Lv = -e^t\, \hat{v} + e^t\, L\hat{v} = -e^t\, \hat{v} + e^t\, (\hat{v} + \hat{F}) = F$$

on $\underline{V}_0$ since L is parabolic. Take $G \in C(\overline{\underline{V}^*})$ with $G(S) = f - v(S)$ on $\underline{V}(S)$ and $G = g - v$ on $\partial \underline{V}^*$. The function $u = v + PG$ clearly solves (3.1). ☐

We now give a sufficient condition for the density of $D(L_1)$ formulated for a general local operator.

Proposition 3.4. *Let $\mu \geq 0$ be a finite regular Borel measure on Ω and $V \in \mathcal{O}(\Omega)$ be relatively compact with $\mu(\partial V) = 0$. Let A be a local operator on Ω such that A_Ω is densely defined and invertible on $C_0(\Omega)$ and $\|A_\Omega^{-1} f\|_{C_0(\Omega)} \leq c \|f\|_{L^p(\Omega,\mu)}$ for $f \in C_0(\Omega)$ and some $p \in [1, \infty)$. Then $\tilde{A}_V$ is densely defined on $C(\overline{V})$.*

Proof. Note that there are relatively compact sets $W_n \in \mathcal{O}(\Omega)$ such that $\partial V \subset W_{n+1} \subset \overline{W_{n+1}} \subset W_n$ and $\mu(W_n) \leq 1/n$ for $n \in \mathbb{N}$. We extend a given $f \in C(\overline{V})$ to a function $\tilde{f} \in C_0(\Omega)$. For $\varepsilon > 0$ there is $\tilde{u} \in D(A_\Omega)$ such that $\|\tilde{f} - \tilde{u}\|_\infty \leq \varepsilon$. Let $\tilde{g} = A\tilde{u}$ and take continuous functions $\alpha_n : \Omega \to [0,1]$ vanishing on W_{n+1} and being equal to 1 off W_n. Set

$\tilde{g}_n = \alpha_n \tilde{g}$ and $v_n = (A_\Omega^{-1}\tilde{g}_n)|\overline{V}$. Then $v_n \in D(\tilde{A}_V)$, $\tilde{g}_n \to \tilde{g}$ in p–norm, and

$$\|f - v_n\|_{C(\overline{V})} \le \|\tilde{f} - \tilde{u}\|_{C_0(\Omega)} + \|A_\Omega^{-1}(\tilde{g} - \tilde{g}_n)\|_{C_0(\Omega)} \le \varepsilon + c\,\|\tilde{g} - \tilde{g}_n\|_{L^p(\Omega,\mu)} \le 2\varepsilon$$

for large n. $\square$

In the remainder of this section we treat the case of the operator L given by (3.2) and (3.4). We first extend [4, Thm.4.1] (combined with [4, Thm.4.10]) to the parabolic case.

Theorem 3.5. *Let L be given by (3.2) and (3.4), where Ω is a bounded open subset of $\mathbb{R}^N$ with boundary of class C^2 and the coefficients belong to $C(\overline{\underline{\Omega}}, \mathbb{R})$, $c \le 0$, and (3.3) holds on $\overline{\underline{\Omega}} = [-1,T] \times \overline{\Omega}$. Take a relatively compact subset $\underline{V} \in \mathcal{O}([0,T] \times \Omega)$ such that $V(t) \neq \emptyset$ for $0 \le t \le T$ and $\partial \underline{V}$ has Lebesgue measure 0 in $\underline{\Omega}$. Then the following assertions are equivalent.*

(a) *$\tilde{L}_{\underline{V}_0}$ is a generator in $C(\overline{\underline{V}})$.*
(b) *$L_{\underline{V}_0}$ is a generator in $C_0(\underline{V}_0)$.*
(c) *$\underline{V}_0$ has a Cauchy barrier for L.*

In this case the conclusions of Theorem 3.3 are true.

Proof. Let $p > N+1$. Using [9, Thm.IV.9.1] we see that $L_{\underline{\Omega}_0}$ is invertible (and densely defined) on $C_0(\underline{\Omega}_0)$ and its inverse maps $(C_0(\underline{\Omega}_0), \|\cdot\|_p)$ continuously into $W^{2,1}_{p,0}(\underline{\Omega}_0) \hookrightarrow C_0(\underline{\Omega}_0)$. So the theorem follows from Proposition 3.4, Theorem 2.3, [14, Thm.4.19] (see the remarks before [14, Thm.6.1]), and Theorem 3.3. $\square$

We briefly indicate another method to verify the density of $D(L_1)$ using L–harmonic functions. Consider $A = \Delta$, $L = \Delta - \frac{d}{dt}$, and $\underline{V} = [0,T] \times (0,1)^2$. Assume that L_1 is not densely defined in $C(\overline{\underline{V}})$. Then there is a regular Borel measure $\mu \neq 0$ such that

$$\int_{\overline{\underline{V}}} u\,d\mu = 0 \qquad \text{for all } \ u \in D(L_1).$$

Since L_0 is densely defined on $C_0(\underline{V}_0)$, the support of μ must be contained in $\partial\underline{V}_0$. The function $u(t,x,y) = (T_0(t)f)(x,y)$ belongs to $C_0(\underline{V})$ and to the kernel of L, where $f \in D(A_0)$ and $T_0(\cdot)$ is the semigroup generated by the Dirichlet Laplacian A_0 on $C_0(V)$. This implies that supp$\mu \subset [0,T] \times \partial V$. Further let $\alpha \in C_c^1(0,1)$, $k \in \mathbb{N}_0$, and $g(t,x,y)$ be equal to $e^{-kt}\alpha(x)$ for $y = 1$ and $(t,x) \in [0,T] \times [0,1]$ and be equal to 0 on the other faces of the lateral boundary. Extend α to a periodic odd function on $\mathbb{R}$ with Fourier coefficients a_n. Define

$$u_m(t,x,y) = e^{-kt} \sum_{n=1}^{m} \frac{a_n}{\sin\sqrt{k - 4\pi^2 n^2}}\, \sin(2\pi n x)\, \sin(y\sqrt{k - 4\pi^2 n^2})$$

for $t \in [0,T]$, $x, y \in \mathbb{R}$, and $m \in \mathbb{N}$. Then $Lu_m = 0$, $u_m = 0$ on the faces $x = 0,1$ and $y = 0$, and $u_m(t,x,1) \to e^{-kt}\beta(x)$ uniformly for t and x. Thus $\int g\,d\mu = 0$. By an approximation argument we now deduce that μ vanishes on $[0,T] \times (0,1) \times \{1\}$. The other faces can be treated in the same way so that μ is supported in the lateral edges. In a last step one sees in a similar manner that this is impossible. Therefore $\mu = 0$ and $D(L_1)$ is dense.

We finally want to study the regularity of the solutions to the homogeneous problem

$$\begin{aligned} Lu &= 0 \qquad \text{on } \underline{V}_s, \\ u &= 0 \qquad \text{on } \partial\underline{V}_s^* \setminus \underline{V}(s), \\ u(s) &= f. \end{aligned} \tag{3.5}$$

for $s \in (S,T)$, $f \in C_0(V(s))$, and the operator L given by (3.2) and (3.4) supposing that $L_0 = L_{\underline{V}_0}$ is a generator on $C_0(\underline{V}_0)$. In [14, Thm.4.13, 6.1] we proved that the solution $u \in C_0(\underline{V}'_s) \cap \bigcap_{p>1} W^{1,2}_{p,loc}(\underline{V}_s)$ of (3.5) is given by $u(t,\cdot) = U(t,s)f$, $t \in [s,T]$, for a variable space propagator $U(t,s) : C_0(V(s)) \to C_0(V(t))$, $S < s \le t \le T$. This means that

$$U(s,s) = I_{C_0(V(s))}, \qquad U(t,r)U(r,s) = U(t,s), \quad \text{and} \tag{3.6}$$

$$(t,s) \mapsto (U(t,s)F(s))^{\#} \quad \text{is continuous in } C_0(\Omega) \tag{3.7}$$

for $S < s \le r \le t \le T$ and $F \in C_0(\underline{V}_0)$, where $g^{\#}$ denotes the extension of $g \in C_0(V)$ by 0 to $C_0(\Omega)$, cf. [14, Def.4.8]. Moreover, $U(t,s)$ is a positive contraction.

In the following we consider initial functions f belonging locally to Slobodeckij spaces $W^\alpha_p(U)$, where U is a bounded open subset of $\mathbb{R}^N$ with a smooth boundary, $\alpha \in [0,2]$, and $1 < p < \infty$. These spaces can be defined, e.g., by Hölder estimates of L^p–type, see [18, §4.4]. Real interpolation of $W^\alpha_p(U)$ yields

$$(W^\alpha_p(U), W^\beta_p(U))_{\theta,p} = W^\gamma_p(U) \tag{3.8}$$

if $\gamma = \alpha(1-\theta) + \beta\theta \notin \mathbb{N}$ and $\theta \in (0,1)$ due to Theorem 4.3.1.1 and formula (2.4.2.16) of [18]. Taking $\alpha = 0$ and $\beta = 2$ in (3.8), [2, Thm.III.4.10.2] leads to

$$W^{1,2}_p([a,b] \times U) = L^p([a,b], W^2_p(U)) \cap W^1_p([a,b], L^p(U)) \hookrightarrow C([a,b], W^{2-\frac{2}{p}}_p(U)) \tag{3.9}$$

if $p \neq 2$. Finally, from [18, Thm.4.6.1, 4.6.2] we deduce

$$W^{2-\frac{2}{q}}_q(U) \hookrightarrow W^{2-\frac{2}{r}}_p(U) \hookrightarrow W^{2-\frac{2}{p}}_p(U) \tag{3.10}$$

if $1 < p < r < q < \infty$. The following result yields in the interior of $\underline{V}'_s$ the regularity at $t = s$ which we can expect in view of the well known results in [9, §IV.9] for regular problems.

Theorem 3.6. *Let L be given by (3.2) and (3.4), where Ω is an open subset of $\mathbb{R}^N$ and the coefficients belong to $C(\underline{\Omega}, \mathbb{R})$, $c \le 0$, and (3.3) holds. Moreover, let $\underline{V} \in \mathcal{O}(\underline{\Omega})$ satisfy $V(t) \neq \emptyset$ for $S \le t \le T$. Assume that $L_{\underline{V}_0}$ is a generator on $C_0(\underline{V}_0)$. Given $s \in (S,T)$ and $f \in C_0(V(s)) \cap \bigcap_{p>1} W^{2-\frac{2}{p}}_{p,loc}(V(s))$, we set $u(t,x) = (U(t,s)f)(x)$ for $(t,x) \in \underline{V}'_s$. Then $u \in C_0(\underline{V}'_s) \cap \bigcap_{p>1} W^{1,2}_{p,loc}(\underline{V}'_s)$.*

Proof. Fix $2 < p < r < q < \infty$, $\delta > 0$, and open, relatively compact subsets U, W, W' of $V(s)$ with smooth boundaries such that $\overline{U} \subseteq W \subseteq \overline{W} \subseteq W'$ and $[s-\delta, s+\delta] \times \overline{W'} \subseteq \underline{V}_0$. Choose $F \in C_0(\underline{V}_0)$ with $\|F\|_\infty \le \|f\|_\infty$, $F(s) = f$, $F(t,x) = f(x)$ for $x \in \overline{W'}$ and $|t-s| \le \delta$. We proceed in two steps.

(1) Take $\tau \in (s-\delta, s)$ and set $\underline{W} = [\tau, \tau+\delta] \times W$ and $\underline{W}' = [\tau, \tau+\delta] \times W'$. Using (3.8), (3.10), (3.9), and the interior apriori estimate [9, (IV.10.12)], we calculate

$$\begin{aligned}
&\|U(s,\tau)F(\tau) - f\|_{W_p^{2-\frac{2}{p}}(W)} \\
&\qquad \le c\,\|U(s,\tau)F(\tau) - f\|_{L^p(W)}^{1-\theta}\,\|U(s,\tau)F(\tau) - f\|^{\theta}_{W_p^{2-\frac{2}{r}}(W)} \\
&\qquad \le c\,\|(U(s,\tau)F(\tau))^{\#} - F(s)^{\#}\|_{C_0(\Omega)}^{1-\theta}\left(\|f\|_{W_q^{2-\frac{2}{q}}(W)} + \|U(s,\tau)F(\tau)\|_{W_q^{2-\frac{2}{q}}(W)}\right)^{\theta} \\
&\qquad \le c\,\|(U(s,\tau)F(\tau))^{\#} - F(s)^{\#}\|_{\infty}^{1-\theta}\left(\|f\|_{W_q^{2-\frac{2}{q}}(W)} + \|U(\cdot,\tau)F(\tau)\|_{W_q^{1,2}(\underline{W})}\right)^{\theta} \\
&\qquad \le c\,\|(U(s,\tau)F(\tau))^{\#} - F(s)^{\#}\|_{\infty}^{1-\theta}\left(\|f\|_{W_q^{2-\frac{2}{q}}(W')} + \|U(\cdot,\tau)F(\tau)\|_{C(\overline{\underline{W'}})}\right)^{\theta} \\
&\qquad \le c\,\|(U(s,\tau)F(\tau))^{\#} - F(s)^{\#}\|_{\infty}^{1-\theta} \qquad (3.11)
\end{aligned}$$

where $\theta = (1-\frac{1}{p})(1-\frac{1}{r})^{-1}$ and the constants c do not depend on τ,

(2) Now let $\tau_n = s - \frac{1}{n}$ for large n, $u_n = U(\cdot,\tau_n)F(\tau_n)$ on $\underline{V}'_s$, $\underline{U} = [s, s+\delta] \times U$, $\underline{W} = [s, s+\delta] \times W$, and $\underline{W}' = [s, s+\delta] \times W'$. By [14, Thm.4.13, 6.3] we have $u_n \in \bigcap_{p>1} W^{1,2}_{p,loc}(\underline{V}'_s)$ since u_n solves (3.5) with initial value $F(\tau_n)$ at time τ_n. Further, $u_n \to u$ in $C_0(\underline{V}'_s)$ due to (3.7). An application of (3.6), [9, (IV.10.12)], and (3.11) yields

$$\begin{aligned}
&\|u_n - u_m\|_{W_p^{1,2}(\underline{U})} = \|U(\cdot,s)\,[U(s,\tau_n)F(\tau_n) - U(s,\tau_m)F(\tau_m)]\|_{W_p^{1,2}(\underline{U})} \\
&\le c\left(\|u_n - u_m\|_{C(\overline{\underline{W}})} + \|U(s,\tau_n)F(\tau_n) - U(s,\tau_m)F(\tau_m)\|_{W_p^{2-\frac{2}{p}}(W)}\right) \\
&\le c\left(\|u_n - u_m\|_{\infty} + \|(U(s,\tau_n)F(\tau_n))^{\#} - F(s)^{\#}\|_{\infty}^{1-\theta} + \|(U(s,\tau_m)F(\tau_m))^{\#} - F(s)^{\#}\|_{\infty}^{1-\theta}\right).
\end{aligned}$$

Thus (u_n) is a Cauchy sequence in $W_p^{1,2}(\underline{U})$ by the strong continuity of $U(t,s)$. As a result, $u_n \to u$ in $W_p^{1,2}(\underline{U})$, and the assertion follows. □

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