High Order Explicit Two-Step Runge-Kutta Methods for Parallel Computers

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Abstract

In this paper we study a class of explicit pseudo two-step Runge-Kutta methods (EPTRK methods) with additional weights v. These methods are especially designed for parallel computers.

We study s-stage methods with local stage order s and local step order $s+2$ and derive a sufficient condition for global convergence order $s+2$ for fixed step sizes. Numerical experiments with 4- and 5-stage methods show the influence of this superconvergence condition.

However, in general it is not possible to employ the new introduced weights to improve the stability of high order methods. We show, for any given s-stage method with extended weights which fullfills the simplifying conditions $B(s)$ and $C(s-1)$, the existence of a reduced method with a simple weight vector which has the same linear stability behaviour and the same order.

Key words: Runge-Kutta methods, parallelism, two-step methods, superconvergence, linear stability
AMS(MOS) subject classification (1991): 65M12, 65M20

1 Introduction

For the numerical solution of systems of first-order ordinary differential equations (ODEs)

$$y' = f(t,y), \quad y(t_0) = y_0, \quad y, f \in \mathbb{R}^n, \tag{1}$$

Fei [5], Cong [2] and Cong et al. [3] have recently investigated a class of parallel explicit pseudo two-step Runge-Kutta methods (EPTRK methods). EPTRK methods compute an approximation $y_m \approx y(t_m)$ by the s-stage scheme

$$Y_m = \mathbb{1} \otimes y_m + h_m (A \otimes I) F(t_{m-1}\mathbb{1} + h_{m-1}c, Y_{m-1}), \tag{2a}$$

$$y_{m+1} = y_m + h_m (b^T \otimes I) F(t_m \mathbb{1} + h_m c, Y_m) \tag{2b}$$

with s (external) stage approximations $Y_m \in \mathbb{R}^{n\times s}$, parameters $A = (a_{ij}) \in \mathbb{R}^{s\times s}$, $c, b \in \mathbb{R}^s$, $\mathbb{1} := (1,\ldots,1)^T$ and step size h_m. F denotes the straightforward extension of f to $\mathbb{R}^{n\times s}$. Here, $\otimes$ denotes the Kronecker tensor product. Notice that the EPTRK method (2) requires only one sequential function evaluation per step on a parallel computer with s processing elements. Numerical

experiments with a variable step size implementation on a shared memory computer have shown that EPTRK methods perform well for non stiff ([3]) and for mildly stiff ([9]) problems.
A more general class of two-step RK methods was introduced by Jackiewicz et al. (see [8] and [1] and the references therein). Although Jackiewicz's methods are constructed not with respect to a parallel implementation it is attractive to borrow the idea of introducing a new weight vector v of s parameters and consider parallel EPTRK methods of the form

$$Y_m = \mathbb{1} \otimes y_m + h_m (A \otimes I) F(t_{m-1}\mathbb{1} + h_{m-1}c, Y_{m-1}), \tag{3a}$$

$$y_{m+1} = y_m + h_m (b^T \otimes I) F(t_m \mathbb{1} + h_m c, Y_m) + h_m (v^T \otimes I) F(t_{m-1}\mathbb{1} + h_{m-1}c, Y_{m-1}). \tag{3b}$$

The additional effort to compute y_{m+1} in (3b) is small and the parallelization of the method is not affected. The aim of this paper is to study the effects of the new parameters v with respect to order and stability.

2 Superconvergence

Definition 2.1. Let $\triangle_{m,0}$ and $\triangle_m$ denote the residuals of the integration scheme (3) obtained by substituting exact solutions $y(t_m)$ and $Y(t_m) := (y(t_m + c_1 h_m), \ldots, y(t_m + c_s h_m))$ of(1), i.e.

$$\triangle_{m,0} = Y(t_m) - \mathbb{1} \otimes y(t_m) - h_m (A \otimes I) Y'(t_{m-1}), \tag{4a}$$

$$\triangle_m = y(t_{m+1}) - h_m (b^T \otimes I) Y'(t_m)) + h_m (v^T \otimes I) Y'(t_{m-1})) \ . \tag{4b}$$

An EPTRK method (3) is of *local step order* p if $\triangle_m = \mathcal{O}(h_m^{p+1})$ and of *local stage order* q if $\triangle_{m,0} = \mathcal{O}(h_m^{q+1})$.

A Taylor series expansion of $y(t)$ in (4) shows that an EPTRK method is of local stage order q and local step order p if the simplifying conditions $C(q)$ and $B(p)$ defined by

$$\left[c, \sigma \frac{c^2}{2}, \sigma^2 \frac{c^3}{3}, \ldots, \sigma^{q-1} \frac{c^q}{q}\right] = A \left[\mathbb{1}, c - \mathbb{1}, (c - \mathbb{1})^2, \ldots, (c - \mathbb{1})^{q-1}\right], \tag{$C(q)$}$$

$$\left[1, \frac{\sigma}{2}, \frac{\sigma^2}{3}, \ldots, \frac{\sigma^{p-1}}{p}\right] = b^T \left[\mathbb{1}, \sigma c, \sigma^2 c^2, \ldots, \sigma^{p-1} c^{p-1}\right] + v^T \left[\mathbb{1}, (c - \mathbb{1}), (c - \mathbb{1})^2, \ldots, (c - \mathbb{1})^{p-1}\right] \tag{$B(p)$}$$

are satisfied. Here $\sigma = h_m / h_{m-1}$ denotes the step size ratio.
By standard techniques it can be shown, that a method satisfying $C(s)$ and $B(s+1)$ converges with order $p = s + 1$, e.g. [10] for the case $v = 0$. One can find methods which fulfill $C(s+1)$, however the c- vector will include large components and the error constants will be large. On the other hand, it is possible to satisfy $B(l)$ with $l > s + 1$. However this will not increase the order of convergence in general. We will show that, for general c with an additional condition on b and v, the order of convergence is $p = s + 2$. We restrict here for simplicity to the case of a constant step size.

Theorem 2.1 (Superconvergence of EPTRK methods). *Let $C(s)$ and $B(s+2)$ be satisfied. Let further hold the condition*

$$(b^T + v^T) \left(\frac{c^{s+1}}{s+1} - A(c-1)^s \right) = 0. \tag{5}$$

Then with starting values of order $s + 2$ the method will converge with order $p = s + 2$.

Proof. Substituting the exact solution into (3) yields

$$\begin{aligned} Y(t_m) &= \mathbb{1} \otimes y(t_m) + h(A \otimes I)F(t_{m-1}\mathbb{1} + hc, Y(t_{m-1})) + \triangle_{0,m} \\ y(t_{m+1}) &= y(t_m) + h(b^T \otimes I)F(t_m\mathbb{1} + hc, Y(t_m)) + h(v^T \otimes I)F(t_{m-1}\mathbb{1} + hc, Y(t_{m-1})) + \triangle_m. \end{aligned} \tag{6}$$

With $C(s)$ and $B(s+2)$ by Taylor expansion follows

$$\begin{aligned} \triangle_{0,m} &= \frac{h^{s+1}}{s!}\left(\frac{c^{s+1}}{s+1} - A(c-1)^s\right) \otimes y^{(s+1)}(t_m) + \mathcal{O}(h^{s+2}), \\ \triangle_m &= \mathcal{O}(h^{s+3}) \ . \end{aligned}$$

Subtracting (3) from (6) yields a recursion for the global error

$$err_{m+1} = y(t_{m+1}) - y_{m+1}.$$

With the standard convergence result $Y(t_m) - Y_m = \mathcal{O}(h^{s+2})$ for $C(s)$ and $B(s+1)$ we get by the mean value theorem and expanding the Jacobians at $(t_m, y(t_m))$

$$\begin{aligned} err_{m+1} &= (I + \mathcal{O}(h))err_m + \mathcal{O}(h)err_{m-1} + \\ &\quad + (b^T + v^T)\left(\frac{c^{s+1}}{s+1} - A(c-1)^s\right)\frac{h^{s+2}}{s!} f_y(t_m, y(t_m)) y^{(s+1)}(t_m) + \mathcal{O}(h^{s+3}). \end{aligned}$$

With (5) follows

$$\|err_{m+1}\| \le (1 + d_1 h)\|err_m\| + d_2 h\|err_{m-1}\| + d_3 h^{s+3}, \ d_1, d_2 d_3 > 0.$$

We can bound $\|err_m\|$ by the solution r_m of

$$r_{m+1} = (1 + d_1 h)r_m + d_2 h r_{m-1} + d_3 h^{s+3}$$

with $r_0 = \|err_0\| = d_4 h^{s+2}$, $r_1 = \|err_1\| = d_5 h^{s+2}$, $d_4, d_5 > 0$, giving finally $err_m = \mathcal{O}(h^{s+2})$. □

3 Numerical Illustration of the Order Results

To discuss the impact of condition (5) for EPTRK methods we construct different sets of coefficients with 4- and 5-stages and apply the resulting methods to three (non stiff) test problems.

Construction of 4-stage Methods

We start with $s = 4$ stages. For any choice of knots c_i a method with $v = 0$ is uniquely determined by $B(4)$ and $C(4)$. The remaining 4 degrees of freedom can be used to satisfy $B(8)$ with c-vector taken as Gaussian collocation points, as done for method gauss4 in Table 1. Here, $E_{s+1} := A(c-1)^s - c^{s+1}/(s+1)$ denotes the local stage error and $e_{s+1} := (b^T + v^T)E_{s+1}$ denotes the residual of condition (5) in Theorem 2.1. Since e_{s+1} does not vanish the global convergence order is only 5.

Global order 6 can be achieved by satisfying $C(s)$, $B(s+2)$ and (5) for given c by computing v or for $v = 0$ by a special choice of c. To attain global order 6 with the same points c_i we choose v as a solution of $e_{s+1} = 0$. But $B(6)$ depends on v and has to be satisfied, too. With the help of Maple V we calculated a solution $v = [0.0, -0.006332901980013884, -0.319483842974888, 0.06964740132900621]$. for this method vgauss4, see Table 1.

Although it is naturally to guarantee condition (5) for arbitrarily chosen knots c_i by introducing a suitable v vector, it is not necessary. By choosing special points c one can find EPTRK methods with $v = 0$ which satisfy (5), $B(s+2)$ as we did for the method n4 with

$$c = [0.1493506562434243,\ 0.6535456428480576,\ 1.123,\ 1.6391116441727]\ .$$

Construction of 5-stage Methods

The method cong5 was proposed in [3]. Due to the small error coefficients this method performs well for non stiff problems. Since $e_{s+1} \neq 0$ the global order is $p = 6$ only, see Table 1. The method cong5 has the knots

$$c = [0.08858795951270395, 0.4094668644407347, 0.7876594617608471, 1, 1.409466864440735]\ . \tag{7}$$

With the choice $v = [0, 0, 0, 0, -0.01842446247125309]$ and the c-vector (7) the method vcong5 fulfills (5), $B(s+2)$ and has global order $p = 7$.
The method n5 has been constructed with $v = 0$ and

$$c = [0.1365941578442505, 0.625, 1.230436842527931, 1.5, 1.6911642569218]\ .$$

These values guarantee (5) and $B(s+2)$ and give a small leading stage error coefficient $\|E_{s+1}\|$.

Table 1: EPTRK methods

	name	conditions	order p	$\|E_{s+1}\|_2$	e_{s+1}
	gauss4	B(8), C(4)	5	1.051	0.2952
4-stage	vgauss4	B(6), C(4)	6	1.051	0
	n4	B(6), C(4)	6	2.334	0
	cong5	B(7), C(5)	6	2.670	0.0475
5-stage	vcong5	B(7), C(5)	7	2.670	0
	n5	B(7), C(5)	7	2.385	0

Test Problems

We consider the following test problems:

- NOFE – a nonlinear problem proposed by Fehlberg [4].

$$y_1' = 2ty_1 \log(\max(y_2, 0.001)) \tag{8a}$$
$$y_2' = -2ty_2 \log(\max(y_1, 0.001)) \tag{8b}$$
$$t \in [0, 5] \tag{8c}$$

  with initial values taken from the exact solution $y_1(t) = \exp(\sin(t^2))$, $\quad y_2(t) = \exp(\cos(t^2))$.

- PROTH – a (non stiff) Prothero-Robinson problem, see [7].

$$y' = \lambda(y - \sin(t)) + \cos(t), \quad t \in [0, 10], \quad \lambda = 0.1, \tag{9}$$

  with exact solution $y(t) = \sin(t)$.

- ORBIT – a two-body orbit problem, see [6].

$$y_1' = y_3, \qquad y_2' = y_4 \tag{10a}$$
$$y_3' = -y_1/r^3, \quad y_4' = -y_2/r^3, \qquad t \in [0,10], \tag{10b}$$

$r = \sqrt{y_1^2 + y_2^2}$, with exact solution $y(t) = [\cos(t), \sin(t), -\sin(t), \cos(t)]$.

We have implemented the EPTRK methods of Table 1 in FORTRAN using double precision and applied them to the test problems. (The code for the method `cong5` can be obtained from our web site http://www.mathematik.uni-halle.de/institute/numerik/software.)
In the figures below we have plotted the number of steps versus the error ERR in the endpoint of the integration, defined by

$$ERR = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{y_i - y_i(t_{\text{end}})}{1 + |y_i(t_{\text{end}})|}\right)^2}\ .$$

Notice, from $ERR \approx Ch^p$ follows $\log(ERR) \approx \log(C) - p \cdot \log(h^{-1})$ and therefore the order of an EPTRK method can be estimated by the slope of the lines in the figures.
The order 6 of the 4-stage methods `n4` and `vgauss4` compared with the order 5 of the method `gauss4` leads to a better accuracy for all step sizes. Due to smaller error constants in the step formula (3b) (not shown in Table 1) the method `n4` with $v = 0$ performs slightly better as `vgauss4` in particular in example PROTH, Fig. 2.
The fact, that `cong5` is of order 6 only can be seen for sufficient small step sizes, i.e. the slope of the lines in the figures is smaller than the slopes of the (nearly parallel) lines of `vcong5` and `n5`. The best overall performance gives the method `n5` with $v = 0$ and small error constants.

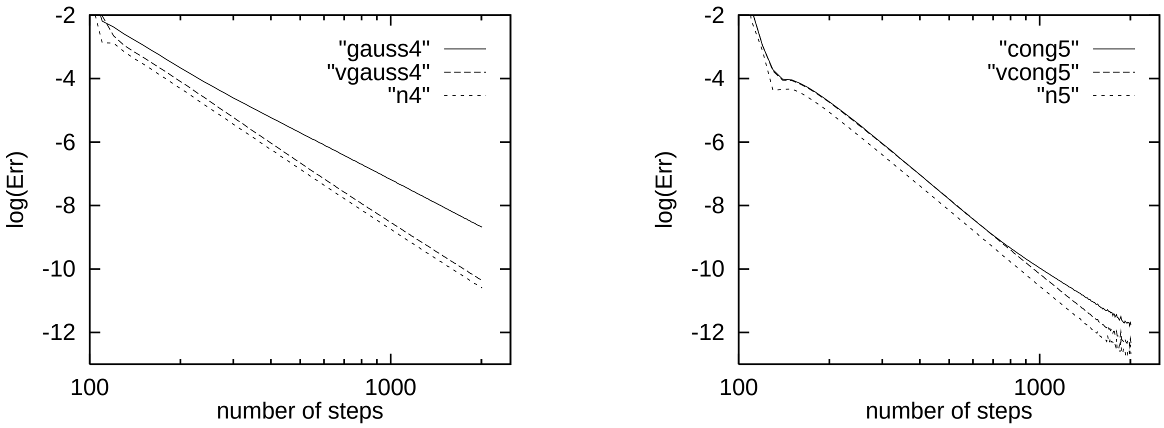

Figure 1: Results for problem NOFE (8)

4 Stability

The stability of EPTRK methods with $v = 0$ has been investigated in [9]. The adaption of the concepts there to the case $v \neq 0$ is straightforward. Here, we start with a short review of the notations:

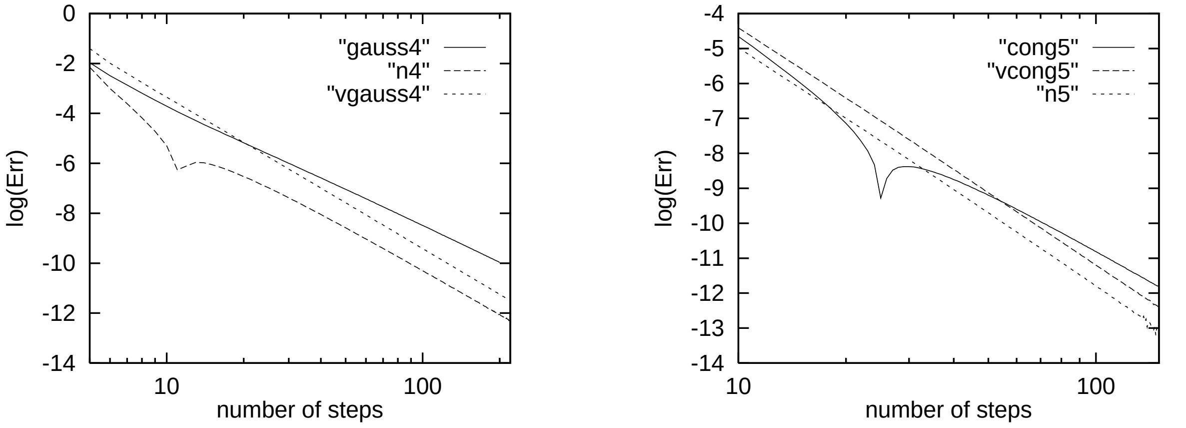

Figure 2: Results for problem PROTH (9)

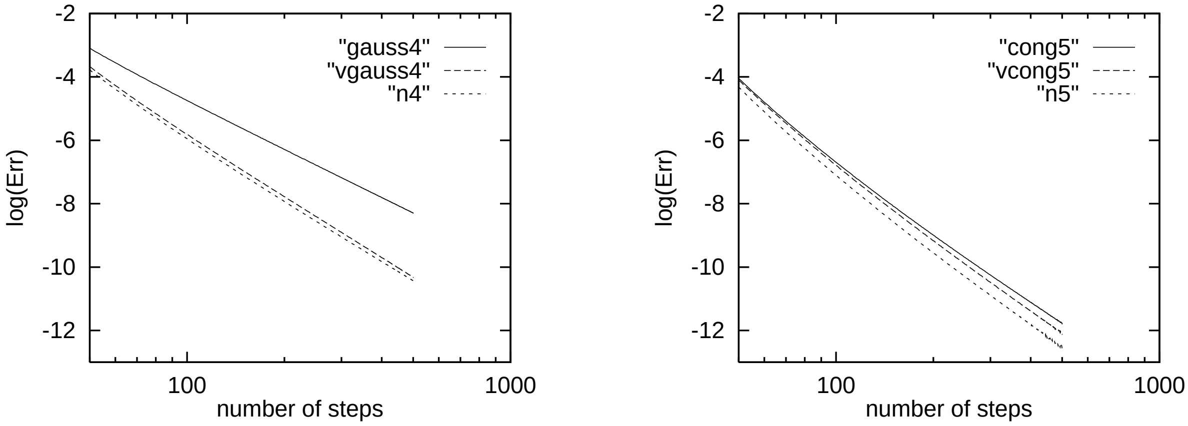

Figure 3: Results for problem ORBIT (10)

We apply the EPTRK methods to the linear test equation $y' = \lambda y, \quad \mathrm{Re}\lambda \leq 0$ with constant step size h. Then the solution fulfills the recursion

$$(Y_m, y_{m+1})^T = M(z)(Y_{m-1}, y_m)^T, \quad z = h\lambda \ , \tag{11}$$

with the amplification matrix

$$M(z) = \begin{pmatrix} zA & \mathbb{1} \\ z^2 b^T A + z v^T & 1 + z b^T \mathbb{1} \end{pmatrix} \ . \tag{12}$$

We define the stability region S^* of an EPTRK method by

$$S^* := \{z : \varrho(M(z)) \leq 1\}$$

and have $\varrho(M(z)) < 1$ if z is an inner point of S^* and hence $(Y_m, y_{m+1}) \to 0$ for $m \to \infty$. We obtain $\varrho(M(z))$ by computing the zeros of the stability polynomial $\varrho(x, z) := \det(xI - M(z))$. But surprisingly, the new parameters v do not give us any new degree of freedom in the corresponding stability polynomial. We have the following theorem:

Theorem 4.1. *For every method* (3) *with* $C(s-1)$ *and* $B(s)$ *there exists a corresponding method with* $v^T = 0$*, satisfying* $C(s-1)$*,* $B(s)$ *and having the same stability polynomial.*

Proof. The proof is given in the appendix of this report. □

A consequence of Theorem 4.1 is that it is not possible to improve the stability of EPTRK methods with a choice $v \neq 0$ for high order methods. However, a nontrivial choice of the v-vector may be useful to conserve the stability for variable step sizes.

5 Conclusion

We have proposed a superconvergence condition (5) in Theorem 2.1 which allows the construction of s-stage EPTRK methods with stage order s and convergence order $s+2$. To satisfy this condition with a given set of knots c_i we have introduced new parameters v into the integration scheme (3). Numerical tests with fixed step size and 4- and 5-stage methods have illustrated the convergence result. Though it has been possible to find special knots c_i which guarantee condition (5) for constant step size with $v = 0$, for variable step sizes we are forced to choose $v \neq 0$. Equation (5) becomes

$$b_m^T \left[\frac{c^{s+1}}{s+1}\sigma_m^{s+1} - \sigma_m A(\sigma_m)(c-1)^s\right] + v_m^T \left[\frac{c^{s+1}}{s+1} - \frac{1}{\sigma_{m-1}}A(\sigma_{m-1})(c-1)^s\right] = 0 \tag{13}$$

with step size ratios $\sigma_m := h_m/h_{m-1}$ and σ_m-dependent RK matrix $A = A(\sigma_m)$ and condition (5) can not longer be satisfied with $v = 0$. Since with $v \neq 0$ the error behavior is more smooth we hope to obtain a more reliable error estimation and step size control in this case. This generalization to variable step sizes will be a topic of further work.
In Theorem 4.1 we have shown that it is not possible to enlarge the stability regions of the EPTRK methods with $B(s)$ and $C(s-1)$ with the new parameters v of the integration scheme. However, the construction in [9] of a EPTRK method based on a given stability polynomial can be simplified with $v \neq 0$.

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6 Appendix

We will prove Theorem 4.1 in the following subsections of this appendix.

6.1 The conditions $C(s-1)$ and $B(s)$

Given a method (3), we start with expressing the conditions $C(s-1)$ and $B(s)$ in a convenient way for prescribed nodes c_i. The i-th unit column vector is denoted by e_i. Further we use the following matrices

$$\begin{aligned}
V_0 :=& (c_i^{j-1})_{i,j=1,\dots,s}\\
V_1 :=& ((c_i-1)^{j-1})_{i,j=1,\dots,s}\\
S :=& \operatorname{diag}(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{s})\\
Q :=& (q_{ij})_{i,j=1,\dots,s}, \quad q_{ij} = \binom{j-1}{i-1}\\
N :=& \begin{pmatrix} 0 & 1 & & 0\\ & \ddots & \ddots & \\ \vdots & & \ddots & 1\\ 0 & \dots & & 0 \end{pmatrix} \in \mathbb{R}^s \times s\\
R =& \sum_{i=1}^{s} (N^T SQ)^{i-1} e_1 e_i^T\\
W =& V_0 R.
\end{aligned}$$

Note that $V_0 = V_1 Q$.

Lemma 6.1. *The matrix R is a regular, upper triangular matrix. The matrix W is regular.*

Proof. SQ is a regular upper triangular matrix, so $N^T SQ$ is a Hessenberg matrix with full lower diagonal. Obviously, the vector $(N^T SQ)^{i-1} e_1$ has zeros below the i-th component and a non-vanishing i-th component. Hence R is a regular upper triangular matrix and $W = V_0 R$ is regular. □

Lemma 6.2. *The matrix A satisfies condition $C(s-1)$ if and only if $W^{-1}AW = K$ with*

$$K = \begin{pmatrix} 0 & 0 & \dots & 0 & p_0\\ 1 & 0 & \dots & 0 & p_1\\ 0 & 1 & \dots & 0 & p_2\\ \vdots & \ddots & \ddots & \vdots & \vdots\\ 0 & & 0 & 1 & p_{s-1} \end{pmatrix}, \tag{14}$$

where the p_i are arbitrary.

Proof. We write the condition $C(s-1)$ in the form

$$0 = (AV_1 - V_0 N^T S) e_k \quad \text{for } k = 1, \dots, s-1.$$

Because Q and R are regular upper triangular matrices, this is equivalent to

$$\begin{aligned}
0 =&(AV_1 - V_0 N^T S)QRe_k \quad \text{ for } k = 1, \ldots, s-1\\
=&AV_0 Re_k - V_0 N^T SQ(N^T SQ)^{k-1} e_1\\
=&AWe_k - V_0 Re_{k+1}\\
=&AWe_k - We_{k+1}.
\end{aligned}$$

Thus K has the structure above with arbitrary last column if and only if A satisfies $C(s-1)$. $\square$

Next we express b^T using the $B(s)$ condition and v^T. Using the matrices defined above, $B(s)$ is given by

$$b^T V_0 + v^T V_1 = \mathbb{1}^T S \tag{15}$$

$$\Leftrightarrow b^T W = -v^T V_0 Q^{-1} R + \mathbb{1}^T SR. \tag{16}$$

6.2 Simplification of the stability matrix

First we transform via

$$\begin{aligned}
M_2(z) :=& \begin{pmatrix} I & 0 \\ -zb^T & 1 \end{pmatrix} M(z) \begin{pmatrix} I & 0 \\ zb^T & 1 \end{pmatrix}\\
=& \begin{pmatrix} z(A + \mathbb{1}b^T) & \mathbb{1} \\ z(b^T + v^T) & 1 \end{pmatrix}
\end{aligned}$$

to eliminate the z^2 expressions in $M(z)$. Next we use the matrix W to obtain

$$\begin{aligned}
M_3(z) :=& \begin{pmatrix} W^{-1} & 0 \\ 0 & 1 \end{pmatrix} M_2(z) \begin{pmatrix} W & 0 \\ 0 & 1 \end{pmatrix}\\
=& \begin{pmatrix} z(K + e_1 b^T W) & e_1 \\ z(b^T W + v^T W) & 1 \end{pmatrix}.
\end{aligned}$$

Note that $W^{-1}\mathbb{1} = e_1$. We get the stability polynomial:

$$\varrho(z,x) = \det(M_3(z) - xI) = \det \begin{pmatrix} z(K + e_1 b^T W) - xI & e_1 \\ z(b^T W + v^T W) & 1-x \end{pmatrix}.$$

We expand the determinant by the last column. In the adjunct to the upper right 1 (from e_1) we exchange the last row $s-1$ times with the row above to move it to the first row:

$$\begin{aligned}
\varrho(z,x) =&(-1)^{s+2}(-1)^{s-1} \det[(I - e_1 e_1^T)(zK - xI) + e_1 z(b^T W + v^T W)]\\
&+ (1-x)\det(z(K + e_1 b^T W) - xI).
\end{aligned}$$

Note that the factor $(-1)^{s+2}$ is the sign of the adjunct in the expansion formula $(1+s+1)$ but the factor $(-1)^{s-1}$ results from the row exchanges in the adjunct. We split the first row of the second determinant to get

$$\begin{aligned}
\varrho(z,x) =& -\det[(I - e_1 e_1^T)(zK - xI) + e_1 xzb^T W + e_1 zv^T W)]\\
&+ (1-x)\det(zK - xI).
\end{aligned}$$

6.3 Computation of the stability polynomial

The matrix K is the companion matrix of the polynomial

$$p(\xi) := \xi^s - p_{s-1}\xi^{s-1} - \cdots - p_1\xi - p_0. \tag{17}$$

Thus

$$\det(zK - xI) = (-z)^s \det(x/zI - K) = (-z)^s p\left(\frac{x}{z}\right) \tag{18}$$

To compute the remaining part of the stability polynomial

$$-\det[(I - e_1e_1^T)(zK - xI) + e_1xzb^TW + e_1zv^TW)]$$

we expand the determinant by the first row. Suppose we delete the first row and the i-th column:

$$\begin{pmatrix} \Box & \Box & \Box & \Box & \Box & \Box & \Box \\ z & -x & & \Box & & & \\ & z & -x & \Box & & & \\ & & z & \Box & & & \\ & & & \Box & -x & & \vdots \\ & & & \Box & z & -x & zp_{s-2} \\ & & & \Box & & z & zp_{s-1} - x \end{pmatrix}. \tag{19}$$

The determinant of the upper left block is z^{i-1}. The lower right block is of the same form as $zK - xI$ but of dimension $s - i$. Its determinant is (see (18)) $(-z)^{s-i}p^i\left(\frac{x}{z}\right)$ where $p^i(\xi)$ is the polynomial

$$p^i(\xi) = \xi^{s-i} - p_{s-1}\xi^{s-i-1} - \cdots - p_{i+1}\xi - p_i.$$

Hence the second determinant is

$$\sum_{i=1}^{s}(-1)^{i+1}z(xb^T + v^T)We_iz^{i-1}(-z)^{s-i}p^i\left(\frac{x}{z}\right)$$

and the stability polynomial is given by

$$\begin{aligned}\varrho(z,x) =& (1-x)(-z)^s p\left(\frac{x}{z}\right) \\ &+ (-z)^s\sum_{i=1}^{s}(xb^T + v^T)We_ip^i\left(\frac{x}{z}\right).\end{aligned}$$

Now we substitute (16) to eliminate b^T

$$\begin{aligned}\varrho(z,x) =& (-z)^s[p\left(\frac{x}{z}\right) + \sum_{i=1}^{s}(v^TWe_i)p^i\left(\frac{x}{z}\right)] \\ &- x(-z)^s[p\left(\frac{x}{z}\right) + \sum_{i=1}^{s}(-\mathbb{1}^TSRe_i + v^TV_0Q^{-1}Re_i)p^i\left(\frac{x}{z}\right)].\end{aligned}$$

We are prepared for the final step:
A method satisfying $C(s-1)$ and $B(s)$ is uniquely determined (for fixed nodes c_i) by the last column p of K and the weights v^T. Our claim ist that for any parameters (p, v^T) there exist parameters $(\tilde{p}, 0)$ that give the same stability function.
We choose

$$\tilde{p}(\xi) = p(\xi) + \sum_{i=1}^{s} (v^T W e_i) p^i(\xi) \tag{20}$$

and show

$$\tilde{p}(\xi) + \sum_{i=1}^{s} (-\mathbb{1}^T S R e_i) \tilde{p}^i(\xi) = p(\xi) + \sum_{i=1}^{s} (-\mathbb{1}^T S R e_i + v^T V_0 Q^{-1} R e_i) p^i(\xi). \tag{21}$$

We write the polynomials p, p^i using the column vectors

$$\begin{aligned} \underline{p} :=& (-p_0, \dots, -p_{s-1}, 1)^T, \quad \underline{\xi} = (1, \xi, \dots, \xi^s)^T. \\ \Rightarrow p(\xi) =& \underline{\xi}^T \underline{p} \\ p^i(\xi) =& \underline{\xi}^T N^i \underline{p} \end{aligned}$$

This allows us to express the polynomials $\tilde{p}$, $\tilde{p}^i$ in a convenient way:

$$\begin{aligned} \tilde{p}(\xi) =& \underline{\xi}^T \left[\underline{p} + \sum_{j=1}^{s} (v^T W e_j) N^j \underline{p} \right] \\ \tilde{p}^i(\xi) =& \underline{\xi}^T N^i \underline{\tilde{p}} = \underline{\xi}^T \left[N^i \underline{p} + \sum_{j=1}^{s} (v^T V_0 R e_j) N^{i+j} \underline{p} \right]. \end{aligned}$$

Substituting the expressions for $\underline{\tilde{p}}$, $\underline{\tilde{p}}^i$ our assertion becomes

$$\begin{aligned} & \underline{\xi}^T \underline{p} + \sum_{i=1}^{s} (v^T V_0 R e_i) \underline{\xi}^T N^i \underline{p} + \sum_{i=1}^{s} (-\mathbb{1}^T S R e_i) \left[\underline{\xi}^T N^i \underline{p} + \sum_{j=1}^{s} (v^T V_0 R e_j) \underline{\xi}^T N^{i+j} \underline{p} \right] \\ =& \underline{\xi}^T \underline{p} + \sum_{i=1}^{s} (-\mathbb{1}^T S R e_i) \underline{\xi}^T N^i \underline{p} + \sum_{i=1}^{s} (v^T V_0 Q^{-1} R e_i) \underline{\xi}^T N^i \underline{p} \end{aligned}$$

which simplifies to

$$\begin{aligned} & \sum_{i=1}^{s} (v^T V_0 R e_i) \underline{\xi}^T N^i \underline{p} + \sum_{i=1}^{s} (-\mathbb{1}^T S R e_i) \sum_{j=1}^{s} (v^T V_0 R e_j) \underline{\xi}^T N^{i+j} \underline{p} \\ & = \sum_{i=1}^{s} (v^T V_0 Q^{-1} R e_i) \underline{\xi}^T N^i \underline{p}. \end{aligned}$$

We compare the coefficients of $\underline{\xi}^T N^i \underline{p}$ for $i = 1, \dots, s-1$ (note that $N^s = 0$) and find:

$$v^T V_0 (I - Q^{-1}) R e_i = \sum_{j=1}^{i-1} v^T V_0 R e_j (\mathbb{1}^T S R e_{i-j}).$$

This identity is verified in Lemma 6.3 below. This completes the proof.
Note that we did not use the fact that the simplified method is based on the same nodes. The stability function of the simplified method is completely determined by the vector $\underline{\tilde{p}}$. This implies that we can choose arbitrary c_i for the method with $v^T = 0$.

Lemma 6.3. *For $i = 1, \dots, s-1$ holds*

$$(I - Q^{-1})Re_i = \sum_{j=1}^{i-1} (\mathbb{1}^T S R e_{i-j}) R e_j.$$

Proof. We prove the lemma by induction:
For $i = 1$ the assumption is valid because of $Re_1 = e_1$, $Q^{-1}e_1 = e_1$.
For $i = k+1$, $1 \le k \le s-2$ we have to prove

$$0 = (I - Q^{-1})Re_{k+1} - \sum_{j=1}^{k} (\mathbb{1}^T S R e_{k+1-j}) R e_j.$$

Substituting $Re_{k+1} = N^T SQRe_k$ and shifting the summation index yields

$$0 = (I - Q^{-1})N^T SQRe_k - \sum_{j=0}^{k-1} (\mathbb{1}^T S R e_{k-j}) R e_{j+1}.$$

We multiply the induction hypothesis (set $i = k$) with $N^T SQ$ and subtract it:

$$\begin{aligned} 0 =& - Q^{-1}N^T SQRe_k + N^T SRe_k - (\mathbb{1}^T SRe_k)e_1 \\ =& - Q^{-1}(N^T SQ - QN^T S + e_1 \mathbb{1}^T S)Re_k \end{aligned}$$

The last component of Re_k vanishes since R is a upper triangular matrix and $k \le s-2$. We show further that the first $s-1$ columns of $[N^T SQ - QN^T S + e_1 \mathbb{1}^T S]$ vanish:
The first row is $0 - \mathbb{1}^T N^T S + \mathbb{1}^T S = e_s^T S$, i.e., the first $s-1$ entries in the first row vanish. For $2 \le i \le s$, $1 \le j \le s-1$ we get

$$\begin{aligned} (N^T SQ)_{ij} =& (SQ)_{i-1,j} \\ =& \frac{1}{i-1}\binom{j-1}{i-2} \\ =& \frac{(j-1)!}{(i-1)!(j-i+1)!} \\ (QN^T S)_{ij} =& (QN^T)_{ij}\frac{1}{j} \\ =& Q_{i,j+1}\frac{1}{j} \\ =& \binom{j}{i-1}\frac{1}{j} \\ =& (N^T SQ)_{ij}. \end{aligned}$$

So the first $s-1$ columns of $[N^T SQ - QN^T S + e_1 \mathbb{1}^T S]$ vanish and hence the product $[N^T SQ - QN^T S + e_1 \mathbb{1}^T S]Re_k$ vanishes. $\square$