SOLVABILITY AND MAXIMAL REGULARITY OF PARABOLIC EVOLUTION EQUATIONS WITH COEFFICIENTS CONTINUOUS IN TIME

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Abstract. We establish maximal regularity of type L^p for a parabolic evolution equation $u'(t) = A(t)u(t) + f(t)$ with $A(\cdot) \in C([0,T], \mathcal{L}(D(A(0)), X))$, and construct the corresponding evolution family on the underlying Banach space X. Our proofs are based on the operator sum method and the use of evolution semigroups. The results are applied to parabolic partial differential equations with continuous coefficients.

1. Introduction

We are concerned with the solvability and maximal regularity of the linear Cauchy problem

$$\begin{aligned} u'(t) &= A(t)u(t) + f(t) \qquad \text{for (a.e.)} \quad 0 \le t \le T, \\ u(0) &= x, \end{aligned} \tag{1.1}$$

where $A(t)$ generates an analytic semigroup on a Banach space X, $f \in L^p([0,T], X)$ for $1 < p < \infty$, $x \in X$, and $T > 0$ is fixed. One of our aims is to remove the usual Hölder condition on $A(\cdot)$ (or $(\lambda - A(\cdot))^{-1}$). So we only assume that

$$D(A(t)) = D(A(0)) =: X_1 \quad \text{and} \quad A(\cdot) \in C([0,T], \mathcal{L}(X_1, X)). \tag{1.2}$$

Since we allow for discontinuous inhomogeneities f, we can only expect to solve (1.1) for a.e. $t \in [0,T]$ with solutions of class

$$u \in C([0,T], X) \cap W^{1,p}_{loc}((0,T], X) \cap L^p_{loc}((0,T], X_1)$$

if $x \in X$ and

$$u \in F := W^{1,p}([0,T], X) \cap L^p([0,T], X_1)$$

if x belongs to the real interpolation space $(X, X_1)_{1-\frac{1}{p},p}$. In particular, in the latter case we show *maximal regularity (of type L^p)*; that is, the existence of a unique solution $u \in F$ of (1.1) satisfying

$$\|u\|_{L^p([0,T],X)} + \|u\|_{L^p([0,T],X_1)} + \|u'\|_{L^p([0,T],X)} \le c_p \left(\|x\|_{1-\frac{1}{p},p} + \|f\|_{L^p([0,T],X)}\right) \tag{1.3}$$

for all $x \in (X, X_1)_{1-\frac{1}{p},p}$ and $f \in L^p([0,T], X)$ and a constant c_p. To achieve this, we suppose that, besides (1.2),

$$\text{the operator } A(s) \text{ has maximal regularity in } L^p \text{ for all fixed } s \in [0,T]. \tag{1.4}$$

Below we discuss sufficient conditions for this hypothesis.

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In order to solve (1.1) for initial values $x \in X$, it is necessary to construct an *evolution family* $U(t,s)$, $0 \le s \le t \le T$, of bounded linear operators on X. This means that

$$U(t,s) = U(t,r)U(r,s), \qquad U(s,s) = Id, \tag{1.5}$$

$$(t,s) \mapsto U(t,s) \quad \text{is strongly continuous} \tag{1.6}$$

for $0 \le s \le r \le t \le T$. Moreover, it will be shown that $U(t,s)$ maps X into real interpolation spaces between X and X_1, satisfies corresponding norm estimates, and solves the homogeneous problem (1.1) with $f = 0$. See Theorem 2.5 for all this.

If $A(t)$ is an elliptic operator on $L^q(\Omega)$, $1 < q < \infty$, hypothesis (1.2) requires only continuity of the coefficients, while (1.4) may require additional regularity with respect to the space variables, see Section 4. So far, no complete characterization of (1.4) is known for elliptic operators on $L^q(\Omega)$. In this case, (1.3) leads to L^p–L^q a priori estimates which are very useful for the investigation of nonlinear equations, compare e.g. [3, 9, 19]. A perturbation result proved in Theorem 3.1 allows to reduce the regularity assumptions on the lower order coefficients to rather weak integrability conditions.

In the case of Hölder continuous $A(\cdot) : [0,T] \to \mathcal{L}(X_1, X)$ and $f : [0,T] \to X$, it is known that there exists an evolution family $U(t,s)$ on X and a unique solution $u \in C^1([0,T], X)$ of (1.1) given by

$$u(t) = U(t,0)x + \int_0^t U(t,s)f(s)\,ds$$

for $x \in X_1$. In addition, $U(t,s)$ maps X into X_1 and is differentiable in $\mathcal{L}(X)$ with $\partial_t U(t,s) = A(t)U(t,s)$ for $t > s$, see e.g. [3, 22, 24]. One also has maximal Hölder regularity which is extensively studied in the monographs [3] and [22]. These results remain essentially valid if the domains $D(A(t))$ depend on time t provided one imposes appropriate conditions on the resolvent of $A(t)$. Among the variety of possible assumptions we only mention the estimate

$$\|A(t)(\lambda - A(t))^{-1}(A(t)^{-1} - A(s)^{-1})\| \le L\,|t-s|^{\alpha}\,|\lambda|^{-\beta} \tag{1.7}$$

for constants $L \ge 0$ and $\alpha, \beta \in (0,1]$ with $\alpha + \beta > 1$, $t, s \in [0,T]$, and λ in a sector with angle greater than $\pi/2$. This condition was introduced by P. Acquistapace and B. Terreni in [1, 2], see also [3, 23] and the references therein.

In the present work we concentrate on the case of time independent domains. Here it is possible to remove the Hölder continuity in time at the prize of increased space regularity. Namely, G. Da Prato and P. Grisvard assumed that, besides (1.2), the domain $D_{A(t)}(\theta+1)$ of the part of $A(t)$ in the interpolation space $X_\theta := D_{A(0)}(\theta)$ is equal to a fixed space $X_{\theta+1}$ and $A(\cdot) \in C([0,T], \mathcal{L}(X_{\theta+1}, X_\theta))$ for some $\theta \in (0,1)$. They obtained solutions $u \in C([0,T], X_{\theta+1})$ of (1.1) with $x \in X_{\theta+1}$ and $f \in C([0,T], X_\theta)$, see [11] and also [12]. Under (almost) the same assumptions, A. Buttu studied in detail maximal Hölder regularity and properties of the corresponding evolution family on interpolation spaces in [6, 7]. Very close to our work is the paper [18] by M. Giga, Y. Giga and H. Sohr, see also [3, III.4.10.10]. They showed maximal L^p–regularity of (1.1) imposing slight variants of (1.2) and (1.4). But we point out that initial values $x \in X$ or the evolution operator on X were not considered in these papers. So the main aim of the present investigation is to construct the evolution family $U(t,s)$ on X and to solve (1.1) for $x \in X$.

Maximal regularity of type L^p has been investigated thoroughly in the autonomous case $A(\cdot) = A_0$. We refer to [3, 14, 21, 25] for a survey and further references. First we recall that by a result of A. Benedek, A.P. Calderón and R. Panzone, [5], hypothesis (1.4) is in fact independent of $p \in (1,\infty)$; that is, if (1.4) is valid for some $p_0 \in (1,\infty)$, then it holds for all $p \in (1,\infty)$, see [14] or [21]. It is known that a densely defined, closed operator A_0 having maximal regularity already generates an analytic semigroup, [14, §2]. De Simon proved in [13] that the converse is true if X is a Hilbert space so that in this case (1.4) is a consequence of (1.2). There are examples of generators of analytic semigroups without maximal regularity if X is not a UMD space, cf. [14, §3]. Further, maximal regularity holds for generators of analytic semigroups on real interpolation spaces due to [10]. More recently, G. Dore and A. Venni showed in [15] maximal regularity in UMD-spaces X for operators having bounded imaginary powers with $\|(-A_0)^{is}\| \le K\, e^{\theta|s|}$ for $s \in \mathbb{R}$ and a constant $0 \le \theta < \pi/2$, see also [3, 25]. Notice that $L^q(\Omega)$ is an UMD-space if $1 < q < \infty$. Finally, A_0 possesses maximal regularity on $L^q(\Omega)$, $1 < q < \infty$, if it generates an analytic semigroup on $L^2(\Omega)$ and satisfies certain heat kernel estimates, [21].

The results of [15] and [21] have been extended to the non-autonomous situation in [23] and [19, 20], respectively, where (1.7) was assumed. The latter condition ensures, in particular, the existence of an evolution family on X due to [1, 2].

Our approach combines the operator sum method as initiated by G. Da Prato and P. Grisvard in [10] and semigroup techniques. In Section 2 we first show that the operator

$$G = -\tfrac{d}{dt} + A(\cdot) \quad \text{with} \quad D(G) = \{f \in W^{1,p}([a,b],X) \cap L^p([a,b],X_1) : f(a) = 0\}$$

is invertible in $L^p([a,b],X)$ if $b-a$ is sufficiently small. Next, we approximate G by the operators $G_n = -\frac{d}{dt} + A_n(\cdot)$ for the Yosida approximations $A_n(t)$ of $A(t)$. The semigroups generated by G_n are uniformly bounded. The proof of this fact is relegated to Section 5. It uses another approximation argument and the Chernoff product formula. So the Trotter–Kato theorem shows that G is a generator. From a result in [26, 27] then follows that the semigroup $T(\cdot)$ generated by G is given by

$$(T(t)f)\,(s) = \begin{cases} U(s,s-t)f(s-t), & a \le s-t \le s \le b, \\ 0, & s-t < a \le s \le b, \end{cases} \tag{1.8}$$

for an evolution family $U(t,s)$ on X and $f \in L^p([a,b],X)$. We call semigroups of this form *evolution semigroups*, see [8, 27] and the references therein for more information. Using the representation of G and simple properties of $T(\cdot)$, one can now verify the asserted regularity of $U(t,s)$ and solve (1.1). In Section 3 we establish a perturbation result for (1.1) which is a consequence of a non–autonomous Miyadera perturbation theorem from [26, 27]. Applications to parabolic partial differential equations are presented in Section 4.

2. The main result

We first state our standing hypotheses on the operators $A(t)$.

(H1) $A(t)$, $0 \le t \le T$, generates an analytic semigroup of negative type on a Banach space X, $D(A(t)) = D(A(0)) =: X_1$, and $A(\cdot) \in C([0,T],\mathcal{L}(X_1,X))$, where X_1 is endowed with the graph norm of $A(0)$.

(H2) For some $p \in (1,\infty)$ and all $f \in L^p([0,T],X)$ and $s \in [0,T]$ there is a unique function $u \in F_0 := \{f \in W^{1,p}([0,T],X) \cap L^p([0,T],X_1) : f(0) = 0\}$ such that

$$u'(t) = A(s)u(t) + f(t) \qquad \text{for a.e. } t \in [0,T]. \tag{2.1}$$

Clearly, (H1) implies that, given $s \in [0,T]$,

$$(\lambda - A(t))^{-1} = (\lambda - A(s))^{-1}\,[1 - (A(t) - A(s))A(0)^{-1}A(0)(\lambda - A(s))^{-1}]^{-1}$$

for $t \in [0,T]$ with $|t-s| \le \delta_s$ and $\lambda \in \Sigma_{\phi_s} := \{\mu \in \mathbb{C} \setminus \{0\} : |\arg\mu| \le \phi_s\} \cup \{0\}$ for some $\pi/2 < \phi_s \le \pi$ and $\delta_s > 0$. So, due to the compactness of $[0,T]$, there are constants $M \ge 0$ and $\pi/2 < \phi \le \pi$ such that

$$\|(\lambda - A(t))^{-1}\|_{\mathcal{L}(X,X_k)} \le \frac{M}{1 + |\lambda|^{1-k}} \tag{2.2}$$

for $\lambda \in \Sigma_\phi$, $t \in [0,T]$, $k = 0,1$, and $X_0 := X$, cf. [11, Lemme 3.5]. In particular, the type of the semigroup $(e^{\tau A(t)})_{\tau \ge 0}$ generated by $A(t)$ does not depend on t. Using (H1) and (2.2) one also sees that the graph norms of the operators $A(t)$ are uniformly equivalent.

As mentioned in the introduction, condition (H2) does not depend on $p \in (1,\infty)$, see e.g. [14, Thm. 4.2], [21, Prop. 2.4], and the references therein. Since $A(s)$ generates a strongly continuous semigroup, the solution $u \in F_0$ of (2.1) is given by

$$u(t) = \int_0^t e^{(t-\tau)A(s)} f(\tau)\, d\tau,$$

see [3, III.1.3.1] or [14, §1]. Due to this representation the mapping $R_s : L^p([0,T],X) \to F_0$; $f \mapsto u$ is closed, and hence bounded. Therefore the solution of (2.1) satisfies

$$\|u\|_F := \|u\|_{L^p([0,T],X)} + \|u'\|_{L^p([0,T],X)} + \|u\|_{L^p([0,T],X_1)} \le c_{s,p}\, \|f\|_{L^p([0,T],X)} \tag{2.3}$$

for a constant $c_{s,p}$ depending on $s \in [0,T]$ and $p \in (1,\infty)$.

Let us fix some notation. For technical reasons, we set $A(t) := A(0)$ for $t \in [-1,0)$, cf. [27, §2, 3]. Notice that (H1) and (H2) still hold for this extension. Given an interval $I = [a,b] \subseteq [-1,T]$, we define the Banach spaces

$$F_I := W^{1,p}(I,X) \cap L^p(I,X_1) \quad \text{and} \quad F_{0,I} := \{f \in F_I : f(a) = 0\}$$

endowed with the canonical norm $\|\cdot\|_F$ and the operators

$$L_I f := -f' + A(\cdot)f,\ f \in F_I, \quad \text{and} \quad G_I f := -f' + A(\cdot)f,\ f \in F_{0,I},$$

on $L^p(I,X)$ equipped with the usual p–norm $\|\cdot\|_p$. In the case $I = [0,T]$ we omit the subscript I. Also, $W_0^{1,p}(I,X) := \{f \in W^{1,p}(I,X) : f(a) = 0\}$. We write $(X,X_1)_{\theta,q}$ for the real interpolation space between X and X_1 of exponent $0 < \theta < 1$ and parameter $1 \le q < \infty$, see e.g. [28, §1.3]. Then

$$\|x\|_{\theta,q} := \|x\| + \left(\int_0^\infty \|t^{1-\theta}\, A(0)e^{tA(0)}x\|^q\, \frac{dt}{t}\right)^{\frac{1}{q}}$$

is an equivalent norm on $(X,X_1)_{\theta,q}$ by, for instance, [28, 1.14.5]. In particular,

$$e^{(\cdot-a)A(0)}x \in F_I \quad \text{and} \quad \|e^{(\cdot-a)A(0)}x\|_F \le c\,\|x\|_{1-\frac{1}{p},p} \quad \text{if} \quad x \in (X,X_1)_{1-\frac{1}{p},p}. \tag{2.4}$$

Recall the continuous and dense embeddings

$$X_1 \overset{d}{\hookrightarrow} (X,X_1)_{\xi,p} \overset{d}{\hookrightarrow} (X,X_1)_{\xi,q} \overset{d}{\hookrightarrow} (X,X_1)_{\eta,1} \overset{d}{\hookrightarrow} X \tag{2.5}$$

for $0 < \eta < \xi < 1$ and $1 \le p \le q < \infty$, see e.g. [28, 1.3.3, 1.6.2]. We further need the continuous embedding

$$F_I \hookrightarrow C(I, (X, X_1)_{1-\frac{1}{p},p}) \tag{2.6}$$

where the corresponding norm estimate does not depend on I, [3, III.4.10.2].

Observe that in (H2) and (2.3) one can replace the interval $[0,T]$ by $I = [a,b] \subseteq [-1,T]$ with a possibly larger but I–independent constant, cf. [14, Thm. 2.5]. We start with a known perturbation argument, cf. [3, III.4.10.10] or [18].

Lemma 2.1. *Assume that (H1) and (H2) hold. Let $p \in (1,\infty)$ and $s \in [-1,T]$. Then there exists a constant δ (depending on p and s) such that the operator G_I defined on $F_{0,I}$ is invertible in $L^p(I,X)$ if $I = [a,b] \subseteq [-1,T]$ with $s \in I$ and $0 < b-a \le \delta$.*

Proof. Let $s \in I = [a,b] \subseteq [-1,T]$. By (H2) the operator $-\frac{d}{dt} + A(s)$ with domain $F_{0,I}$ is invertible in $L^p(I,X)$. Notice that we have

$$G_I f = [Id - (A(s) - A(\cdot))(-\tfrac{d}{dt} + A(s))^{-1}]\,(-\tfrac{d}{dt} + A(s))f$$

for $f \in F_{0,I}$. Further, (H1) and (2.3) yield

$$\|(A(s) - A(\cdot))(-\tfrac{d}{dt} + A(s))^{-1} f\|_p \le \sup_{r,r' \in I} \|A(r) - A(r')\|_{\mathcal{L}(X_1,X)}\, c_{s,p}\, \|f\|_p$$

for $f \in L^p([0,T],X)$. This easily implies the assertion. □

In order to show that G_I is a generator, we use the Yosida approximations $A_n(t) = nA(t)(n - A(t))^{-1}$ of $A(t)$ and the operator $G_{n,I} = -\frac{d}{dt} + A_n(\cdot)$ defined on $W_0^{1,p}(I,X)$. Since $A_n(\cdot) \in C([-1,T],\mathcal{L}(X))$, there is an evolution family $U_n(t,s)$, $-1 \le s \le t \le T$, such that

$$\tfrac{\partial}{\partial t} U_n(t,s) = A_n(t)U_n(t,s) \quad \text{and} \quad \tfrac{\partial}{\partial s} U_n(t,s) = -U_n(t,s)A_n(s)$$

in $\mathcal{L}(X)$ for $t \ge s$. This easily implies that $G_{n,I}$ generates the evolution semigroup $T_{n,I}(\cdot)$ on $L^p(I,X)$ corresponding to $U_n(t,s)$, cf. (1.8). It is further clear that

$$\lim_{n\to\infty} \|G_{n,I} f - G_I f\|_p = \lim_{n\to\infty} \|(A_n(\cdot) - A(\cdot))f\|_p = 0 \tag{2.7}$$

for $f \in F_{0,I}$. The next result will be proved in Section 5.

Lemma 2.2. *If (H1) holds, then $\|U_n(t,s)\| \le C$ for a constant $C \ge 0$, $n \in \mathbb{N}$, and $-1 \le s \le t \le T$. In particular, $\|T_{n,I}(t)\| \le C$ for $n \in \mathbb{N}$, $t \ge 0$, and intervals $I \subseteq [-1,T]$.*

Combining the above facts we now establish the existence of the evolution family $U(t,s)$.

Proposition 2.3. *Assume that (H1) and (H2) hold. Let $I = [a,b] \subseteq [0,T]$ be given as in Lemma 2.1. Then there is an evolution family $U(t,s)$, $0 \le s \le t \le T$, on X such that G_I generates the corresponding evolution semigroup*

$$(T_I(t)f)(s) = \begin{cases} U(s,s-t)f(s-t), & a \le s-t \le s \le b, \\ 0, & s-t < a \le s \le b, \end{cases}$$

on $L^p(I,X)$ and G_I has the inverse

$$(G_I^{-1} f)(t) = -\int_a^t U(t,s)f(s)\,ds, \qquad t \in I,\ f \in L^p(I,X).$$

Moreover, $T_n(t)$ converges strongly to $T(t)$ as $n \to \infty$, where $T_n(\cdot)$ and $T(\cdot)$ are the evolution semigroups on $L^p([0,T],X)$ induced by $U_n(t,s)$ and $U(t,s)$, respectively.

Proof. Let $[a,b] \subseteq [-1,T]$ be as in Lemma 2.1. Due to Lemma 2.1, Lemma 2.2, and (2.7), the Trotter–Kato theorem, [24, Thm. 3.4.5], shows that G_I generates a strongly continuous semigroup $T_I(\cdot)$ on $L^p(I,X)$ and $T_{n,I}(t)$ converges strongly to $T_I(t)$ as $n \to \infty$. Clearly, $F_{0,I}$ is dense in $C_0((a,b],X)$, and for $\varphi \in C_c^1(a,b]$ and $f \in F_{0,I}$ we have $\varphi f \in F_{0,I}$ and $G_I(\varphi f) = \varphi G_I f - \varphi' f$. By virtue of [27, Thm. 2.4], this implies that $T_I(\cdot)$ is an evolution semigroup given by an evolution family $U_I(t,s)$, $a < s \le t \le b$, on X. Using the convergence of $T_{n,I}(\cdot)$ to $T_I(\cdot)$, one easily sees that setting $U(t,s) := U_I(t,s)$ for $a < s \le t \le b$ defines an evolution family $U(t,s)$, $0 \le s \le t \le T$, on X. Since $T_I(t) = 0$ for large t, the inverse of G_I is given by

$$(G_I^{-1} f)(t) = -\int_0^\infty (T_I(\tau)f)(t)\,d\tau = -\int_a^t U(t,s)f(s)\,ds$$

for $t \in I$ and $f \in L^p(I,X)$. (Notice that the first equality holds for a.e. $t \in I$ due to e.g. [27, Lemma 3.2], but the functions on the left and right hand side are continuous in t.)

To show the last assertion, take $0 = t_0 < t_1 < \cdots < t_m = T$ and $0 < t < t_k - t_{k-1}$ such that the intervals $I_k = [t_{k-1}, t_k]$ and $I_k^t = [t_k - t, t_k + t]$ satisfy the assumptions of Lemma 2.1. For $f \in L^p([0,T],X)$, one obtains

$$\begin{aligned} \|T(t)f - T_n(t)f\|_p^p = &\sum_{k=1}^{m-1} \|(T_{I_k}(t) - T_{I_k,n}(t))\, f|I_k\|_p^p + \|(T_{I_k^t}(t) - T_{I_k^t,n}(t))\, f|I_k^t\|_p^p \\ &+ \|(T_{I_m}(t)f - T_{I_m,n}(t))\, f|I_m\|_p^p \end{aligned}$$

so that $T_n(t)f \to T(t)f$ as $n \to \infty$ for small $t \ge 0$. The semigroup property and the uniform boundedness of $T_n(t)$ then imply convergence for all $t \ge 0$. $\square$

The following lemma provides most of the required regularity properties of $U(t,s)$.

Lemma 2.4. *Assume that (H1) and (H2) hold. Then the operators $U(t,s)$, $0 \le s \le t \le T$, obtained in Proposition 2.3 also yield an evolution family on $(X,X_1)_{\theta,q}$ for $\theta \in (0,1)$ and $q \in [1,\infty)$. Moreover, the function $u = U(\cdot,s)x$ belongs to $F_{[s,T]}$, $L_{[s,T]}u = 0$, and u satisfies $\|u\|_F \le c\,\|x\|_{1-\frac{1}{p},p}$ for $x \in (X,X_1)_{1-\frac{1}{p},p}$, $p \in (1,\infty)$, $s \in [0,T)$, and a constant c not depending on s and x.*

Proof. (1) For $s \in [0,T)$ and $p \in (1,\infty)$, let $\delta = \delta(s,p)$ be given by Lemma 2.1. Set $I = [s,b] := [s, \min\{s+\delta, T\}]$. We define

$$\begin{aligned} f_n(t) &= U_n(t,s)x - e^{(t-s)A_n(0)}x, & f(t) &= U(t,s)x - e^{(t-s)A(0)}x, \\ g_n(t) &= (A_n(0) - A_n(t))e^{(t-s)A_n(0)}x, & g(t) &= (A(0) - A(t))e^{(t-s)A(0)}x, \end{aligned} \tag{2.8}$$

for $x \in X_1$, $t \in I$, and $n \in \mathbb{N}$. It is straightforward to see that

$$f_n(t) = -\int_s^t U_n(t,\tau)g_n(\tau)\,d\tau = (G_{n,I}^{-1}\, g_n)(t)$$

and that g_n converges to g in $L^p(I,X)$. Then (2.7), Lemma 2.2, and Proposition 2.3 imply that f_n tends in $L^p(I,X)$ to the function

$$I \ni t \mapsto (G_I^{-1} g)(t) = -\int_s^t U(t,\tau)g(\tau)\,d\tau.$$

On the other hand, we have $e^{tA_n(0)}x \to e^{tA(0)}x$ as $n \to \infty$. Proposition 2.3, Lemma 2.2 and the dominated convergence theorem yield

$$\int_0^T \int_0^{T-\tau} \|U_n(\tau+t,\tau)x - U(\tau+t,\tau)x\|^p \, dt \, d\tau = \int_0^T \|T_n(t)h - T(t)h\|_p^p \, dt \quad \longrightarrow \quad 0$$

as $n \to \infty$, where $h \equiv x$. Consequently, for all $x \in X_1$ and a.e. $s \in [0,T]$, there exists a subsequence such that $f_{n_k}(t)$ converges to $f(t)$ for a.e. $t \in [s,b]$, and hence, $f = G_I^{-1}$, i.e.,

$$U(t,s)x = e^{(t-s)A(0)}x - \int_s^t U(t,\tau)g(\tau)\, d\tau$$

for a.e. $t \in [s,b]$. By continuity these equations hold for all $s \in [0,T]$ and $t \in [s,b]$.

Next, let $x \in (X,X_1)_{1-\frac{1}{p},p}$ in (2.8). Approximating x in $(X,X_1)_{1-\frac{1}{p},p}$ by $x_k \in X_1$, one easily deduces that $f = G_I^{-1}g$ also holds for $x \in (X,X_1)_{1-\frac{1}{p},p}$. As a result, $u := U(\cdot,s)x = f + e^{(\cdot-s)A(0)}x$ belongs to F_I and

$$L_I u = g + (A(\cdot) - A(0))e^{(\cdot-s)A(0)}x = 0$$

for $x \in (X,X_1)_{1-\frac{1}{p},p}$. Moreover, (2.6), Lemma 2.1, (2.4), and (H1) imply the estimate

$$\begin{aligned}\|U(t,s)x\|_{1-\frac{1}{p},p} &\le c_1 \|U(\cdot,s)x\|_F \le c_1 (\|G_I^{-1}g\|_F + \|e^{(\cdot-s)A(0)}x\|_F) \\ &\le c_2 (\|g\|_p + \|x\|_{1-\frac{1}{p},p}) \le c_3 \|x\|_{1-\frac{1}{p},p}\end{aligned}$$

for $x \in (X,X_1)_{1-\frac{1}{p},p}$, $t \in I$, and constants c_k depending on I and p but not on t and x.

(2) Now cover $[0,T]$ by finitely many intervals of the above type. Using (2.6) and part (1), we see that $U(\cdot,s)x$ belongs to the kernel of $L_{[s,T]}$ and satisfies

$$\|U(t,s)x\|_{1-\frac{1}{p},p} \le c\,\|U(\cdot,s)x\|_F \le \tilde{c}\,\|x\|_{1-\frac{1}{p},p} \tag{2.9}$$

for $x \in (X,X_1)_{1-\frac{1}{p},p}$, $0 \le s \le t \le T$, and constants independent of t, s, and x. Finally, from (2.5), (2.9), and the strong continuity of $U(t,s)$ on X, we deduce the strong continuity of $U(t,s)$ on the spaces $(X,X_1)_{\theta,q}$, $\theta \in (0,1)$, $q \in [1,\infty)$, using the reiteration theorem [28, 1.10.2]. □

We can now prove our main result.

Theorem 2.5. *Assume that (H1) and (H2) hold. Let $p \in (1,\infty)$. Then, for $x \in (X,X_1)_{1-\frac{1}{p},p}$ and $f \in L^p([0,T],X)$, there exists a unique function $u \in W^{1,p}([0,T],X) \cap L^p([0,T],X_1)$ satisfying*

$$\begin{aligned} u'(t) &= A(t)u(t) + f(t) \qquad \text{for a.e. } 0 \le t \le T, \\ u(0) &= x, \end{aligned} \tag{2.10}$$

and we have the estimate

$$\|u\|_{L^p([0,T],X)} + \|u\|_{L^p([0,T],X_1)} + \|u'\|_{L^p([0,T],X)} \le c_p (\|x\|_{1-\frac{1}{p},p} + \|f\|_{L^p([0,T],X)}) \tag{2.11}$$

for a constant c_p independent of x and f. The solution u is given by

$$u(t) = U(t,0)x + \int_0^t U(t,s)f(s)\, ds, \qquad 0 \le t \le T, \tag{2.12}$$

for an evolution family $U(t,s)$, $0 \le s \le t \le T$, on X. If $x \in X$ and $f \in L^p([0,T],X)$, then (2.12) gives the unique solution of (2.10) in the space $F_{loc} := C([0,T],X) \cap$

$W^{1,p}_{loc}((0,T],X) \cap L^p_{loc}((0,T],X_1)$. *The operators $U(t,s)$ also yield an evolution family on the real interpolation space $(X,X_1)_{\theta,q}$ for $\theta \in (0,1)$ and $q \in [1,\infty)$. Moreover, $U(t,s)$ maps X into $(X,X_1)_{1-\frac{1}{p},p}$ and there is a constant $\tilde{c}_p$ such that*

$$\|U(t,s)x\|_{1-\frac{1}{p},p} \le \tilde{c}_p\,(t-s)^{\frac{1}{p}-1}\,\|x\| \tag{2.13}$$

for $1 < p < \infty$ and $0 \le s < t \le T$. Finally, the evolution semigroup on $L^p([0,T],X)$ corresponding to $U(t,s)$ is generated by the operator $Gf = -f' + A(\cdot)f$ with $D(G) = W^{1,p}_0([0,T],X) \cap L^p([0,T],X)$.

Proof. Let $U(t,s)$, $0 \le s \le t \le T$, be the evolution family obtained in Proposition 2.3. Fix $f \in L^p([0,T],X)$, $x \in (X,X_1)_{1-\frac{1}{p},p}$, and $p \in (1,\infty)$. Cover $[0,T]$ by finitely many intervals $I_k = [t_{k-1},t_k]$ with $t_0 = 0$ and $t_n = T$ such that the conclusion of Proposition 2.3 holds for I_k. Let $G_k = G_{I_k}$, $L_k = L_{I_k}$, and $g_k = g|I_k$ for $g \in L^p([0,T],X)$. We define on $L^p(I_k,X)$, $k = 1,\cdots,n$, the functions

$$v_1 = -G_1^{-1}f_1\,,\quad v_2 = -G_2^{-1}f_2 + U(\cdot,t_1)v_1(t_1),\quad \cdots\quad, \; v_n = -G_n^{-1}f_n + U(\cdot,t_{n-1})v_{n-1}(t_{n-1})\,.$$

These functions belong to F_{I_k} and $L_kv_k = -f_k$ by virtue of Proposition 2.3, Lemma 2.4, and (2.6). Therefore, the function u given by

$$u(t) := U(t,0)x + v_k(t) = U(t,0)x + \int_0^t U(t,s)f(s)\,ds, \quad t \in I_k,$$

belongs to F and solves (2.10). To show uniqueness, let $w \in F_0$ satisfy $w'(t) = A(t)w(t)$. Then $w_1 \in D(G_1)$ and $G_1w_1 = 0$ so that $w_1 = 0$ since G_1 is invertible. Proceeding to the right, we derive $w = 0$.

As a result, the operator $G = G_{[0,T]}$ with domain $D(G) = F_0$ in $L^p([0,T],X)$ is bijective and has the bounded inverse given by $Rf(t) = -\int_0^t U(t,s)f(s)\,ds$. In particular, G is closed and its graph norm is equivalent to $\|\cdot\|_F$ because of the inverse mapping theorem. Together with Lemma 2.4 this implies the estimate (2.11). As in the proof of Proposition 2.3 one sees that R is also the inverse of the generator of the evolution semigroup $T(\cdot)$ on $L^p([0,T],X)$ induced by $U(t,s)$. So the last assertion follows.

Given $0 \le s < t \le T$, take the absolutely continuous function $\varphi = \varphi_{t,s}$ which is equal to 0 on $[0,s]$ and to 1 on $[t,T]$ and is linear on (s,t). Notice that $\|\varphi'\|_p = (t-s)^{\frac{1}{p}-1}$. Define $u = \varphi(\cdot)\,U(\cdot,s)x$ for $x \in X$, where we let $U(r,s) := 0$ for $0 \le r < s$. Then, $(T(h)u)\,(r) = \varphi(r-h)\,U(r,s)x$ which implies

$$u \in D(G) = F_0 \quad \text{and} \quad Gu = -\varphi'(\cdot)\,U(\cdot,s)x. \tag{2.14}$$

Using (2.6) and (2.11), we now obtain

$$\|U(t,s)x\|_{1-\frac{1}{p},p} \le c_1\,\|u\|_F \le c_2\,\|\varphi'(\cdot)\,U(\cdot,s)x\|_p \le c_3\,(t-s)^{\frac{1}{p}-1}\,\|x\|$$

for constants c_k depending on p. From (2.14) we also derive that $U(\cdot,0)x$ belongs to the kernel of $L_{[\varepsilon,T]}$ for all $\varepsilon > 0$. So (2.12) gives a solution of (2.10) in F_{loc} if $x \in X$. It remains to show uniqueness of such solutions. Let $w \in F_{loc}$ satisfy (2.10) with $x = 0$ and $f = 0$. Then the restriction $w|[\varepsilon,T] \in F_{[\varepsilon,T]}$ solves (2.10) with $f = 0$ and initial value $w(\varepsilon)$ at initial time ε. The first part of the proof implies that $w(t) = U(t,\varepsilon)w(\varepsilon)$ for all $t \ge \varepsilon > 0$. Hence, $w = 0$ by the continuity of w. □

3. A perturbation result

In this section we consider the perturbed problem

$$\begin{aligned} u'(t) &= A(t)u(t) + B(t)u(t) + f(t) \qquad \text{for a.e. } 0 \le t \le T, \\ u(0) &= x, \end{aligned} \tag{3.1}$$

where we suppose that the operators $A(t)$ satisfy the conditions (H1) and (H2). For the perturbations $B(t)$ we require that

(B) $B(t) \in \mathcal{L}((X,X_1)_{1-\frac{1}{q},q}, X)$ for some $q \in [1,\infty)$, $B(\cdot)x$ is measurable for $x \in (X,X_1)_{1-\frac{1}{q},q}$, and $\|B(t)\|_{\mathcal{L}((X,X_1)_{1-\frac{1}{q},q},X)} \le \beta(t)$ for a.e. $t \in [0,T]$ and a function $\beta \in L^r[0,T]$ with $r \in (q,\infty]$.

Here we let $(X,X_1)_{0,1} := X$. Condition (B) can be verified for lower order terms of parabolic partial differential equations with coefficients belonging to suitable Lebesgue spaces, cf. Section 4. We now show that under these assumptions the problem (3.1) has essentially the same properties as (2.10).

Theorem 3.1. *Assume that (H1), (H2), and (B) hold. Let $p \in (1, \frac{r}{q})$. Then, for $x \in (X,X_1)_{1-\frac{1}{p},p}$ and $f \in L^p([0,T],X)$, there exists a unique function $u \in W^{1,p}([0,T],X) \cap L^p([0,T],X_1)$ satisfying (3.1) and the estimate (2.11). Moreover, u is given by*

$$u(t) = U_B(t,0)x + \int_0^t U_B(t,s)f(s)\,ds, \qquad 0 \le t \le T, \tag{3.2}$$

for an evolution family $U_B(t,s)$, $0 \le s \le t \le T$, on X. If $x \in X$ and $f \in L^p([0,T],X)$, then (3.2) gives the unique solution of (3.1) in the space $F_{loc} = C([0,T],X) \cap W^{1,p}_{loc}((0,T],X) \cap L^p_{loc}((0,T],X_1)$. The operators $U_B(t,s)$ also yield an evolution family on the real interpolation space $(X,X_1)_{\theta,\tau}$ for $\theta \in (0, 1-\frac{q}{r})$ and $\tau \in [1,\infty)$. Moreover, $U_B(t,s)$ maps X into $(X,X_1)_{1-\frac{1}{p},p}$ and there is a constant $\tilde{c}_p$ such that

$$\|U_B(t,s)x\|_{1-\frac{1}{p},p} \le \tilde{c}_p\,(t-s)^{\frac{1}{p}-1}\,\|x\|$$

for $x \in X$, $1 < p < \frac{r}{q}$, and $0 \le s < t \le T$. Finally, the evolution semigroup on $L^p([0,T],X)$ corresponding to $U_B(t,s)$ is generated by the operator $G_B f = -f' + A(\cdot)f + B(\cdot)f$ with $D(G_B) = W^{1,p}_0([0,T],X) \cap L^p([0,T],X_1)$.

Proof. (1) Let $p \in (1,\frac{r}{q})$ and $f \in F_0 = W^{1,p}_0([0,T],X) \cap L^p([0,T],X_1)$. Set $\frac{1}{q'} = 1 - \frac{1}{q}$. Since $\frac{1}{q'} < 1 - \frac{p}{r}$, we can estimate

$$\begin{aligned}
\|B(\cdot)f\|_p &\le \left(\int_0^T \beta(t)^p\,\|f(t)\|^p_{\frac{1}{q'},q}\,dt\right)^{\frac{1}{p}} \\
&\le \|\beta\|_r \left(\int_0^T \|f(t)\|^{p(1-\frac{p}{r})^{-1}}_{\frac{1}{q'},q}\,dt\right)^{\frac{1}{p}(1-\frac{p}{r})} \\
&\le \|\beta\|_r \left(\int_0^T \|f(t)\|_X^{\frac{p}{q}(1-\frac{p}{r})^{-1}}\,\|f(t)\|_{X_1}^{\frac{p}{q'}(1-\frac{p}{r})^{-1}}\,dt\right)^{\frac{1}{p}(1-\frac{p}{r})} \\
&\le \|\beta\|_r\,\|f\|^{\frac{1}{q}}_{\frac{p}{q}(\frac{1}{q}-\frac{p}{r})^{-1}}\,\|f\|^{\frac{1}{q'}}_{L^p([0,T],X_1)} \\
&\le \|\beta\|_r\,\|f\|_\infty^{\frac{p}{r}}\,\|f\|_p^{\frac{1}{q}-\frac{p}{r}}\,\|f\|^{\frac{1}{q'}}_{L^p([0,T],X_1)}\,.
\end{aligned}$$

Here we have used Hölder's inequality in the second and fourth line, interpolation for the third estimate, and the Riesz-Thorin convexity theorem for the last one. Due to $F \hookrightarrow L^\infty([0,T],X)$ and Young's inequality, we further obtain

$$\begin{aligned}
\|B(\cdot)f\|_p &\le c_1\,\|\beta\|_r \left(\varepsilon^{-1}\,\|f\|_p^{\frac{1}{q}-\frac{p}{r}}\,\varepsilon\,\|f\|_F^{1-(\frac{1}{q}-\frac{p}{r})}\right) \\
&\le c_1\,\|\beta\|_r \left(\alpha\,\varepsilon^{-\frac{1}{\alpha}}\,\|f\|_p + (1-\alpha)\,\varepsilon^{\frac{1}{1-\alpha}}\,\|f\|_F\right) \\
&\le c_2\varepsilon^{-\frac{1}{\alpha}}\,\|f\|_p + c_3\,\varepsilon^{\frac{1}{1-\alpha}}\,\|f\|_F
\end{aligned} \tag{3.3}$$

for constants c_k, $\alpha := \frac{1}{q} - \frac{p}{r} \in (0,1)$, and all $\varepsilon > 0$. Let $G = -\frac{d}{dt} + A(\cdot)$ with domain F_0 in $L^p([0,T],X)$, cf. Theorem 2.5. Then (3.3) and (2.11) yield

$$\begin{aligned}
\|B(\cdot)(\lambda - G)^{-1}f\|_p &\le c_2\varepsilon^{-\frac{1}{\alpha}}\,\|(\lambda-G)^{-1}f\|_p + c_3\,\varepsilon^{\frac{1}{1-\alpha}}\,\|(\lambda-G)^{-1}f\|_F \\
&\le c_4\,(\varepsilon^{-\frac{1}{\alpha}}\,\lambda^{-1} + \varepsilon^{\frac{1}{1-\alpha}})\,\|f\|_p \le \frac{1}{2}\,\|f\|_p
\end{aligned}$$

for $f \in L^p([0,T],X)$, where we have chosen first a sufficiently small $\varepsilon > 0$ and then a sufficiently large $\lambda > 0$ to derive the last inequality. As a result, the operator $\lambda-(G+B(\cdot))$ with domain F_0 is invertible in $L^p([0,T],X)$ for $1 < p < \frac{r}{q}$ and large λ.

(2) Now let $1 - \frac{1}{q} + \frac{1}{r} < \frac{1}{p} < 1$. (Notice that $\frac{q}{r} \le 1 - \frac{1}{q} + \frac{1}{r}$.) As in the preceding section we set $A(t) = A(0)$, $B(t) = 0$, and $\beta(t) = 0$ for $t \in [-1,0)$. Using (B) and (2.13) we estimate

$$\begin{aligned}
\int_s^{s+t} \|B(\tau)U(\tau,s)x\|^p\,d\tau &\le c_1 \int_s^{s+t} \beta(\tau)^p\,(\tau-s)^{-\frac{p}{q'}}\,\|x\|^p\,d\tau \\
&\le c_1\,\|\beta\|_r^p \left(\int_0^t \tau^{-\frac{p}{q'}(1-\frac{p}{r})^{-1}}\,d\tau\right)^{1-\frac{p}{r}} \|x\|^p \\
&= c_2\,t^{1-\frac{p}{r}-\frac{p}{q'}}\,\|x\|^p =: c(t)^p\,\|x\|^p
\end{aligned}$$

for $s, s{+}t \in [-1,T]$ and $x \in X$. Hence, $c(t) < 1$ for sufficiently small $t > 0$. Moreover, $B(\cdot)$ is G-bounded by (3.3). Therefore the non–autonomous Miyadera perturbation theorem [27, Thm. 3.4] shows the existence of an evolution family $U_B(t,s)$, $0 \le s \le t \le T$, on X such that the corresponding evolution semigroup on $L^p([0,T],X)$ is generated by $G{+}B(\cdot)$ with domain F_0. In addition, [27, Thm. 3.4] gives

$$U_B(t,s)x = U(t,s)x + \int_s^t U_B(t,\tau)B(\tau)U(\tau,s)x\,d\tau \tag{3.4}$$

for $0 \le s \le t \le T$ and $x \in \bigcup_{-1<r<s} U(s,r)X$.

(3) The evolution family $U_B(t,s)$ induces an evolution semigroup on all spaces $L^p([0,T],X)$, $1 \le p < \infty$. Denote the corresponding generator by G_B^p. Due to parts (1) and (2) we have $G_B^{p_0} = -\frac{d}{dt} + A(\cdot) + B(\cdot)$ with $D(G_B^{p_0}) = F_0^{p_0}$ if $1 - \frac{1}{q} + \frac{1}{r} < \frac{1}{p_0} < 1$. (For a moment we express the dependence on p explicitely.) Notice that G_B^p is the part of $G_B^{p_0}$ in $L^p([0,T],X)$ for $p \ge p_0$. Take $p \in (1, \frac{r}{q})$. Part (1) implies that G_B^p extends $(-\frac{d}{dt} + A(\cdot) + B(\cdot), F_0^p)$ and that there exists $\lambda \in \rho(G_B^p) \cap \rho(-\frac{d}{dt} + A(\cdot) + B(\cdot))$. Hence, $G_B^p = -\frac{d}{dt} + A(\cdot) + B(\cdot)$ on F_0^p for $1 < p < \frac{r}{q}$. Also, the graph norm of G_B^p is equivalent to $\|\cdot\|_F$ by virtue of (3.3) and the inverse mapping theorem. As in the proof of Lemma 2.4

one sees that G_B has the inverse

$$\left(G_B^{-1}f\right)(t) = -\int_0^t U_B(t,s)f(s)\,ds.$$

Moreover, $u := U_B(\cdot,s)x$ belongs to $F^p_{[s+\varepsilon,T]}$, $-u' + A(\cdot)u + B(\cdot)u = 0$, and

$$\|U_B(t,s)x\|_{1-\frac{1}{p},p} \le c_p\,(t-s)^{\frac{1}{p}-1}\,\|x\|$$

for $x \in X$, $0 \le s < t \le T$, a constant c_p, and $1 < p < \frac{r}{q}$, compare the proof of Theorem 2.5.

It remains to study the behaviour of $U_B(t,s)$ on real interpolation spaces. Let $g(t) = (A(0) - A(t))e^{(t-s)A(0)}x$ for $0 \le s < t \le T$ and $x \in (X,X_1)_{1-\frac{1}{p},p}$. Then the function $u(t) = e^{(t-s)A(0)}x - (G^{-1}_{[s,T]}g)(t)$ belongs to the kernel of $L_{[s,T]}$ by (2.4). Hence, $u(t) = U(t,s)x$ due to Theorem 2.5. From (3.3), (H1), and (2.4) we deduce

$$\|B(\cdot)G^{-1}_{[s,T]}g\|_p \le c_1\,\|g\|_p \le c_2\,\|x\|_{1-\frac{1}{p},p}.$$

One further obtains

$$\|e^{tA(0)}x\|_{1-\frac{1}{q},q} \le c_3\,t^{\frac{1}{q}-\frac{1}{p}}\,\|x\|_{1-\frac{1}{p},q} \le c_4\,t^{\frac{1}{q}-\frac{1}{p}}\,\|x\|_{1-\frac{1}{p},p}$$

for $1 < p < q < \infty$, $x \in (X,X_1)_{1-\frac{1}{p},p}$, and $0 < t \le T$ by [22, Prop. 2.2.9] and (2.5). Thus,

$$\begin{aligned}
\|B(\cdot)e^{(\cdot-s)A(0)}x\|_p &\le c_4\left(\int_s^T \beta(t)^p\,(t-s)^{p(\frac{1}{q}-\frac{1}{p})}\,\|x\|^p_{1-\frac{1}{p},p}\,dt\right)^{\frac{1}{p}}\\
&\le c_4\,\|\beta\|_r\,\|x\|_{1-\frac{1}{p},p}\left(\int_0^T t^{(\frac{p}{q}-1)(1-\frac{p}{r})^{-1}}\,dt\right)^{\frac{1}{p}-\frac{1}{r}}\\
&= c_5\,\|x\|_{1-\frac{1}{p},p}
\end{aligned}$$

for $p < q < r$. The case $1 \le q \le p$ can be handled similarly. Consequently, $\|B(\cdot)U(\cdot,s)x\|_p \le c_6\,\|x\|_{1-\frac{1}{p},p}$. A straightforward approximation argument then shows that (3.4) holds for every $x \in (X,X_1)_{1-\frac{1}{p},p}$. We can rewrite this identity as

$$U_B(t,s)x = U(t,s)x - \left(G^{-1}_{B,[s,T]}B(\cdot)U(\cdot,s)x\right)(t).$$

As a result, $U_B(\cdot,s)x$ belongs to the kernel of $(L+B(\cdot))^p_{[s,T]}$ if $1 < p < \frac{r}{q}$ and $x \in (X,X_1)_{1-\frac{1}{p},p}$ and $U_B(t,s)$ is uniformly bounded on $(X,X_1)_{1-\frac{1}{p},p}$. Now one can complete the proof as in the case of Theorem 2.5. $\square$

Remark 3.2. *Notice that Theorem 3.1 yields maximal regularity of type L^p not for all values of $p \in (1,\infty)$ if $B(\cdot)$ is unbounded in time, i.e., $r < \infty$. The following simple example shows that our upper bound for p is almost optimal if $q = 1$. Take $X = \mathbb{C}$, $A(t) = -1$, $B(t) = (1-t)^{-\frac{1}{2}}$ for $t \in [0,1]$ so that (B) holds for $q = 1$ und each $r \in (1,2)$. Let u be the solution of* (3.1) *for $x = 0$ and $f \equiv 1$. Then, $u' \in L^p[0,T]$ if and only if $1 \le p < 2$. It is known that maximal L^p–regularity does not depend on p in the autonomous case, see e.g.* [14, 21], *and in the non-autonomous case for more regular $A(t)$,* [20].

4. Applications to parabolic initial boundary value problems

To obtain an idea of the strength of our main results, consider the parabolic system

$$\begin{aligned} \partial_t u(t,x) &= -\mathcal{A}(t,x)u(t,x) + \mathcal{B}(t,x)u(t,x) + f(t,x), \qquad 0 \le t \le T,\ x \in \mathbb{R}^n, \\ u(0,x) &= u_0(x), \qquad x \in \mathbb{R}^n, \end{aligned} \tag{4.1}$$

in $X = L^q(\mathbb{R}^n, \mathbb{C}^N)$, $1 < q < \infty$, where

$$\mathcal{A}(t,x) = (-1)^m \sum_{|\alpha|=2m} a_\alpha(t,x)\, D^\alpha \quad \text{and} \quad \mathcal{B}(t,x) = \sum_{|\beta|<2m} b_\beta(t,x)\, D^\beta$$

for some $m, n, N \in \mathbb{N}$. Here $D^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ denotes a spatial derivative in multi-index notation. The coefficients of the principal part $\mathcal{A}(t,x)$ are assumed to be of class

$$a_\alpha \in BUC([0,T] \times \mathbb{R}^n, \mathbb{C}^{N\times N})$$

and to be uniformly (K,θ)–elliptic in the sense of [4, 16]. This means that there are constants $\theta \in [0, \frac{\pi}{2})$ and $K \ge 1$ such that

$$\sum_{|\alpha|=2m} \|a_\alpha\|_\infty \le K, \quad \sigma(\tilde{\mathcal{A}}(t,x,\xi)) \subset \Sigma_\theta \setminus \{0\}, \quad \text{and} \quad |\tilde{\mathcal{A}}(t,x,\xi)^{-1}| \le K$$

for $t \in [0,T]$, $x, \xi \in \mathbb{R}^n$ with $|\xi| = 1$, where

$$\tilde{\mathcal{A}}(t,x,\xi) := \sum_{|\alpha|=2m} a_\alpha(t,x)\, \xi^\alpha$$

denotes the principal symbol of $\mathcal{A}$. We define the L^q–realization $A(t)$ of $\mathcal{A}(t,x)$ by

$$(A(t)u)(x) := -\mathcal{A}(t,x)u(x), \quad x \in \mathbb{R}^n,\ t \in [0,T], \qquad D(A(t)) := W^{2m,q}(\mathbb{R}^n, \mathbb{C}^N).$$

Let $\theta < \theta' < \frac{\pi}{2}$. Due to [16, Thm. 6.1], there is a constant $w > 0$ such that $A(t) - w$ generates an analytic semigroup of negative type and $w - A(t)$ admits bounded imaginary powers with $\|(w - A(t))^{is}\| \le Ne^{\theta'|s|}$ for $s \in \mathbb{R}$ and a constant $N \ge 1$, see also [4, §9]. By the Dore–Venni Theorem [15, Thm. 3.2], we then obtain maximal regularity of type L^p for the operator $A(t)$, i.e., (H2) holds. Hypothesis (H1) is easily verified. Observe that no extra regularity of the coefficients $a_\alpha(t,x)$ besides BUC is needed here.

Thus we may apply Theorem 2.5 to conclude that for each fixed $p, q \in (1,\infty)$ the problem (4.1) with $\mathcal{B} = 0$ has maximal regularity of type $L^p - L^q$ as well. More precisely, for each $f \in L^p([0,T], L^q(\mathbb{R}^m, \mathbb{C}^N))$ and each $u_0 \in (X, X_1)_{1-\frac{1}{p},p} \cong B_{q,p}^{2m(1-\frac{1}{p})}(\mathbb{R}^n, \mathbb{C}^N)$, a Besov space (see Remark 4 in [28, 2.4.2]), there is a unique solution

$$u \in W^{1,p}([0,T], L^q(\mathbb{R}^n, \mathbb{C}^N)) \cap L^p([0,T], W^{2m,q}(\mathbb{R}^n, \mathbb{C}^N))$$

of (4.1) and estimate (2.11) is valid. Moreover, there is an evolution family $U(t,s)$ on $X = L^q(\mathbb{R}^n, \mathbb{C}^N)$ for (4.1) which satisfies the interpolation estimate (2.13) and for each $u_0 \in X$ and $f \in L^p([0,T], X)$ there is unique solution

$$u \in C([0,T], X) \cap W^{1,p}_{loc}((0,T], X) \cap L^p_{loc}((0,T], W^{2m,q}(\mathbb{R}^n, \mathbb{C}^N))$$

of (4.1).

To include lower order terms, i.e. $\mathcal{B} \ne 0$, we may use Theorem 3.1. On $L^q(\mathbb{R}^n, \mathbb{C}^N)$ we define

$$(B(t)u)(x) := \mathcal{B}(t,x)u(x), \quad x \in \mathbb{R}^n,\ t \in [0,T], \qquad D(B(t)) := W^{2m,q}(\mathbb{R}^n, \mathbb{C}^N),$$

assuming that, for given $p, q \in (1,\infty)$, the coefficients belong to class

$$b_\beta \in L^{r_k}([0,T], L^{s_k}(\mathbb{R}^n, \mathbb{C}^N))$$

for $|\beta| = k < 2m$, where $r_k, s_k \in (1,\infty]$ satisfy

$$\frac{p}{r_k} + \frac{n}{2ms_k} + \frac{k}{2m} < 1, \quad r_k > p, \quad s_k > q.$$

Set $\sigma_k := s_k/q$ and $\frac{1}{\sigma_k'} := 1 - \frac{1}{\sigma_k}$ and choose $\rho_k > 1$ such that

$$1 - \frac{n}{2mq\sigma_k} - \frac{k}{2m} > \frac{1}{\rho_k} > \frac{p}{r_k}.$$

By 2.3.2(4a), 2.3.3, and 2.8.1(12) in [28] we have the continuous embedding

$$B_{q,\rho_k}^{2m(1-\frac{1}{\rho_k})}(\mathbb{R}^n, \mathbb{C}^N) \hookrightarrow W^{k,q\sigma_k'}(\mathbb{R}^n, \mathbb{C}^N)$$

which yields, for $|\beta| = k$ and $u \in (X, X_1)_{1-\frac{1}{\rho_k},\rho_k}$,

$$\|b_\beta(t,\cdot)\, D^\beta u\|_{L^q} \le \|b_\beta(t,\cdot)\|_{L^{q\sigma_k}} \|u\|_{W^{k,q\sigma_k'}} \le c\, \|b_\beta(t,\cdot)\|_{L^{s_k}} \|u\|_{1-\frac{1}{\rho_k},\rho_k}.$$

As a result, the k–order part of $B(t)$ satisfies (B) with q replaced by ρ_k and r by r_k. Since $1 < p < p_0 := \min\{\frac{r_k}{\rho_k} : k = 0, \cdots, 2m-1\}$ by assumption, a slight variant of Theorem 3.1 applies, and we obtain again maximal regularity of type $L^p - L^q$ for $1 < p < p_0$ and the existence of an evolution family $U(t,s)$ on $X = L^q(\mathbb{R}^n, \mathbb{C}^N)$.

As another application of our main results, consider a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$, a differential operator

$$\mathcal{A}(t,x) = -\sum_{k,l=1}^n a_{kl}(t,x)\, \partial_k \partial_l + \sum_{k=1}^n a_k(t,x)\, \partial_k + a_0(t,x),$$

and a boundary operator

$$\mathcal{B}(x) = \sum_{k=1}^n b_k(x)\, \partial_k + b_0(x).$$

Here we assume that

$$a_{kl}, a_k, a_0 \in C([0,T] \times \overline{\Omega}, \mathbb{R}), \quad a_{kl}(t,\cdot), a_k(t,\cdot), a_0(\cdot) \in C^\rho(\overline{\Omega}), \quad b_k, b_0 \in C^{1+\rho}(\partial\Omega, \mathbb{R}),$$

that $\mathcal{A}(t,x)$ is uniformly elliptic, i.e.,

$$\sum_{k,l=1}^n a_{kl}(t,x)\, \xi_k \xi_l \ge c\, |\xi|^2$$

for all $x \in \Omega$, $0 \le t \le T$, $\xi \in \mathbb{R}^n$, and a constant $c > 0$, and that

$$\sum_{k=1}^n b_k(x)\, \nu_k(x) \ge c_0 > 0, \quad x \in \partial\Omega,$$

where $\nu(x)$ denotes the outer normal at $x \in \partial\Omega$. Then the L^q–realization $A(t)$ of $\mathcal{A}$ with boundary conditions $\mathcal{B} = 0$ is given by

$$(A(t)u)(x) := -\mathcal{A}(t,x)u(x), \quad x \in \Omega,\ t \in [0,T],$$
$$D(A(t)) := W_{\mathcal{B}}^{2m,q} = \{u \in W^{2,q}(\Omega) : \mathcal{B}u = 0\}.$$

It is wellknown that $A(t)$ generates an analytic semigroup on $X = L^q(\Omega)$, cf. [22, Thm. 3.1.3], which satisfies Gaussian bounds, see [17, Thm. VI.3.1]. Therefore hypothesis (H2) holds due to [21, Thm. 3.1]. Condition (H1) is easy to check, but notice that we have to restrict the boundary operator to the time independent case since otherwise $D(A(t))$ would depend on t. As a consequence of Theorem 2.5, the parabolic second order initial boundary value problem

$$\begin{aligned} \partial_t u(t,x) &= -\mathcal{A}(t,x)u(t,x) + f(t,x), \qquad 0 \le t \le T,\ x \in \Omega, \\ \mathcal{B}(x)u(t,x) &= 0, \qquad 0 \le t \le T,\ x \in \partial\Omega, \\ u(0,x) &= u_0(x), \qquad x \in \Omega, \end{aligned}$$

has maximal regularity of type $L^p - L^q$, and it exists the corresponding evolution family $U(t,s)$ on $L^q(\Omega)$. Observe that by means of the perturbation result Theorem 3.1 the regularity of the lower order terms a_k, a_0 can be relaxed considerably, but we are not going into details here.

5. Proof of Lemma 2.2

Throughout this section we assume that condition (H1) holds. Recall that (H1) implies the uniform resolvent estimate (2.2) and that the graph norms of $A(t)$ are uniformly equivalent. In this section we want to show uniform boundedness of the evolution families $U_n(t,s)$ solving the Cauchy problem for the Yosida approximations $A_n(t) = nA(t)(n - A(t))^{-1}$. To that purpose, we use the integrals $\int_s^t A(\tau)x\,d\tau$ defined for $x \in X_1$ and $0 \le s < t \le T$. Set $\omega(t) := \sup_{|s-r|\le t} \|A(s) - A(r)\|_{\mathcal{L}(X_1,X)}$.

Lemma 5.1. *Assume that (H1) holds. Let $0 < t \le \delta$ with $\omega(\delta) \le \frac{1}{2M}$ and $0 \le s - t < s \le T$. Then the operator $\frac{1}{t}\int_{s-t}^s A(\tau)\,d\tau$ defined on X_1 generates an analytic semigroup on X satisfying (2.2) with constants ϕ and $2M$. Moreover,*

$$\left\| \left(\lambda - \frac{1}{t}\int_{s-t}^s A(\tau)\,d\tau\right)^{-1} - \left(\lambda - \frac{1}{t}\int_{r-t}^r A(\tau)\,d\tau\right)^{-1} \right\| \le \frac{4M^2}{1+|\lambda|}\,\omega(|s-r|)$$

for $\lambda \in \Sigma_\phi$ and $s-t, r-t, s, r \in [0,T]$.

Proof. Notice that (H1) and (2.2) yield, for $x \in X_1$ and $\lambda \in \Sigma_\phi$,

$$\lambda - \frac{1}{t}\int_{s-t}^s A(\tau)x\,d\tau = \left(Id - \frac{1}{t}\int_{s-t}^s (A(\tau) - A(s))(\lambda - A(s))^{-1}\,d\tau\right)(\lambda - A(s))x,$$

$$\left\| \frac{1}{t}\int_{s-t}^s (A(\tau) - A(s))(\lambda - A(s))^{-1}x\,d\tau \right\| \le M\omega(t)\,\|x\| \le \frac{1}{2}\,\|x\|.$$

This implies the first assertion. The proof of the second one is straightforward. ☐

In the sequel, let $0 \le s - t < s \le T$ and $0 < t \le \delta$ with $\omega(\delta) \le \frac{1}{2M}$. We further introduce the Yosida approximations

$$B_n(s,t) = \frac{n}{t}\int_{s-t}^s A(\tau)\,d\tau\,\left(n - \frac{1}{t}\int_{s-t}^s A(\tau)\,d\tau\right)^{-1} = n^2\left(n - \frac{1}{t}\int_{s-t}^s A(\tau)\,d\tau\right)^{-1} - nId,$$

$n \in \mathbb{N}$, of the above integrals. Lemma 5.1 implies that

$$\|e^{\tau B_n(s,t)}\| \le K \qquad \text{and} \qquad \|B_n(s,t) - B_n(r,t)\| \le 4nM^2\,\omega(|s-r|) \tag{5.1}$$

for $n \in \mathbb{N}$ and a constant K. We also need the following properties.

Lemma 5.2. *Let $0 < t \le \delta$, $n \in \mathbb{N}$, $s, r, s-t, r-t \in [0,T]$, and $\tau \ge 0$. Then*

1. $\|\exp(\tau t B_n(s,t)) - \exp(\tau t B_n(r,t))\| \le 4nM^2K^2\,\tau t\,\omega(|s-r|)$ *and*
2. $[\frac{d}{dt}\exp(tB_n(s,t))]_{|t=0} = A_n(s)$ *in $\mathcal{L}(X)$ and uniformly in s.*

Proof. The first assertion follows from (5.1) and

$$\begin{aligned} e^{\tau t B_n(s,t)} - e^{\tau t B_n(r,t)} &= \int_0^{\tau t} \frac{\partial}{\partial\sigma}\left[e^{\sigma B_n(s,t)}\, e^{(\tau t-\sigma)B_n(r,t)}\right] d\sigma \\ &= \int_0^{\tau t} e^{\sigma B_n(s,t)}\,[B_n(s,t) - B_n(r,t)]\, e^{(\tau t-\sigma)B_n(r,t)}\, d\sigma. \end{aligned}$$

To show the second assertion, we observe that

$$\|B_n(s,t)\| \le n2M \sup_{0\le t\le T} \|A(t)\|_{\mathcal{L}(X_1,X)} =: cn$$

Together with (5.1), this yields

$$\|e^{\alpha B_n(s,t)} - Id\| \le \int_0^1 \|\alpha\, B_n(s,t)\, e^{\tau\alpha B_n(s,t)}\|\, d\tau \le ncK\alpha \tag{5.2}$$

for $\alpha > 0$. Using the identity

$$\frac{1}{t}\,(e^{tB_n(s,t)} - Id) - A_n(s) = \int_0^1 (B_n(s,t)\, e^{\tau t B_n(s,t)} - A_n(s))\, d\tau,$$

we infer from (5.1) and (5.2) that

$$\begin{aligned} &\|\tfrac{1}{t}\,(e^{tB_n(s,t)} - Id) - A_n(s)\| \\ &\le \int_0^1 \|e^{\tau t B_n(s,t)}\|\,\|B_n(s,t) - A_n(s)\|\, d\tau + \int_0^1 \|e^{\tau t B_n(s,t)} - Id\|\,\|A_n(s)\|\, d\tau \\ &\le K\,\|B_n(s,t) - A_n(s)\| + n^2c^2K\,t. \end{aligned}$$

So it remains to show that $\|B_n(s,t) - A_n(s)\| \to 0$ as $t \to 0$ uniformly in s. In fact,

$$\begin{aligned} B_n(s,t) - A_n(s) =& \frac{n}{t}\int_{s-t}^{s} (A(\tau) - A(s))\, d\tau \left(n - \frac{1}{t}\int_{s-t}^{s} A(\tau)\, d\tau\right)^{-1} \\ &+ nA(s)\left(n - \frac{1}{t}\int_{s-t}^{s} A(\tau)\, d\tau\right)^{-1} \frac{1}{t}\int_{s-t}^{s} (A(\tau) - A(s))\, d\tau\, (n - A(s))^{-1}, \end{aligned}$$

and Lemma 5.1 gives

$$\|B_n(s,t) - A_n(s)\| \le 2nM\,\omega(t) + ncM\,\omega(t). \qquad \square$$

For intervals $I = [a, a+\delta] \subseteq [0,T]$ we now establish a uniform bound for the semigroups $T_{n,I}(\cdot)$ on $L^p(I,X)$ generated by $G_n = -\frac{d}{dt} + A_n(\cdot)$ with domain $W_0^{1,p}(I,X)$, see Section 2. Lemma 2.2 then follows from this estimate. To that purpose, we define bounded operators $V_{n,I}(t)$, $n \in \mathbb{N}$, on $L^p(I,X)$ by setting

$$(V_{n,I}(t)f)(s) = \begin{cases} e^{tB_n(s,t)}\, f(s-t), & a \le s-t \le s \le a+\delta, \\ 0, & s-t < a \le s \le s+\delta, \end{cases}$$

and $V_{n,I}(0) = Id$. Notice that $V_{n,I}(t) = 0$ for $t > \delta$. We will use the operators $V_{n,I}(t)$ in order to approximate $T_{n,I}(t)$ in the sense of Chernoff's product formula. One of the conditions to apply this formula is a straightforward consequence of Lemma 5.2.

Lemma 5.3. *We have* $\frac{1}{t}(V_{n,I}(t)f - f) \to G_{n,I}f$ *as* $t \to 0$ *for* $n \in \mathbb{N}$ *and* $f \in W_0^{1,p}(I,X)$.

Further, we have to estimate powers of $V_{n,I}(t)$. So let us compute

$$\begin{aligned}(V_{n,I}(t)^m f)(s) &= e^{tB_n(s,t)}\,(V_{n,I}(t)^{m-1}f)(s-t) = \cdots = \\ &= e^{tB_n(s,t)} \cdots e^{tB_n(s-(m-1)t,t)} f(s-mt) \\ &=: \Pi_{n,I}(s,t,m)\, f(s-mt)\end{aligned}$$

for $m \in \mathbb{N}$ and $s - mt \geq a$. If $s - mt < a$, we set $\Pi_{n,I}(s,t,m) := 0$. To handle these products we renorm the space X by

$$\|x\|_{s,t,n} := \sup_{\tau \geq 0} \|e^{-\varepsilon\tau}\, e^{\tau t B_n(s,t)} x\|,$$

where $\varepsilon > 0$ is fixed. From (5.1) and Lemma 5.2(1) one easily derives the following properties.

Lemma 5.4. *Let* $0 < t \leq \delta$ *as in Lemma 5.1,* $s, s-t \in [0,T]$*, and* $n \in \mathbb{N}$*. Then* $\|\cdot\|_{s,t,n}$ *is a norm on* X *satisfying*

1. $\|x\| \leq \|x\|_{s,t,n} \leq K\|x\|$,
2. $\|e^{-\varepsilon}\, e^{tB_n(s,t)}x\|_{s,t,n} \leq \|x\|_{s,t,n}$,
3. $|\, \|x\|_{s,t,n} - \|x\|_{s-t,t,n}\,| \leq nC_\varepsilon\, t\omega(t)\|x\|_{s-t,t,n}$ *if* $s - 2t \in [0,T]$,

where C_ε *is a constant only depending on* $\varepsilon > 0$.

Lemma 5.5. *Under the above assumptions we have*

1. $\|e^{-\varepsilon m}\, \Pi_{n,I}(s,t,m)\| \leq K \exp(nC_\varepsilon\, \omega(\delta)\, mt)$ *and*
2. $\limsup_{m\to\infty} \sup_{s\in I} \|e^{-\varepsilon m}\, \Pi_{n,I}(s, \frac{t}{m}, m)\| \leq K$

Proof. (1) We only have to consider the case $s - mt \geq a$. From Lemma 5.4 we obtain

$$\begin{aligned}\|e^{-\varepsilon m}\, \Pi_{n,I}(s,t,m)x\| &\leq \left\|e^{-\varepsilon}\, e^{tB_n(s-t,t)} \cdots e^{-\varepsilon}\, e^{tB_n(s-(m-1)t,t)}x\right\|_{s,t,n} \\ &\leq (1 + nC_\varepsilon\omega(t)t)\left\|e^{-\varepsilon}\, e^{tB_n(s-t,t)} \cdots e^{-\varepsilon}\, e^{tB_n(s-(m-1)t,t)}x\right\|_{s-t,t,n} \\ &\leq e^{nC_\varepsilon\omega(t)t}\left\|e^{-\varepsilon}\, e^{tB_n(s-t,t)} \cdots e^{-\varepsilon}\, e^{tB_n(s-(m-1)t,t)}x\right\|_{s-t,t,n} \\ &\leq \cdots \leq \\ &\leq e^{nC_\varepsilon\omega(t)t(m-1)}\,\|e^{-\varepsilon}\, e^{tB_n(s-(m-1)t,t)}x\|_{s-(m-1)t,t,n} \\ &\leq K\, e^{nC_\varepsilon\omega(t)tm}\,\|x\| \leq K\, e^{nC_\varepsilon\omega(\delta)tm}\,\|x\|.\end{aligned}$$

(2) After replacing t by $\frac{t}{m}$, step (1) of the proof yields

$$\|e^{-\varepsilon m}\, \Pi_{n,I}(s, \tfrac{t}{m}, m)x\| \leq K \exp(nC_\varepsilon\, \omega(\tfrac{t}{m})t) \;\to\; K \quad \text{as} \quad m \to \infty. \qquad \square$$

Proof of Lemma 2.2. Given $I = [a, a+\delta] \subseteq [0,T]$ with $\omega(\delta) \leq \frac{1}{2M}$, let $V_{n,I}(t)$, $G_{n,I}$, and $T_{n,I}(t)$ be defined as above on $L^p(I,X)$. Lemma 5.5 yields

$$\|(e^{-\varepsilon}V_{n,I}(t))^m\| \leq K\, e^{\omega t m}$$

for $t \geq 0$ and $m \in \mathbb{N}$ and a constant $\omega \geq 0$ independent of m and t. Moreover, by Lemma 5.3 we have

$$\frac{d}{dt}\,(e^{-\varepsilon}V_{n,I}(t)f)_{|t=0} = e^{-\varepsilon}\,G_{n,I}f$$

for $f \in W_0^{1,p}(I,X)$. Since $G_{n,I}$ generates $T_{n,I}(\cdot)$, Chernoff's product formula, see e.g. [24, Cor. 3.5.4], implies

$$T_{n,I}(e^{-\varepsilon}t)f = \lim_{m\to\infty} e^{-\varepsilon m}V_{n,I}(\tfrac{t}{m})^m f$$

for $f \in L^p(I,X)$ and $t \geq 0$. Consequently,

$$\|T_{n,I}(e^{-\varepsilon}t)\| \leq \limsup_{m\to\infty}\,\sup_{s\in I}\,\|e^{-\varepsilon m}\,\Pi_n(s,\tfrac{t}{m},m)\| \leq K.$$

Letting $\varepsilon \to 0$, we see that the corresponding evolution family $U_n(t,s)$ satisfies $\|U_n(s,s-t)\| \leq K$ for $0 \leq s-t \leq s \leq T$ and $0 \leq t \leq \delta$. So we may take $C = K^l$ for a natural number $l \geq T/\delta$. □

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