On the geometry of Baer subplanes in $PG(2,4)$

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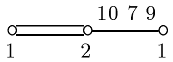
Abstract

Let $\mathbb{C}$ be an $L_3(4)$-orbit of Baer subplanes in $PG(2,4)$. We define a graph Γ on $\mathbb{C}$ where two subplanes are called adjacent if their intersection contains only one point. We determine Γ and the action of $L_3(4)$ is analysed in detail. It is shown that Γ is the collinearity graph of a geometry of Buekenhout [4] with diagram



We also give a detailed construction of this geometry. Moreover, we prove that, if $L_3(4)$ and $L_3(4) : 2_1$ are automorphism groups, this geometry is determined up to isomorphism by its diagram.

1 Introduction

Let $\mathcal{P} = PG(2,4)$ be the (unique) projective plane of order four and let $G \leq Aut\mathcal{P}$ with $G \simeq L_3(4)$. Let $\mathbb{B}$ be the set of all Baer subplanes in $\mathcal{P}$, i.e. sets of seven points and seven lines whose induced incidence structure is isomorphic to $PG(2,2)$. The relation

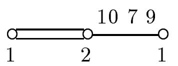
$$\mathcal{B} \sim \mathcal{B}' \text{ if and only if } |\mathcal{B} \cap \mathcal{B}'| \text{ is odd}$$

is an equivalence relation on $\mathbb{B}$. The equivalence classes are also orbits of the action of G on $\mathbb{B}$ (see e.g. [9]). Let $\mathbb{C}$ be one of these classes. Then we prove the following theorem:

Theorem 1.1 *Let Γ be the graph defined as follows. The vertices of Γ are the Baer subplanes of $\mathcal{P}$ in one orbit $\mathbb{C}$ of G. Two subplanes are said to be adjacent if their intersection contains a single point. Then G is a flag-transitive automorphism group of Γ. If $\mathcal{B}$ is a vertex of Γ, the group $G_{\mathcal{B}}$ has orbits of size one, 42, 21 and 56, respectively. Denote by $\Gamma(x;\mathcal{B})$ the $G_{\mathcal{B}}$-orbit of size x on $\mathbb{C}$. Then we have the following parameters in Γ:*

1. $d(\Gamma) = 2$,
2. $\Gamma(\mathcal{B}) = \Gamma(42;\mathcal{B})$,
3. *for* $\mathcal{B}' \in \Gamma(\mathcal{B})$ *we have* $|\Gamma(\mathcal{B},\mathcal{B}')| = 8$, $|\Gamma(\mathcal{B}') \cap \Gamma(21;\mathcal{B})| = 9$ *and* $|\Gamma(\mathcal{B}') \cap \Gamma(56;\mathcal{B})| = 24$,
4. *for* $\mathcal{B}' \in \Gamma(21;\mathcal{B})$ *we have* $|\Gamma(\mathcal{B}') \cap \Gamma(\mathcal{B})| = 18$, $|\Gamma(\mathcal{B}') \cap \Gamma(21;\mathcal{B})| = 8$ *and* $|\Gamma(\mathcal{B}') \cap \Gamma(56;\mathcal{B})| = 16$,
5. *for* $\mathcal{B}' \in \Gamma(56;\mathcal{B})$ *we have* $|\Gamma(\mathcal{B}') \cap \Gamma(\mathcal{B})| = 18$, $|\Gamma(\mathcal{B}') \cap \Gamma(21;\mathcal{B})| = 6$ *and* $|\Gamma(\mathcal{B}') \cap \Gamma(56;\mathcal{B})| = 18$.

In [4], F. Buekenhout constructs a geometry $\mathcal{G}$ with the following diagram D:



The groups G and $A \simeq L_3(4) : 2$ (G extended by a unitary polarity) act flag-transitively on $\mathcal{G}$. We show the following theorem:

Theorem 1.2 *Let $\mathcal{G}$ and $\mathcal{G}'$ be two geometries with diagram D such that the groups G and A act as flag-transitive automorphism groups on $\mathcal{G}$ and $\mathcal{G}'$. If $\mathcal{G}$ and $\mathcal{G}'$ have 120 points, then $\mathcal{G} \simeq \mathcal{G}'$. Moreover, Γ is isomorphic to the collinearity graph of $\mathcal{G}$.*

The paper is organized as follows. In the next section, we recall basic facts on hyperovals, Baer subplanes and unitals further needed. For basic definitions on geometries and diagrams see [5]. In the third section, we define certain sets of Baer subplanes in $\mathcal{P}$ called butterflies. These butterflies are used in the fourth section to determine parameters of the graph Γ. In the final section, we construct a geometry $\mathcal{G}(\Gamma)$ such that its collinearity graph is isomorphic to Γ. We show that any geometry $\mathcal{G}$ fulfilling the properties in Theorem 1.2 is isomorphic to $\mathcal{G}(\Gamma)$.

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2 Baer subplanes, hyperovals and unitals

2.1 Hyperovals

Let $P = PG(2,q)$ be the desarguesian projective plane over $GF(q)$. A set $\mathcal{O}$ of $q+1$ points in P is called an *oval* if no three points in $\mathcal{O}$ are collinear. By definition, there is exactly one tangent t of $\mathcal{O}$ (i.e. a line t with $|\mathcal{O} \cap t| = 1$) through every point of $\mathcal{O}$.

If q is odd, then $\mathcal{O}$ is *maximal*: For every point $p \in P - \mathcal{O}$, the set $\{p\} \cup \mathcal{O}$ contains three collinear points. If q is even, then all tangents of $\mathcal{O}$ meet in a point p, the *nucleus* of $\mathcal{O}$. Then the set $\mathcal{H} := \{p\} \cup \mathcal{O}$ is also a set in which no three points are collinear. We call $\mathcal{H}$ a *hyperoval* in P. Moreover, $\mathcal{H}$ is maximal in the above sense. By construction, $|l \cap \mathcal{H}| \in \{0,2\}$ for every line l of P.

Let now $\mathcal{P} = PG(2,4)$ and $G \simeq L_3(4)$. Then $\mathcal{P}$ has hyperovals. Each hyperoval $\mathcal{H}$ consists of six points and there are 15 lines meeting $\mathcal{H}$ in two points and six meeting in no points. These six lines form a hyperoval of the dual projective plane. If $\mathcal{Q}$ is a quadrangle in $\mathcal{P}$, then there is a unique hyperoval through $\mathcal{Q}$ (see e.g. [11]). Therefore the group $PGL_3(4)$ acts transitively on the set of all 168 hyperovals in $\mathcal{P}$. According to [9], G has three orbits of size 56 on that set. These orbits can be characterized as classes of the following equivalence relation:

$$\mathcal{H}_1 \sim \mathcal{H}_2 :\Leftrightarrow |\mathcal{H}_1 \cap \mathcal{H}_2| \text{ is even.}$$

Let G be the group $L_3(4)$. Then, for a hyperoval $\mathcal{H}$ in $\mathcal{P}$, we have $G_{\mathcal{H}} \simeq A_6$ and $G_{\mathcal{H}}$ is a maximal subgroup of G.

2.2 Baer subplanes

An involution $i \in P\Gamma L_3(q^2) - PGL_3(q^2)$ is called a *Baer involution*. The set of fixed points and lines of i forms a subplane of $P = PG(2,q^2)$ (with the induced incidence relation) isomorphic to $PG(2,q)$, called a *Baer subplane* of P. If H denotes the group $L_3(q^2)$, then $H_{\mathcal{B}} \simeq L_3(q)$ for a Baer subplane $\mathcal{B}$.

If $\mathcal{B}$ is a Baer subplane of P, then every line of P meets $\mathcal{B}$ either in one or $q+1$ points. We call the lines l with $|l \cap \mathcal{B}| = q+1$ *$\mathcal{B}$-lines*. If $\mathcal{B}$ and $\mathcal{B}'$ are two Baer subplanes of P and l is line of P such that l is a $\mathcal{B}$-line and a $\mathcal{B}'$-line, then we say that l is a line in $\mathcal{B} \cap \mathcal{B}'$. This does not imply $|\mathcal{B} \cap \mathcal{B}' \cap l| = q+1$ and the following theorem holds:

Theorem 2.1 *[2], [12] For any two Baer subplanes $\mathcal{B}$ and $\mathcal{B}'$ of $PG(2,q^2)$ the intersection $\mathcal{B} \cap \mathcal{B}'$ contains the same number of points and lines.*

In [6], this theorem has been generalized for Baer subspaces of projective spaces $PG(d,q^2)$.

Let now $\mathcal{Q}$ be a quadrangle in P. Then there is a unique Baer subplane through $\mathcal{Q}$. Thus, $PGL_3(q^2)$ acts transitively on the set of all Baer subplanes.

If $\mathcal{Q}$ is a quadrangle in $\mathcal{P}$, then the Baer subplane through $\mathcal{Q}$ can be constructed as follows: Denote by $\mathcal{D}$ the set of the three diagonal points of $\mathcal{Q}$ (i.e. the intersection of opposite 'edges' and the 'diagonals' of $\mathcal{Q}$). By FANO's theorem, the three points in $\mathcal{D}$ are collinear. Therefore, $\mathcal{Q} \cup \mathcal{D}$ is a Baer subplane. By construction, if l denotes the $\mathcal{P}$-line through the points in $\mathcal{D}$, then $\bar{\mathcal{D}} \cup \mathcal{Q}$, with $\bar{\mathcal{D}} := l - \mathcal{D}$, is a hyperoval. Thus, the set of all 360 Baer subplanes in $\mathcal{P}$ splits into three orbits of size 120 under the action of G. According to [11], the orbits can also be characterized as classes of an equivalence relation, namely the following:

$$\mathcal{B}_1 \sim \mathcal{B}_2 :\Leftrightarrow |\mathcal{B}_1 \cap \mathcal{B}_2| \text{ is odd.}$$

Let now $\mathbb{C}$ be one of theses equivalence classes, then it is shown in [8], using certain results of [11] and [10], that there are exactly 42 members of $\mathbb{C}$ intersecting a given subplane $\mathcal{B} \in \mathbb{C}$ in a single point, 21 intersect $\mathcal{B}$ in three collinear points and 56 in a triangle.

2.3 Unitals

Again $P = PG(2,q^2)$ denotes the desarguesian plane over $GF(q^2)$. Let $\mathcal{U}$ be a set of q^3+1 points of P such that $\mathcal{U}$ does not contain a full line. If every line meets $\mathcal{U}$ either in one or $q+1$ points, $\mathcal{U}$ is called a *unital* in P.

Denote by $[x,y,z]$ the homogenous coordinates of the points of P. Then the absolute points of the equation $xx^q + yy^q + zz^q = 0$ form a unital.

By construction, $\mathcal{U}$ together with the lines meeting $\mathcal{U}$ in $q+1$ points is an $S(2,q+1,q^3+1)$ Steiner system. Therefore, if $\mathcal{U}$ is a unital in $\mathcal{P} = PG(2,4)$, $\mathcal{U}$ is an $S(2,3,9)$, hence, $\mathcal{U}$ is isomorphic to $AG(2,3)$. If $G \simeq L_3(4)$, we have $G_{\mathcal{U}} \simeq \mathbb{Z}_3^2 : Q_8$ and $G_{\mathcal{U}}$ is maximal in G.

According to [1], a unital in $\mathcal{P}$ can be constructed as follows: Let $\mathcal{H}$ be a hyperoval in $\mathcal{P}$ and let $\mathcal{D}$ be a triangle on $\mathcal{H}$. Then the nine points on the 'edges' of $\mathcal{D}$ outside $\mathcal{H}$ form a unital $\mathcal{U}$. Thus, $\mathcal{D}$ belongs to a parallel class of $\mathcal{U}$. Let $\mathcal{D}' := \mathcal{H} - \mathcal{D}$. Then $\mathcal{D}'$ belongs to another parallel class of the same unital $\mathcal{U}$. Moreover, the set $\mathcal{H}' := \mathcal{P} - (\mathcal{H} \cup \mathcal{U})$ is a hyperoval which splits uniquely into two triangles determined by the two remaining parallel classes of $\mathcal{U}$.

Conversely, if $\mathcal{U}$ is a unital in $\mathcal{P}$, its parallel classes determine four triangles in its complement. The choice of each pair of these triangles forms a hyperoval [11].

3 Butterflies in $PG(2,4)$

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two Baer subplanes in $\mathcal{P}$ intersecting in three collinear points. Denote by z the line in $\mathcal{B}_1 \cap \mathcal{B}_2$ containing all points of $\mathcal{B}_1 \cap \mathcal{B}_2$.

Lemma 3.1 $G_{\mathcal{B}_1,\mathcal{B}_2} \simeq D_8$.

Proof. Clearly, we have $zG_{\mathcal{B}_1,\mathcal{B}_2} = \{z\}$. Thus, $G_{\mathcal{B}_1,\mathcal{B}_2}$ is isomorphic to a subgroup of S_4. Since there are exactly 21 subplanes intersecting $\mathcal{B}_1$ in three collinear points, we have $|G_{\mathcal{B}_1,\mathcal{B}_2}| \geq 8$. Assume that $x \in G_{\mathcal{B}_1,\mathcal{B}_2}$ with $o(x) = 3$. Let $z \neq z'$ be a line in $\mathcal{B}_1 \cap \mathcal{B}_2$. Then z' containes at least one point of $\mathcal{B}_1 \cap \mathcal{B}_2$. Since $< x >$ acts transitively on $z \cap \mathcal{B}_1 = z \cap \mathcal{B}_2$, $\mathcal{B}_1 \cap \mathcal{B}_2$ contains at least four lines contradicting 2.1. □

Corollary 3.2 *$G_{\mathcal{B}}$ acts transitively on the Baer subplanes intersecting $\mathcal{B}$ in three collinear points.* □

$\mathcal{B}_1$ and $\mathcal{B}_2$ are also Baer subplanes of the dual plane intersecting in three lines. Since the three points in $\mathcal{B}_1 \cap \mathcal{B}_2$ are collinear, the three lines in $\mathcal{B}_1 \cap \mathcal{B}_2$ are concurrent and we hold the following:

Corollary 3.3 *There is a unique point p in $\mathcal{B}_1 \cap \mathcal{B}_2$ such that all three lines of $\mathcal{B}_1 \cap \mathcal{B}_2$ contain p. The remaining lines through p are tangents of both, $\mathcal{B}_1$ and $\mathcal{B}_2$.* □

Lemma 3.4 *There is one and only one pair of Baer subplanes $(\mathcal{B}'_1, \mathcal{B}'_2)$ with the following properties:*

1. *$\mathcal{B}'_1 \cap \mathcal{B}'_2$ contains three collinear points,*
2. *$p \in \mathcal{B}'_1 \cap \mathcal{B}'_2$ and z contains all three points in $\mathcal{B}'_1 \cap \mathcal{B}'_2$,*
3. *$\mathcal{B}_i \cap \mathcal{B}'_j = \{p\}$, $i, j \in \{1, 2\}$.*

Proof. Denote by $\mathcal{H}_i$ the (unique) hyperoval through $\mathcal{B}_i - z$. Then $\mathcal{H}_1 \cap \mathcal{H}_2$ consists of the two points of z not in $\mathcal{B}_1 \cap \mathcal{B}_2$. By [7], there exists a unique pair $(\mathcal{H}'_1, \mathcal{H}'_2)$ of hyperovals with the following properties:

1. $|\mathcal{H}'_1 \cap \mathcal{H}'_2| = 2$,
2. $\mathcal{H}_i \cap \mathcal{H}'_j = \emptyset$, $i, j \in \{1, 2\}$.

Denote by t_1 and t_2 the two common tangents of $\mathcal{B}_1$ and $\mathcal{B}_2$ through p. Then $t_i \cap \mathcal{H}_j = \emptyset$. The set $\{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}'_1, \mathcal{H}'_2\}$ covers 20 points of $\mathcal{P}$. Assume $p \in \mathcal{H}'_1$. Denote by p' the point outside $\{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}'_1, \mathcal{H}'_2\}$. Then $p' \in t_1$. Thus $t_2 \subset \mathcal{H}'_1 \cup \mathcal{H}'_2$, a contradiction. Therefore $p \notin \mathcal{H}'_i$. Together with $|\mathcal{P} - (\mathcal{H}_1 \cup \mathcal{H}_2)| = 11$ this yields $\mathcal{H}'_i \cap \mathcal{B}_j \neq \emptyset$ and therefore, by [11], $|\mathcal{H}'_i \cap \mathcal{B}_j| = 2$. Moreover, $\mathcal{H}'_i \cap \mathcal{B}_j \subset z$, thus, $\mathcal{H}'_1 \cap \mathcal{B}_j = \mathcal{H}'_2 \cap \mathcal{B}_j$ and $\mathcal{H}'_1 \cap \mathcal{H}'_2$ consists of the two points in $\mathcal{B}_1 \cap \mathcal{B}_2$ different from p. Let $\mathcal{Q}'_i$ be the quadrangle $\mathcal{H}'_i - z = \mathcal{H}'_i - (\mathcal{H}'_1 \cap \mathcal{H}'_2)$. Then $\mathcal{Q}'_i$ defines a (unique) Baer subplane $\mathcal{B}'_i$. By construction, the pair $(\mathcal{B}'_1, \mathcal{B}'_2)$ has the desired properties. □

Definition 3.5 The set $\mathcal{S} = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}'_1, \mathcal{B}'_2\}$ is called a *butterfly* $\mathcal{S}$ in $\mathcal{P}$ and (p, z) is called the *central pair* of $\mathcal{S}$. The set $\mathcal{W} = \{\mathcal{B}_i, \mathcal{B}'_i\}$ is said to be a *wing* of $\mathcal{S}$.

By construction, a butterfly $\mathcal{S}$ is uniquely determined by one of the following data:

- $\mathcal{B}_1$ and $\mathcal{B}_2$,
- $\mathcal{B}_1$ and $\mathcal{B}'_1$ or
- $\mathcal{B}_1$ and (p, z).

Lemma 3.6 *If $\mathbb{C}$ is a class (G-orbit) of Baer subplanes in $\mathcal{P}$, then $\mathbb{C}$ defines exactly 2520 wings and 630 butterflies.*

Proof. Let $\mathbb{W}$ be the set of all wings, $\mathbb{S}$ that of all butterflies defined by $\mathbb{C}$ and $\mathcal{B} \in \mathbb{C}$. Since there are 42 members of $\mathbb{C}$ intersecting $\mathcal{B}$ in a single point, we have $|\mathbb{W}| = \frac{120 \cdot 42}{2} = 2520$. A butterfly $\mathcal{S}$ has exactly four wings, so $|\mathbb{S}| = \frac{2520}{4} = 630$ since any wing defines a unique butterfly. □

Lemma 3.7 *Let $\mathcal{W}$ be a wing, then $G_{\mathcal{W}} \simeq Q_8$.*

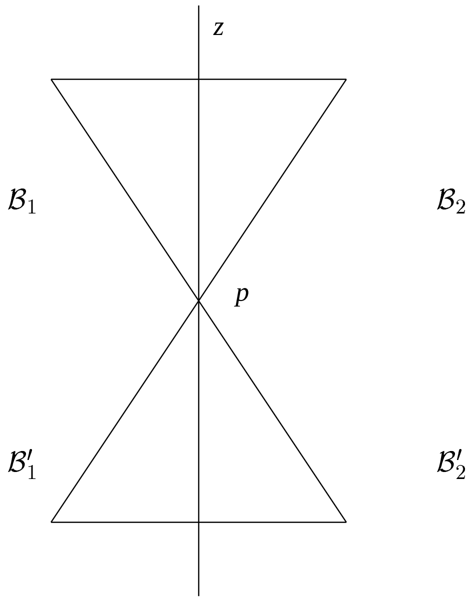


Figure 1: A butterfly

Proof. Let $\mathcal{W} = \{\mathcal{B}, \mathcal{B}'\}$ and let z be the unique line in $\mathcal{B} \cap \mathcal{B}'$. Let $\mathcal{Q} = \mathcal{B} - z$, $\mathcal{Q}' = \mathcal{B}' - z$. Then $\mathcal{Q}$ and $\mathcal{Q}'$ define two disjoint hyperovals $\mathcal{H}$ and $\mathcal{H}'$. Therefore, $G_{\mathcal{W}} \leq G_{\mathcal{H} \cup \mathcal{H}'} \simeq \mathbb{Z}_3^2 : Q_8$. If p denotes the single point in $\mathcal{B} \cap \mathcal{B}'$, then $G_{\mathcal{W}} \leq G_{p,z} \simeq \mathbb{Z}_2^4 : A_4$. Hence, $G_{\mathcal{W}}$ is isomorphic to a subgroup of S_3 or Q_8. By the previous lemma, $|\mathbb{W}| = 2520$, so $|G_{\mathcal{W}}| \geq 8$. Thus, $G_{\mathcal{W}} \simeq Q_8$. □

Lemma 3.8 *Let $\mathcal{B} \in \mathbb{C}$ and let $\mathcal{W} = \{\mathcal{B}, \mathcal{B}'\}$ be a wing through $\mathcal{B}$. Then $G_{\mathcal{B},\mathcal{W}} \simeq \mathbb{Z}_4$. Moreover, $G_{\mathcal{B}}$ is transitive on the wings through $\mathcal{B}$.*

Proof. By the previous lemma, $G_{\mathcal{B},\mathcal{W}}$ is a subgroup of Q_8. Since $G_{\mathcal{W}}$ interchanges the hyperovals $\mathcal{H}$ and $\mathcal{H}'$ constructed as above, it interchanges $\mathcal{B}$ and $\mathcal{B}'$. Thus, $G_{\mathcal{B},\mathcal{W}} \simeq \mathbb{Z}_4$. □

4 Determination of the parameters of Γ

Let $\mathbb{C}$ be an orbit of Baer subplanes of $G \simeq L_3(4)$ in $\mathcal{P} = PG(2,4)$. We define Γ to be the following graph: The vertices of Γ are the members of $\mathbb{C}$ where two subplanes $\mathcal{B}$ and $\mathcal{B}'$ are called adjacent if their intersection contains a single point. By this definition, the edges of Γ correspond to the wings of butterflies defined by the subplanes in $\mathbb{C}$. We determine the parameters of Γ.

Lemma 4.1 *$G_{\mathcal{B}}$ has four orbits of size one, 42, 21 and 56, respectively, on the vertices of* Γ

Proof. By 3.2, $G_{\mathcal{B}}$ acts transitively on the members of $\mathbb{C}$ intersecting $\mathcal{B}$ in three collinear points, giving an orbit of size 21. Lemma 3.8 shows that $G_{\mathcal{B}}$ is transitive on the neighbourhood of $\mathcal{B}$ in Γ yielding an orbit of size 42. Let now $\mathcal{B}$ and $\mathcal{B}'$ intersect in a triangle $\mathcal{D}$. The transitivity of $G_{\mathcal{B}}$ on the triangles in $\mathcal{B}$ yields two members of $\mathbb{C}$ intersecting $\mathcal{B}$ in $\mathcal{D}$. By 2.1, $\mathcal{B}$ and $\mathcal{B}'$ share three lines, namely the three 'edges' of $\mathcal{D}$. Since $\mathcal{B}$ has 28 triangles, we hold $G_{\mathcal{B},\mathcal{D}} \simeq S_3$. Let $i \in G_{\mathcal{B},\mathcal{D}}$ fix an edge l of $\mathcal{D}$. Then i fixes the remaining point p in $(\mathcal{B}-\mathcal{D}) \cap l$. Hence $i \in G_p \cap G_l \simeq 2^4 : A_4$. As i interchanges the two points q and q' of $\mathcal{D}$ on l, it interchanges the two points in $\mathcal{D} - \{p, q, q'\}$. Therefore, $i \notin G_{\mathcal{B},\mathcal{B}'} \simeq \mathbb{Z}_3$. Trivially, $\{\mathcal{B}\}$ is an orbit of $G_{\mathcal{B}}$ and the lemma is proved. □

Denote by $\Gamma(x;\mathcal{B})$ the $G_{\mathcal{B}}$-orbit of size x on $\mathbb{C}$. Then $\Gamma(\mathcal{B}) = \Gamma(42;\mathcal{B})$. $\Gamma(21;\mathcal{B})$ contains exactly the subplanes intersecting $\mathcal{B}$ in three collinear points, $\Gamma(56;\mathcal{B})$ those intersecting $\mathcal{B}$ in a triangle.

Lemma 4.2 $d(\Gamma) = 2$.

Proof. Since $G_{\mathcal{B}}$ is maximal in G, Γ is connected. Let $\mathcal{B}$ and $\mathcal{B}'$ intersect in three collinear points. Then $\mathcal{B}$ and $\mathcal{B}'$ define a butterfly $\mathcal{S}$, thus, $|\Gamma(\mathcal{B},\mathcal{B}')| \geq 2$.

Let $\mathcal{B}$ and $\mathcal{B}'$ intersect in a triangle $\mathcal{D} = \{p_1, p_2, p_3\}$. Since Γ is connected and $G_{\mathcal{B}}$ is transitive on the members of $\mathbb{C}$ intersecting $\mathcal{B}$ in three collinear points and on those intersecting $\mathcal{B}$ in a triangle, we hold $d(\mathcal{B},\mathcal{B}') \in \{2,3\}$. Assume that $d(\mathcal{B},\mathcal{B}') = 3$. Then there exists a path $\mathcal{B} \sim \mathcal{B}_1 \sim \mathcal{B}_2 \sim \mathcal{B}'$ where $\mathcal{B}$ and $\mathcal{B}_2$ intersect in three collinear points and $|\mathcal{B}' \cap \mathcal{B}_1| = 3$. Again, by the transitivity of $G_{\mathcal{B}'}$ on all subplanes in $\mathbb{C}$ intersecting $\mathcal{B}'$ in a triangle, $\mathcal{B}' \cap \mathcal{B}_1$ consists of three collinear points. Also, $\mathcal{B}$ and $\mathcal{B}_2$ define a butterfly $\mathcal{S}$ with central pair (p, z). Then w.l.o.g., $\mathcal{B}_1$ also belongs to $\mathcal{S}$, thus $\mathcal{B} \cap \mathcal{B}_1 = \{p\} = \mathcal{B}_1 \cap \mathcal{B}_2$. By 2.1, $\mathcal{B} \cap \mathcal{B}'$ consists of three lines, the three 'edges' of $\mathcal{D}$. Set $\{q_i\} := (p_j p_l \cap \mathcal{B}') - \mathcal{B}$, $\{i, j, l\} = \{1, 2, 3\}$. Then q_1, q_2 and q_3 are three collinear points. If $p \in \mathcal{D}$, then two of the three edges belong to $\mathcal{B}'$ and to $\mathcal{B}_2$, so $|\mathcal{B}' \cap \mathcal{B}_2| \geq 2$, a contradiction. Therefore, $p \notin \mathcal{D}$ and consequently,

$p \notin \mathcal{B}'$. Thus, $\mathcal{B}' \cap \mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$, and, since $\{q_1, q_2, q_3\}$ is the only line of $\mathcal{B}'$ in the complement of $\mathcal{D}$, $\mathcal{B}' \cap \mathcal{B}_1 = \{q_1, q_2, q_3\}$. Clearly, $q_1 q_2$ is not a $\mathcal{B}$-line, thus, $q_1 q_2 \neq z$. Let $\mathcal{B}_1'$ be the fourth subplane in $\mathcal{S}$. Then again we hold $\mathcal{B}' \cap \mathcal{B}_1' = \{q_1, q_2, q_3\}$. But $\mathcal{B}' \cap \mathcal{B}_1' \subset z$, a contradiction. $\square$

Lemma 4.3 *Let $\mathcal{B}' \in \Gamma(21; \mathcal{B})$. Then $|\Gamma(\mathcal{B}, \mathcal{B}')| = 18$. Moreover, $\Gamma(\mathcal{B}, \mathcal{B}')$ splits into three G-orbits of size two, eight and eight.*

Proof. Since $\mathcal{B}$ and $\mathcal{B}'$ intersect in three collinear points, they define a butterly $\mathcal{S}$ with central pair (p, z). Set $\mathcal{B} \cap \mathcal{B}' = \{p, p_1, p_2\}$. Let $\mathcal{N}$ be a neighbour of $\mathcal{B}$ with $\mathcal{B} \cap \mathcal{N} = \{p\}$. Then $\mathcal{N} \in \Gamma(\mathcal{B}')$ if and only if $\mathcal{N}$ belongs to $\mathcal{S}$. By 3.8, $|G_{\mathcal{B}, \mathcal{B}', \mathcal{N}}| \leq 4$, so $\mathcal{N}$ is not fixed by $G_{\mathcal{B}, \mathcal{B}'}$ and we hold a G-orbit of size two.

If $\mathcal{N} \in \Gamma(\mathcal{B})$ with $p_1 \in \mathcal{N} \cap \mathcal{B}$, then $\mathcal{N}$ and $\mathcal{B}'$ share the two $\mathcal{B}$-tangents through $\mathcal{N} \cap \mathcal{B}$, thus, $\mathcal{N} \notin \Gamma(\mathcal{B}')$.

Let now $q \in \mathcal{B} - z$. Then there are six neighbours of $\mathcal{B}$ through q such that any $\mathcal{B}$-line is in two of them. Set $g_i := p_i q$ for $i = 1, 2$. Then $l_i := q \cup (g_i - \mathcal{B})$ defines two neigbours $\mathcal{N}_{i,1}$ and $\mathcal{N}_{i,2}$ of $\mathcal{B}$ through q. Furthermore, we have $l_i \cap \mathcal{B}' = \emptyset$. We shall see, that $\mathcal{N}_{i,j} \in \Gamma(\mathcal{B}')$. Clearly, we have $|\mathcal{N}_{i,j} \cap \mathcal{B}'| \geq 1$. Now $\mathcal{N}_{i,j}$ contains the points of l_i and each two points of the $\mathcal{B}$-tangents t_1 and t_2 through q. Therefore, we hold $\mathcal{N}_{i,j} \cap \mathcal{B}' \subset (t_1 \cup t_2) \cap \mathcal{B}'$. By 3.3, t_1 and t_2 are also $\mathcal{B}'$-tangents, we have $|\mathcal{N}_{i,j} \cap \mathcal{B}'| \leq 2$.

It remains to show that $G_{\mathcal{B}, \mathcal{B}', \mathcal{N}_{i,j}} = 1$. Let $\{q'\} = \mathcal{B}' \cap \mathcal{N}_{i,j}$. Then $G_{\mathcal{B}, \mathcal{B}', \mathcal{N}_{i,j}} \leq G_{\mathcal{B}, \mathcal{B}', q, q'}$. Now $G_{\mathcal{B}, \mathcal{B}', q, q'} \leq G_{p,q}$ and so $G_{\mathcal{B}, \mathcal{B}', q, q'} \leq O_2(G_{pq})$ since $G_{\mathcal{B}, \mathcal{B}'} \simeq D_8$. Thus $G_{\mathcal{B}, \mathcal{B}', q, q'} = 1$. $\square$

Lemma 4.4 *Let $\mathcal{B}' \in \Gamma(\mathcal{B})$. Then $|\Gamma(\mathcal{B}, \mathcal{B}')| = 8$ and $\Gamma(\mathcal{B}, \mathcal{B}')$ splits into two G-orbits of size four.*

Proof. Again, $\mathcal{B}$ and $\mathcal{B}'$ define a butterfly $\mathcal{S}$ with central pair (p, z). If $\mathcal{N} \in \Gamma(\mathcal{B})$ with $\mathcal{N} \cap \mathcal{B} = \{p\}$, then $\mathcal{N}$ and $\mathcal{B}'$ share the two $\mathcal{B}$-tangents through p and therefore, $\mathcal{N} \notin \Gamma(\mathcal{B}')$. Assume that $p \neq p_1 \in \mathcal{N} \cap \mathcal{B}$ is on z. We can assume that z is not a $\mathcal{N}$-line (otherwise $\mathcal{N} \notin \Gamma(\mathcal{B}')$ follows immediately). Let z_1 be the line in $\mathcal{B} \cap \mathcal{N}$. Then z_1 is a $\mathcal{B}'$-tangent through some point p_1' in $\mathcal{N} \cap \mathcal{B}'$. Let now z_1' denote a line in $\mathcal{N} \cap \mathcal{B}'$ through p_1'. Thus, z_1' is a $\mathcal{B}$-tangent through some point q in $\mathcal{B}$. We can assume that p_1' is the only point of z_1' in $\mathcal{N} \cap \mathcal{B}'$, otherwise $\mathcal{N} \notin \Gamma(\mathcal{B}')$. Then $z_1' \cap \mathcal{B}$ is in $\mathcal{N}$ or $z_1' \cap \mathcal{B} = \{p\}$. Clearly, $z_1' \cap \mathcal{B} \neq \{p_1\}$, so $q = p$. Denote by t_1 and t_2 the two $\mathcal{B}$-tangents through p_1. Therefore, the lines z_1, t_1 and t_2 are $\mathcal{N}$-lines. Assume that t_i is a $\mathcal{B}'$-line. Then $t_i \cap z = \{p_1\}$ is a point in

$\mathcal{B}'$, a contradiction. So t_1 and t_2 are also $\mathcal{B}'$-tangents. Also, z_1' and t_1', the $\mathcal{B}'$-tangent through p_1' different from z_1, are $\mathcal{N}$-lines. Since $\mathcal{N} \in \Gamma(\mathcal{B})$, these are $\mathcal{B}$-tangents. Let $p_2 \in \mathcal{N}$ such that $p_2 \in z_1$, $p_2 \notin \mathcal{B} \cup \mathcal{B}'$. The remaining two $\mathcal{N}$-lines s_1 and s_2 pass through p_2. Again, since $\mathcal{N} \in \Gamma(\mathcal{B})$, s_1 and s_2 are $\mathcal{B}$-tangents. Assume that s_1 is a $\mathcal{B}'$-line. Then $|\mathcal{N} \cap \mathcal{B}'| = 3$ and s_2 is a $\mathcal{B}'$-line as well. Thereby, $s_1 \cap s_2 = \{p_2\}$ is a point of $\mathcal{B}'$, a contradiction. Thus, we have $\mathcal{N} \in \Gamma(\mathcal{B}')$. Since any Baer subplane is uniquely determined by a quadrangle, we find that $\mathcal{N}$ is determined by p_1, z_1 and z_1'. Hence, we hold eight such subplanes in $\Gamma(\mathcal{B}, \mathcal{B}')$. Now $G_{\mathcal{B},\mathcal{B}'} \simeq \mathbb{Z}_4$ acts transitively on the quadrangle $\mathcal{B} - z$. Thus, $G_{\mathcal{B},\mathcal{B}',\mathcal{N}} \leq G_{\mathcal{B},\mathcal{B}',p_1} \simeq \mathbb{Z}_2$. But the involution in $G_{\mathcal{B},\mathcal{B}',p_1}$ interchanges the two lines of $\mathcal{B}$ through p_1 different from z. This proves $G_{\mathcal{B},\mathcal{B}',\mathcal{N}} = 1$.

Assume now that $\mathcal{N} \in \Gamma(\mathcal{B})$, $\mathcal{N} \cap \mathcal{B} = \{q\}$ with $q \notin z$. Again, z_1 denotes the unique $\mathcal{B}$-line in $\mathcal{N} \cap \mathcal{B}$. Thus, $z_1 \neq z$ and z_1 is a $\mathcal{B}'$-tangent through some point q'. Since $q' \in \mathcal{N}$, we have $q' \neq p$. Assume that $q' \in z$. Then $z \cap z_1 = \{q'\}$ is in $\mathcal{B}$, a contradiction. Let z_1' be a line in $\mathcal{N} \cap \mathcal{B}'$ through q'. As above, we can assume that q' is the only point of z_1' in $\mathcal{N} \cap \mathcal{B}'$ and it follows that $p \in z_1'$. This proves $p \notin z_1$. Therefore, we hold the following $\mathcal{N}$-lines: z_1, t_1 and t_2 (the two $\mathcal{B}$-tangents through q), z_1' and t_1' (the remaining $\mathcal{B}'$-tangent through q'). The lines z_1' and t_1' are again $\mathcal{B}$-tangents since $\mathcal{N} \in \Gamma(\mathcal{B})$. Since $q \notin z$, we find $t_i \cap z$ is a point of $\mathcal{B}'$ and at least one of t_1 and t_2 is a $\mathcal{B}'$-tangent (otherwise $t_1 \cap t_2$ is in $\mathcal{B}'$). Let g be one of the four $\mathcal{B}'$-lines intersecting z in a point different from p. Then g is a $\mathcal{B}$-tangent through some point not on z. Since at least one of the $\mathcal{B}$-tangents through such a point must be a $\mathcal{B}'$ tangent and there are eight of these lines, either t_1 or t_2 is a $\mathcal{B}'$-line and $\mathcal{N} \notin \Gamma(\mathcal{B}')$. $\square$

Lemma 4.5 *Let $\mathcal{B}' \in \Gamma(21; \mathcal{B})$. Then $|\Gamma(\mathcal{B}') \cap \Gamma(21; \mathcal{B})| = 8$. Moreover, $G_{\mathcal{B},\mathcal{B}'}$ has two orbits of size four on $\Gamma(\mathcal{B}') \cap \Gamma(21; \mathcal{B})$.*

Proof. Again, $\mathcal{B}$ and $\mathcal{B}'$ define a butterfly $\mathcal{S}$ with central pair (p, z). Set $\mathcal{B} \cap \mathcal{B}' = \{p, p_1, p_2\}$ and let z_1 and z_2 be the lines through p in $\mathcal{B} \cap \mathcal{B}'$ different from z. Then the remaining $\mathcal{B}'$-lines are the two $\mathcal{B}$-tangents through p_1 and p_2.

Let now $\mathcal{L} \in \Gamma(21; \mathcal{B})$. Since any $\mathcal{B}$-line contains at least one of the points p, p_1 or p_2, we have that at least one of these points is in $\mathcal{L} \cap \mathcal{B}$. If $\mathcal{L} \in \Gamma(\mathcal{B}')$, then p, p_1 or p_2 is the only point in $\mathcal{L} \cap \mathcal{B}'$.

Assume $p \in \mathcal{L} \cap \mathcal{B}'$. Since $\mathcal{L} \in \Gamma(21; \mathcal{B})$, $\mathcal{B}$ and $\mathcal{L}$ define a butterfly $\mathcal{S}'$ with central pair (p', z'). If $p = p'$, then z, z_1 and z_2 are $\mathcal{L}$-lines, hence, $\mathcal{L} \notin$

$\Gamma(\mathcal{B}')$. Thus $p \neq p'$. We can assume that $p_1, p_2 \notin \mathcal{L}$, otherwise $\mathcal{L} \notin \Gamma(\mathcal{B})'$. Therefore, $p' \neq p_i$, $i = 1, 2$. Set $\{q\} = pp' \cap \mathcal{B} - \{p, p'\}$. Then $q \in \mathcal{L}$. Then the $\mathcal{L}$-lines are: $pp' = z_i$ for exactly one i, the lines p_ip' for $i = 1, 2$ (these are $\mathcal{B}'$-tangents through p_i), and the two $\mathcal{B}$-tangents through p and through q. If t is any $\mathcal{B}$-tangent through q, then t is also a $\mathcal{B}'$-tangent. Thus, $\mathcal{L} \in \Gamma(\mathcal{B}')$.

Assume that $p_1 \in \mathcal{L} \cap \mathcal{B}'$. Then we can assume $p \notin \mathcal{L}$. Again, let $\mathcal{S}'$ be the butterfly defined by $\mathcal{B}$ and $\mathcal{L}$ and let (p', z') be its central pair. Assume first that $p' = p_1$. Then the $\mathcal{L}$-lines are: z, z' and the three $\mathcal{B}$-line through p_1 (except of z these are $\mathcal{B}'$-tangents), and the two tangents through the two remaing points on z' in $\mathcal{B} \cap \mathcal{L}$. Again, only z is a line in $\mathcal{L} \cap \mathcal{B}'$ and $\mathcal{L} \in \Gamma(\mathcal{B}')$.

Thus, it remains to analyse the situation where $p' \neq p_1$. Then the $\mathcal{L}$-lines are the three $\mathcal{B}$-lines through p', namely pp', $p_1p' = z'$ and p_2p', the two $\mathcal{B}$-tangents t_1 and t_2 through p_1, and the two $\mathcal{B}$ tangents through $z' \cap \mathcal{B} - \{p_1, p'\}$. But t_1 and t_2 are also $\mathcal{B}'$-lines, hence, $\mathcal{L} \notin \Gamma(\mathcal{B}')$.

This shows that $|\Gamma(\mathcal{B}') \cap \Gamma(21; \mathcal{B})| = 8$.

The proof so far shows that there are at least two orbits of $G_{\mathcal{B},\mathcal{B}'}$ on $\Gamma(\mathcal{B}') \cap \Gamma(21; \mathcal{B})$. Let $\mathcal{L} \in \Gamma(\mathcal{B}') \cap \Gamma(21; \mathcal{B})$ with $\mathcal{L} \cap \mathcal{B}' = \{p\}$. Since $G_{\mathcal{B},\mathcal{B}'}$ is transitive on the four points in $\mathcal{B} - z$, we have $|\mathcal{L}G_{\mathcal{B},\mathcal{B}'}| = 4$. Let now $\mathcal{L} \in \Gamma(\mathcal{B}') \cap \Gamma(21; \mathcal{B})$ with $\mathcal{L} \cap \mathcal{B}' = \{p_1\}$. Then $G_{\mathcal{B},\mathcal{B}',p_1} \simeq \mathbb{Z}_2^2$ still interchanges the two $\mathcal{B}$-lines through p_1 different from z. Again, we hold $|\mathcal{L}G_{\mathcal{B},\mathcal{B}'}| = 4$. $\square$

Lemma 4.6 *Let $\mathcal{B}' \in \Gamma(56; \mathcal{B})$. Then $|\Gamma(\mathcal{B}, \mathcal{B}')| = 18$. Moreover, $\Gamma(\mathcal{B}, \mathcal{B}')$ splits into six orbits of size three.*

Proof. Let $\mathcal{B}'' \in \Gamma(\mathcal{B})$. Since $d(\Gamma) = 2$, we have $|\Gamma(\mathcal{B}'') \cap \Gamma(56; \mathcal{B})| = 42 - |\Gamma(\mathcal{B}'') \cap \Gamma(\mathcal{B})| - |\Gamma(\mathcal{B}'') \cap \Gamma(21; \mathcal{B})| = 42 - 8 - 18 = 24$, so $|\Gamma(\mathcal{B}, \mathcal{B}')| = 18$. Let $P := \mathcal{B} \sim \mathcal{B}'' \sim \mathcal{B}'$ be a path from $\mathcal{B}$ to $\mathcal{B}'$. Then, $G_P = G_{\mathcal{B},\mathcal{B}''} \cap G_{\mathcal{B},\mathcal{B}''} \simeq \mathbb{Z}_4 \cap \mathbb{Z}_3$ proving the lemma. $\square$

Lemma 4.7 *Let $\mathcal{B}' \in \Gamma(56; \mathcal{B})$, $\mathcal{B}'' \in \Gamma(21; \mathcal{B})$. Then $|\Gamma(\mathcal{B}') \cap \Gamma(21; \mathcal{B})| = 6$, $|\Gamma(\mathcal{B}'') \cap \Gamma(56; \mathcal{B})| = 16$ and $|\Gamma(\mathcal{B}') \cap \Gamma(56; \mathcal{B})| = 18$. Moreover, $\Gamma(\mathcal{B}') \cap \Gamma(56; \mathcal{B})$ splits into six orbits of size three and $\Gamma(\mathcal{B}') \cap \Gamma(21; \mathcal{B})$ splits into two orbits of size 3.*

Proof. By the previous lemmas, we hold $|\Gamma(\mathcal{B}') \cap \Gamma(21; \mathcal{B})| = 6$, $|\Gamma(\mathcal{B}'') \cap \Gamma(56; \mathcal{B})| = 16$ and $|\Gamma(\mathcal{B}') \cap \Gamma(56; \mathcal{B})| = 18$. Assume a path $\mathcal{B} \sim \mathcal{B}_1 \sim \mathcal{B}' \sim \mathcal{B}_2$ with $\mathcal{B}_2 \in \Gamma(56; \mathcal{B})$. Then $|G_{\mathcal{B},\mathcal{B}',\mathcal{B}_2}| \in \{1, 3\}$. If $G_{\mathcal{B},\mathcal{B}',\mathcal{B}_2} = G_{\mathcal{B},\mathcal{B}'} = G_{\mathcal{B},\mathcal{B}_2}$, then also $\mathcal{B} \cap \mathcal{B}' = \mathcal{B} \cap \mathcal{B}_2$, a contradiction. Assume that $\mathcal{B}' \sim \mathcal{B}''$. Then

$G_{\mathcal{B},\mathcal{B}',\mathcal{B}''} = G_{\mathcal{B},\mathcal{B}'} \cap G_{\mathcal{B},\mathcal{B}''} \simeq \mathbb{Z}_3 \cap D_8 = 1$ and the lemma is proved. $\square$

This proves Theorem 1.1. $\square$

5 The construction of the Buekenhout geometry

Again, $\mathbb{C}$ is a G-orbit on Baer subplanes in $\mathcal{P}$. By [11], there exists a G-orbit $\mathbb{H}$ on hyperovals such that $|\mathcal{B} \cap \mathcal{H}|$ is odd for all $\mathcal{B} \in \mathbb{C}$ and $\mathcal{H} \in \mathbb{H}$. Let $\mathcal{U}$ be a unital in $\mathcal{P}$. Then $G_{\mathcal{U}} \simeq \mathbb{Z}_3^2 : Q_8$. As in section 2.3, the complement of $\mathcal{U}$ in $\mathcal{P}$ splits into four triangles defined by the parallel classes in $\mathcal{U}$. The union of any two of these triangles is a hyperoval. We choose two triangles $\mathcal{D}_1$ and $\mathcal{D}_2$ of these such that $\mathcal{H}_1 := \mathcal{D}_1 \cup \mathcal{D}_2 \in \mathbb{H}$. Let $H := G_{\mathcal{U}} \cap G_{\mathcal{H}_1}$. Then $H \simeq \mathbb{Z}_3^2 : \mathbb{Z}_4$ acts transitively on $\mathcal{H}_1$. Set $\mathcal{H}_2 := \mathcal{P} - (\mathcal{U} \cup \mathcal{H}_1)$. Then $\mathcal{H}_2 \in \mathbb{H}$. We define

$$\mathbb{B} := \{\mathcal{B} \in \mathbb{C} : \mathcal{B} \cap \mathcal{H}_1 = \mathcal{D}_i, i \in \{1,2\}\}.$$

Lemma 5.1 *$G_{\mathbb{B}} = H$ and H acts transitively on $\mathbb{B}$. For $\mathcal{B} \in \mathbb{B}$ we have $H_{\mathcal{B}} \simeq S_3$.*

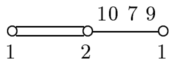
Proof. $G_{\mathcal{U}}$ contains a unique subgroup $\bar{H} \simeq \mathbb{Z}_3^2 : \mathbb{Z}_2$. Thus, $\bar{H}$ fixes all four triangles setwise, especially, $\mathcal{D}_i\bar{H} = \mathcal{D}_i$ for $i \in \{1,2\}$. Moreover, H acts transitively on $\mathcal{H}_1$, hence, $\mathcal{D}_1 H = \{\mathcal{D}_1, \mathcal{D}_2\}$ and $H \leq G_{\mathbb{B}}$. Since G acts transitively on the set of triangles in $\mathcal{P}$, we have $\frac{120 \cdot 28}{1120} = 3$ members of $\mathbb{C}$ through $\mathcal{D}_i$. This yields $|\mathbb{B}| = 6$. Fix $\mathcal{B} \in \mathbb{B}$ with $\mathcal{B} \cap \mathcal{H}_1 = \mathcal{D}_1$. Then $\mathcal{B} - \mathcal{D}_1$ contains a $\mathcal{B}$-line l such that any edge of $\mathcal{D}_1$ contains exactly one point of $\mathcal{B}$ on l. This shows that $\mathcal{U} \cap \mathcal{B} = l$, hence $|\mathcal{B} \cap \mathcal{H}_2| = 1$. Clearly, 3^2 does not divide $|H_l|$ so $H_{\mathcal{B}} \simeq S_3$ and H acts transitively on $\mathbb{B}$. Let M be a maximal subgroup of G conatining H. Then $M = G_{\mathcal{U}}$ or $M = G_{\mathcal{H}_i} \simeq A_6$ for $i \in \{1,2\}$ and H is maximal in M. Since $G_{\mathcal{D}_i} = \bar{H}$, $|\mathcal{D}_i G_{\mathcal{U}}| = 4$ and $G_{\mathcal{U}}$ does not stabilize $\mathbb{B}$. Now $G_{\mathcal{H}_2}$ acts transitively on the complement of $\mathcal{H}_2$ in $\mathcal{P}$, thus $G_{\mathbb{B}} \neq G_{\mathcal{H}_2}$. Finally, $G_{\mathcal{H}_1}$ acts 4-transitively on $\mathcal{H}_1$ proving that $G_{\mathbb{B}} = H$. $\square$

Lemma 5.2 *Γ induces a dual 3×3-grid on $\mathbb{B}$ on which H acts flag-transitively.*

Proof. Let $\mathcal{B}_i \in \mathbb{B}$ with $\mathcal{B}_i \cap \mathcal{H}_1 = \mathcal{D}_i$, $i \in \{1,2\}$. Since $\mathcal{B}_1, \mathcal{B}_2 \in \mathbb{C}$, we have $|\mathcal{B}_1 \cap \mathcal{B}_2| \in \{1,3\}$. If $|\mathcal{B}_1 \cap \mathcal{B}_2| = 3$, then $\mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{U}$ contain at least two points p_1 and p_2. Let now $< x >$ be a complement of $O_3(H)$ in H such that $\mathcal{B}_1 x = \mathcal{B}_2$. Therefore, $p_1 p_2 x = p_1 p_2$. But x interchanges parallel classes of $\mathcal{U}$, a contradiction. So $|\mathcal{B}_1 \cap \mathcal{B}_2| = 1$. Define a point-line geometry Δ as follows: Points (resp. lines) of Δ are the elements of $\mathbb{B}$ intersecting $\mathcal{H}_1$ in $\mathcal{D}_2$ (resp. in $\mathcal{D}_2$). A point of Δ is incident with a line, if their intersection contains a single point of $\mathcal{P}$. Then Δ is a digon. Obviously, x is an automorphism of the incidence graph of Δ which itself is a subgraph of Γ proving the lemma. $\square$

Set $\mathcal{G}(\Gamma)$ to be the following geometry of rank three. The points of $\mathcal{G}(\Gamma)$ are the elements of $\mathbb{C}$, the lines are the wings defined by $\mathbb{C}$ and the planes are the 560 sets in $\mathbb{B}G$ with the natural incidence. By construction, $\mathcal{G}(\Gamma)$ is residually connected and G acts as a flag-transitive automorphism group on $\mathcal{G}(\Gamma)$.

On easily computes that the residue of a wing in $\mathcal{G}(\Gamma)$ is a digon containing each two points and planes. According to [3], the residue of a point in $\mathcal{G}(\Gamma)$ is a $(10, 7, 9)$-gon. So $\mathcal{G}(\Gamma)$ has the following Buekenhout diagram:



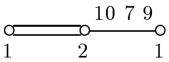
The corresponding amalgam is the following: Let $F := \{\mathcal{B}, \mathcal{W}, \mathbb{B}\}$ be a chamber of $\mathcal{G}(\Gamma)$ then $G_{\mathcal{B}} \simeq L_2(7)$, $G_{\mathcal{W}} \simeq Q_8$ and $G_{\mathbb{B}} \simeq \mathbb{Z}_3^2 : \mathbb{Z}_4$. Furthermore, $G_{\mathcal{B},\mathcal{W}} \simeq \mathbb{Z}_4$, $G_{\mathcal{B},\mathbb{B}} \simeq S_3$, $G_{\mathcal{W},\mathbb{B}} \simeq \mathbb{Z}_4$ and $G_F \simeq \mathbb{Z}_2$.

Lemma 5.3 *Let i be a unitary polarity of $\mathcal{P}$ (i.e. an involution in $OutG$ fixing a unital). Set $A :=< G, i >$. Then A is a flag-transitive automorphism group of both Γ and $\mathcal{G}(\Gamma)$. If $F := \{\mathcal{B}, \mathcal{W}, \mathbb{B}\}$ is a chamber of $\mathcal{G}(\Gamma)$, then $A_{\mathcal{B}} \simeq PGL_2(7)$, $A_{\mathcal{W}} \simeq Q_8 * \mathbb{Z}_4$, $A_{\mathbb{B}} \simeq (\mathbb{Z}_3^2 : \mathbb{Z}_4) \times \mathbb{Z}_2$, $A_{\mathcal{B},\mathcal{W}} \simeq D_8$, $A_{\mathcal{B},\mathbb{B}} \simeq D_{12}$, $A_{\mathcal{W},\mathbb{B}} \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$ and $A_F \simeq \mathbb{Z}_2^2$.*

Proof. Let i be a unitary polarity of $\mathcal{P}$ and let $\mathcal{U}$ be the unital fixed by i. Then $C_G(i) = G_{\mathcal{U}}$. If $H \leq G_{\mathcal{U}}$ with $H \simeq \mathbb{Z}_3^2 : \mathbb{Z}_4$, then $[H, i] = 1$ and i preserves all three G-orbits on hyperovals and Baer subplanes. This shows that $A \leq Aut\mathcal{G}(\Gamma)$ and thereby, $A \leq Aut\Gamma$. If $F := \{\mathcal{B}, \mathcal{W}, \mathbb{B}\}$ is a chamber of $\mathcal{G}(\Gamma)$, then $[G_{\mathcal{B}}, i] \neq 1$, so $A_{\mathcal{B}} \simeq PGL_2(7)$. By construction, $A_{\mathbb{B}} \simeq (\mathbb{Z}_3^2 : \mathbb{Z}_4) \times \mathbb{Z}_2$. Since $|A_{\mathcal{W}} : G_{\mathcal{W}}| = |A_{\mathcal{W},\mathbb{B}} : G_{\mathcal{W},\mathbb{B}}| = 2$, we hold $A_{\mathcal{W},\mathbb{B}} \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$ and $A_F \simeq \mathbb{Z}_2^2$. This implies $A_{\mathcal{B},\mathcal{W}} \simeq D_8$. Therefore, $A_{\mathcal{W}} \simeq Q_8 * \mathbb{Z}_4$ and $A_{\mathcal{B},\mathbb{B}} \simeq \mathbb{Z}_2 \times S_3 \simeq D_{12}$. $\square$

Now we are in the situation to prove Theorem 1.2.

Theorem 5.4 *Let $\mathcal{G}$ be a geometry on which G and A act as flag-transitive automorphism groups. Assume that $\mathcal{G}$ has the following diagram:*



Suppose that $\mathcal{G}$ has 120 points. Then $\mathcal{G} \simeq \mathcal{G}(\Gamma)$.

Proof. Let p be a point of $\mathcal{G}$. Then $|G : G_p| = |A : A_p| = 120$ and therefore, $G_p \simeq L_2(7)$ and $A_p \simeq PGL_2(7)$. For a line l of $\mathcal{G}$ through p, the diagram implies $|G_l| = 8$, $|A_l| = 16$, $|G_p : G_{p,l}| = |A_p : A_{p,l}| = 42$. Since G and A act flag-transitively on $\mathcal{G}$, the collinearity graph of $\mathcal{G}$ is isomorphic to Γ. Since $\mathcal{G}$ 120 points, we have $\frac{120 \cdot 28}{6}$ planes in $\mathcal{G}$. This shows that for a plane e we find $|G_e| = 36$ and $|A_e| = 72$ acting flag-transitively on a dual 3×3-grid. Therefore, G_e (resp. A_e) contains a 3-Sylow subgroup of G (resp. A). If M is a maximal subgroup of G (resp. A) containing G_e (resp. A_e), then M is isomorphic to $\mathbb{Z}_3^2 : Q_8$ or A_6 (resp. $(\mathbb{Z}_3^2 : Q_8) \times \mathbb{Z}_2$ or M_{10}). This implies $G_e \simeq \mathbb{Z}_3^2 : \mathbb{Z}_4$ and $A_e \simeq (\mathbb{Z}_3^2 : \mathbb{Z}_4) \times \mathbb{Z}_2$. So $\mathcal{G}$ and $\mathcal{G}(\Gamma)$ have isomorphic amalgams since in G and A there is only one way to amalgamate these groups in the given manner. $\square$

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