Finite Element Solution of an Improperly Posed Elliptic Cauchy Problem

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Abstract

An improperly posed Cauchy problem for an elliptic differential equation of second order is stated in a weak form by two methods. The existence and uniqueness of the solution of the corresponding variational problems is shown. It is demonstrated that there can be numerical difficulties arising in the finite dimensional treatment by finite element methods. The question how the triangulation must be chosen is investigated. We get the result that it must have a certain regularity. Independent of this, the corresponding finite dimensional linear system is ill–conditioned. Furthermore some error estimats and a convergence result are given.

1 Introduction

There are several methods of treating Cauchy problems for second order elliptic (and also the corresponding parabolic) differential equations, see e.g. [2], [8], [9]. In this paper, among others the numerical solution of problems of this type by means of finite elements is investigated. To be specific, we consider a bounded open region $\Omega \subset \mathbb{R}^n, n = 2$ or 3, with sufficiently smooth boundary Γ which ist decomposed into two non empty parts Γ_0 and Γ_1 such that $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\Gamma_0 \cap \Gamma_1 = \emptyset$. Furthermore, let γ_1 be a non empty subset of Γ_1, $\gamma_1 \subset \Gamma_1$. Γ_0 and γ_1 are assumed to be closed sets and of positive measure. With these preliminaries we define formally the incorrect Cauchy problem for an elliptic differential equation of second order in divergence form :

Problem 1: Find $u: \ \Omega \to \mathbb{R}$ such that

$$-div(a_0 \nabla u) + b_0 u = f \quad \text{on} \ \Omega \tag{1.1}$$

$$u = \varphi \quad \text{on} \ \gamma_1 \tag{1.2}$$

$$g + a_0 \frac{\partial u}{\partial n} = 0 \quad \text{on} \ \Gamma_1. \tag{1.3}$$

u must be thought of as an element of a suitable linear space (see the weak formulation below where spaces are specified). The given data a_0, b_0, g, f and φ are assumed to be independent of u and sufficiently smooth. Furthermore, let $a_0 > 0$, $b_0 \geq 0$. $\partial/\partial n$ on a surface part means the derivative with respect to the outer normal. There are many applications of the above problem. Especially, we mention the electrocardiology, see e.g. [4], where Problem P1 is a mathematical model of the electric potential u of the heart.
With the given expositions, Problem 1 is the classical formulation of an elliptic Cauchy problem. Note that on the surface part Γ_0 no boundary condition is given. On γ_1 two boundary conditions must hold. As is well known these problems are (in the sense of Hadamard) improperly posed [8]. There is a large literature about these. Especially we mention the work of Tarkhanov [9] which discusses several analytic questions (existence, solvability, uniqueness,...).

2 Variational formulation with two linear spaces

We introduce two linear spaces $H_{\Gamma_0} \subset H^1(\Omega)$ and $H_{\gamma_1} \subset H^1(\Omega)$ by

$$H_{\Gamma_0} := \{v \in H^1(\Omega);\ v = 0 \quad \text{on} \quad \Gamma_0\,\}, \tag{2.1}$$

$$H_{\gamma_1} := \{v \in H^1(\Omega);\ v = 0 \quad \text{on} \quad \gamma_1\}. \tag{2.2}$$

(Where possible standard notation is used, cf. [3].) These spaces are closed linear sub-spaces of $H^1(\Omega)$. On $H^1(\Omega) \times H^1(\Omega)$ we define the bilinear form

$$a(u,v) := (a_0 \nabla u, \nabla v) + (b_0 u, v), \qquad u, v \in H^1(\Omega), \tag{2.3}$$

and on $H^1(\Omega)$ the linear form

$$f(v) := (f,v) - < g, v >_{\Gamma_1}, \quad g \in H^{1/2}(\Gamma), \quad v \in H^1(\Omega). \tag{2.4}$$

$(\cdot,\cdot)$ means the $L^2(\Omega)-$scalar product, and $< g, v >_{\Gamma_1}$ denotes the integral

$$\int_{\Gamma_1} g v ds, \quad g \in H^{1/2}(\Gamma), \quad v \in H^1(\Omega),$$

where ds is the (scalar) surface element. Other surface integrals are defined analogously. For simplicity we identify in this article the dual and the primal spaces (e.g. we write $g \in H^{1/2}(\Gamma)$ instead of $g \in H^{-1/2}(\Gamma)$). Defining the space $V := H_{\Gamma_0} \times H^{1/2}(\Gamma)$, a second bilinear form $b(u;(v,\mu)) := a(u,v) + < u, \mu >_{\gamma_1}$ on $H^1(\Omega) \times V$ and a second linear form $F((v,\mu)) := f(v) + < \varphi, \mu >_{\gamma_1}$ on V, we can state

Problem 2: Find $u \in H^1(\Omega)$ such that

$$b(u;(v,\mu)) = F((v,\mu)) \quad \forall\, (v,\mu) \in V. \tag{2.5}$$

It is easily shown that this variational formulation formally reproduces the classical equations given in Problem 1. Setting now (and in the following) $\varphi = 0$ we also have the equivalence of Problem 2 and of the following

Problem 3: Determine $u \in H_{\gamma_1}$ such that

$$a(u,v) = f(v) \quad \forall\, v \in H_{\Gamma_0}. \tag{2.6}$$

This is one of the basic problems in this paper. For numerical approximations we also need the finite dimensional form of this problem. With suitable finite dimensional linear spaces $H_{\gamma_1 h} \subset H_{\gamma_1}$ and $H_{\Gamma_0 h} \subset H_{\Gamma_0}$ we introduce also

Problem 4: Find $u_h \in H_{\gamma_1 h}$ such that

$$a(u_h, v_h) = f(v_h) \quad \forall\, v_h \in H_{\Gamma_0 h}. \tag{2.7}$$

2.1 Theoretical results

Some theoretical results (among others concerning the existence and uniqueness of solutions) for elliptic Cauchy problems will be mentioned. The solvability of ill–posed Cauchy problems of the type stated in Problem 1 (especially for the Laplace equation) is investigated in, e.g., [8, Sections 5,6] and extensively by Tarkhanov [9]. The Cauchy problem is solvable in certain spaces. For the Laplace equation and defined Sobolew spaces this is proved under some assumptions in [9, Chapter 12]. In [8, pp.33–35] the continuous dependence (stability) in a certain norm sense of the solution on the Cauchy data (φ and g in Problem 1) is proved.

Existence and uniqueness of solutions of the variational problems given above can be investigated, e. g., by means of the following Theorem due to Babuška [1] (notice that the well-known Lax-Milgram-Theorem can not be applied here):

Theorem: Let H_1, H_2 be two Hilbert spaces. Let $B(\cdot,\cdot)$ be a bilinear form on $H_1 \times H_2$. Assume

$$|\, B(u,v)\,| \leq c_1 \parallel u \parallel_{H_1} \parallel v \parallel_{H_2}, \quad \forall\, u \in H_1, \forall v \in H_2, \quad c_1 < \infty, \tag{2.8}$$

$$\sup_{\|u\|_{H_1} \leq 1} |\, B(u,v)\,| \geq c_2 \parallel v \parallel_{H_2}, \quad \forall\, v \in H_2, \quad c_2 > 0, \tag{2.9}$$

$$\sup_{\|v\|_{H_2} \leq 1} |\, B(u,v)\,| \geq c_3 \parallel u \parallel_{H_1}, \quad \forall\, u \in H_1, \quad c_3 > 0. \tag{2.10}$$

Further let $F_0 \in H_2'$ be a linear form on H_2.
Then there is axactly one element $u_0 \in H_1$ such that

$$B(u_0, v) \;=\; F_0(v) \quad \forall\, v \in H_2$$

and

(2.11) $$\| u_0 \|_{H_1} \leq \frac{1}{c_3} \| F_0 \|_{H_2'} .$$

In our case it is $B(\cdot,\cdot) = a(\cdot,\cdot)$ and $F_0(\cdot) = f(\cdot)$. For Problems 3 and 4 we must choose $H_1 = H_{\gamma_1}$, $H_2 = H_{\Gamma_0}$ and $H_1 = H_{\gamma_1 h}$, $H_2 = H_{\Gamma_0 h}$, respectively. First we consider the existence and uniqueness of solutions for the basic Problem 3.

We note that the bilinear form $a(\cdot,\cdot)$, eq. (2.3), is $H^1(\Omega)-$elliptic and continuous with positive constants α_0 and α_1, respectively (a_0 and b_0 are assumed to be continuous in $\overline{\Omega}$):

(2.12) $$\begin{aligned} a(v,v) \;&\geq\; \alpha_0 \| v \|^2 \quad \forall v \in H^1(\Omega), \qquad \| \cdot \| \;\equiv\; \| \cdot \|_{H^1(\Omega)}, \\ \alpha_0 &= \min_{x\in\Omega}\{a_0(x),\; b_0(x)\}. \end{aligned}$$

(2.13) $$| \, a(u,v) \, | \leq \; \alpha_1 \| u \| \| v \| \quad \forall \, u,v \in H^1(\Omega).$$

To apply the foregoing Theorem to Problem 3 a few definitions and hypotheses are needed.

Definition:

D1. For any $v \in H_{\Gamma_0}$ there is an element $\psi_v \in H^1(\Omega)$, $\psi_v \neq v$, with $\psi_v \,|_{\gamma_1} = v \,|_{\gamma_1}$.

D2. For any $v \in H_{\Gamma_0}$ a linear form $\overline{f}_v(\cdot)$ is defined on $H^1(\Omega)$. Simple examples are $\overline{f}_v(\overline{v}) := K_1(v,\overline{v})$, $\overline{f}_v(\overline{v}) = K_1(v,\overline{v})_{L^2}/(K_2 + \| v \|_{L^2} \cdot \| \overline{v} \|_{L^2})$ and $\overline{f}_v(\overline{v}) := K_1(v,\overline{v}) + K_2 < v,\overline{v} >_{\Gamma_{sc}}$ $\forall \, \overline{v} \in H^1(\Omega)$. $(\cdot,\cdot)$ can be the scalar product on $L^2(\Omega)$ or $H^1(\Omega)$. K_1, K_2 are arbitrary parameters. $\Gamma_{sc} \subset \Gamma$ is a boundary part which can be chosen suitably.

D3. For given $v \in H_{\Gamma_0}$, $\overline{w}_{v,\kappa} \in H^1(\Omega)$ is defined as solution of the variational equation

(2.14) $$a(\overline{w}_{v,\kappa},\overline{v}) + \kappa < \overline{w}_{v,\kappa},\overline{v} >= -a(\psi_v,\overline{v}) + \overline{f}_v(\overline{v}) \quad \forall \quad \overline{v} \in H^1(\Omega).$$

κ is a non negative parameter. Especially, at the end (of the proof for the following Lemma) the limit $\kappa \to +\infty$ is of interest. For large κ the boundary term $\kappa < \overline{w}_{v,\kappa},\overline{v} >$ forces the solution $\overline{w}_{v,\kappa}$ to have "small" boundary values (cf. e.g. [7]). This follows from the equivalence of the given variational equation and the minimization problem in $H^1(\Omega)$

$$a(\overline{v},\overline{v}) + \kappa < \overline{v},\overline{v} > +2\left(a(\psi_v,\overline{v}) - \overline{f}_v(\overline{v})\right) \to \min .$$

Remark: For the purpose of the proof of the following Lemma, in eq. (2.14) the term $\kappa < \overline{w}_{v,\kappa},\overline{v} >$ could be replaced, e.g., by $\kappa < \overline{w}_{v,\kappa},\overline{v} >_{\gamma_1}$ as well.

Lemma 2.1: Suppose Γ is sufficiently smooth such that the following assumptions hold:

A1. There is a positive constant K_{11} such that $\| \overline{f}_v \|^* \leq K_{11} \| v \| \; \forall \, v \in H_{\Gamma_0}$ where K_{11} is independent of v and $\| \cdot \|^*$ is the norm on the dual space of $H^1(\Omega)$.

A2. For any $v \in H_{\Gamma_0}$, ψ_v (according to D1) and $\overline{f}_v$ can be chosen such that

$$\lim_{\kappa \to +\infty} \left[-\kappa < \overline{w}_{v,\kappa}, v > + \overline{f}_v(v) \right] \leq 0.$$

$\overline{w}_{v,\kappa}$ is defined in D3.

A3. There is a finite positive constant β_0 (independent of v) such that

$$\| \psi_v \| \leq \beta_0 \| v \|, \; v \in H_{\Gamma_0}.$$

Then there is a unique solution $u \in H_{\gamma_1}$ of Problem 3.

Remark: The crucial point is hypothesis A2. For example, if we would use $\overline{f}_v(\overline{v}) = K_1(v, \overline{v})$, then $\overline{w}_{v,\kappa}$ as a solution of eq. (2.14) is a function of K_1 and A2 can be reduced to the following problem: determine (for large κ) a real parameter K_1 (assumed to exist) such that $K_1 \leq \frac{\kappa < \overline{w}_{v,\kappa}(K_1), v>}{(v,v)}$, $v \neq 0$.

Proof: It is shown that the assumptions (2.8)–(2.10) of the Theorem of Babuška are satisfied. Since $a(\cdot,\cdot)$ and $f(\cdot)$ are continuous on $H^1(\Omega) \times H^1(\Omega)$ and $H^1(\Omega)$, respectively, we only must show that the ineqs. (2.9) and (2.10) are valid. These read here

$$\sup_{u \in H_{\gamma_1}, \|u\| \leq 1} | \, a(u,v) \, | \geq c_2 \| v \| \quad \forall v \in H_{\Gamma_0}, \tag{2.15}$$

$$\sup_{v \in H_{\Gamma_0}, \|v\| \leq 1} | \, a(u,v) \, | \geq c_3 \| u \| \quad \forall u \in H_{\gamma_1}. \tag{2.16}$$

We prove the first of these inequalities. The second can be obtained from an analogous argument (see below) since ineq. (2.16) can be generated from ineq. (2.15) by the change $u \rightleftharpoons v$, $H_{\gamma_1} \rightleftharpoons H_{\Gamma_0}$ (notice that $a(\cdot,\cdot)$ is symmetric). Now we consider ineq. (2.15).
We use the inclusions $H_{\gamma_1} \subset H^1(\Omega)$ and $H_{\Gamma_0} \subset H^1(\Omega)$. Let an arbitrary (fixed) $v \in H_{\Gamma_0}$ be given. If $v = 0$ then (2.15) is true. Therefore we consider $v \neq 0$. If $v \,|_{\gamma_1} = 0$ then it is also $v \in H_{\gamma_1}$, and it follows immediately

$$\sup_{\|u\| \leq 1} | \, a(u,v) \, | = \sup_{\|u\| > 0} \frac{| \, a(u,v) \, |}{\| u \|} \geq \frac{a(v,v)}{\| v \|} \geq c_2 \| v \| \quad \text{with} \quad c_2 = \alpha_0.$$

For this v, ineq. (2.15) is satisfied. Suppose now $v \,|_{\gamma_1} \neq 0$.
Write $u \in H_{\gamma_1}$ in the form $u = v - w$, $w \in H^1(\Omega)$, with $w \,|_{\gamma_1} = v \,|_{\gamma_1}$. It follows

$$\sup_{\|u\| \leq 1} | \, a(u,v) \, | = \sup_{\|u\| > 0} \frac{| \, a(u,v) \, |}{\| u \|} = \sup_{w \in H^1(\Omega)} \frac{| \, a(v-w,v) \, |}{\| v - w \|} = \sup_{w \in H^1(\Omega)} \frac{| \, a(v,v) - a(w,v) \, |}{\| v - w \| \, .}$$

Here and in the following it is always understood that $\| v - w \| > 0$ and $w |_{\gamma_1} = v |_{\gamma_1}$.
As is obvious, it would be sufficient to have an element $w^* \in H^1(\Omega)$ such that

$$w^* |_{\gamma_1} = v |_{\gamma_1}, \quad a(w^*, v) \leq 0 \quad \text{and} \quad \| v \| \geq c \| v - w^* \| > 0, \quad c > 0.$$

From this we would obtain

$$\frac{| a(v,v) - a(w^*, v) |}{\| v - w^* \|} \quad \geq \quad c \frac{a(v,v)}{\| v \|} \quad \geq \quad \alpha_0 c \| v \| .$$

Since

$$\sup_{w \in H^1(\Omega)} \frac{| a(v,v) - a(w,v) |}{\| v - w \|} \quad \geq \quad \frac{| a(v,v) - a(w^*, v) |}{\| v - w^* \|}$$

we arrive at

$$\sup_{u \in H_{\gamma_1}, \|u\| \leq 1} | a(u,v) | \quad \geq \quad \alpha_0 c \| v \| \qquad v \in H_{\Gamma_0},$$

i.e. ineq. (2.15) for that v. Now we construct such an element w^* with the help of $\overline{w}_{v,\kappa}$ given in D3. Define w^* by $w^* := \overline{w}_{v,\kappa} + \psi_v$ where ψ_v is characterized in D1. Since $v \in H_{\Gamma_0} \subset H^1(\Omega)$, eq. (2.14) gives

$$a(\overline{w}_{v,\kappa} + \psi_v, v) = -\kappa < \overline{w}_{v,\kappa}, v > + \overline{f}_v(v)$$

and from assumption A2 we get

$$\sup_{w \in H^1(\Omega), \|v - w\| > 0} \frac{| a(v,v) - a(w,v) |}{\| v - w \|} \geq \frac{a(v,v)}{\| v - \psi_v - \overline{w}_{v,\kappa} \|} \tag{2.17}$$

To estimate the denominator on the right, in eq. (2.14) the special case $\overline{v} = \overline{w}_{v,\kappa}$ is taken. Using ineqs. (2.12) ansd (2.13) we find

$$\alpha_0 \| \overline{w}_{v,\kappa} \| \leq \alpha_0 \| \overline{w}_{v,\kappa} \| + \kappa \frac{\| \overline{w}_{v,\kappa} \|^2_{L^2(\Gamma)}}{\| \overline{w}_{v,\kappa} \|} \leq \alpha_1 \| \psi_v \| + \| \overline{f}_v \|^* .$$

This and the assumption A3 yield

$$\| v - w^* \| = \| v - \psi_v - \overline{w}_{v,\kappa} \| \leq \| v \| + \| \psi_v \| + \| \overline{w}_{v,\kappa} \| \leq \left(1 + \beta_0 + \frac{\alpha_1 \beta_0}{\alpha_0} \right) \| v \| + \frac{1}{\alpha_0} \| \overline{f}_v \|^* .$$

Thus under assumption A1 there is a constant $c > 0$ (c independent of v) such that

$$\frac{1}{\| v - w^* \|} \geq \frac{c}{\| v \|}.$$

This is used in ineq. (2.17) and the result is that ineq. (2.15) is satisfied.

As mentioned, ineq. (2.16) can be proved under analogous assumptions. The main steps will be sketched: Construct (see D1) for given $u \in H_{\gamma_1}$ an element $\varphi_u \in H^1(\Omega)$, $\varphi_u \neq u$, $\varphi_u |_{\Gamma_0} = u |_{\Gamma_0}$. Choose (see D2) a suitable functional, say, $\underline{f}_u$ on $H^1(\Omega)$. As in D3 determine $\underline{w}_{u,\kappa} \in H^1(\Omega)$ such that

$$a(\underline{w}_{u,\kappa}, \overline{v}) + \kappa < \underline{w}_{u,\kappa}, \overline{v} >= -a(\varphi_u, \overline{v}) + \underline{f}_u(\overline{v}) \quad \forall \quad \overline{v} \in H^1(\Omega).$$

The analogous assumption A1 is $\| \underline{f}_u \|^* \leq K_{22} \| u \| \;\; \forall u \in H_{\gamma_1},\; K_{22} > 0$. Assumption A2 is replaced by

$$\lim_{\kappa \to +\infty} \left[-\kappa < \underline{w}_{u,\kappa}, u > + \underline{f}_u(u) \right] \;\leq\; 0, \quad u \in H_{\gamma_1}.$$

In A3 we write $\| \varphi_u \| \leq \beta_1 \| u \|$, $u \in H_{\gamma_1}$, $\beta_1 > 0$. In ineq. (2.16) for any $u \in H_{\gamma_1}$, $v \in H_{\Gamma_0}$ is written $v = u - W$, $W \in H^1(\Omega)$, with $W \mid_{\Gamma_0} = u \mid_{\Gamma_0}$. Choosing the special element $W^* := \underline{w}_{u,\kappa} + \varphi_u$ and replacing now w^* by W^*, estimates analogous to those given above can be done to obtain ineq. (2.16). Under this assumptions this inequality is verified. This proves the Lemma. $\Box$

As will be seen in the next subsection, Problem 4 is of special interest. Using the foregoing Lemma 2.1 existence and uniqueness of a solution can be proved for it. However, the existence depends for the Cauchy problem on the triangulation, see Lemma 2.2. Hence, we shall first consider a typical finite element solution of Problem 4.

2.2 Numerical solution by FEM

For simplicity we consider a problem in two dimensions, $\Omega \subset \mathbb{R}^2$. A typical example is shown in the Fig. 1.

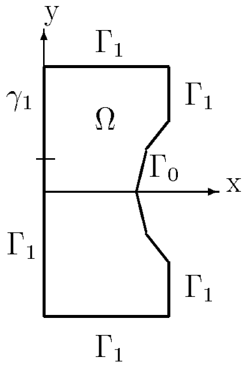

Fig. 1
The bounded open set Ω used as a typical example.
$\Gamma = \Gamma_0 \cup \Gamma_1, \quad \gamma_1 \subset \Gamma_1$.

In order to simplify the following furthermore we assume Ω to be a polygon. Let $\overline{\Omega}$ be triangulated by triangels and let the number auf nodes on $\overline{\Omega}$ be M. $p\,(p \geq 1)$ nodes are assumed to be on each of the boundary parts γ_1 and Γ_0. The numbering is chosen as follows:

$$(2.18) \qquad P_1, ..., P_p \in \gamma_1; \qquad P_{p+1}, ..., P_{2p} \in \Gamma_0, \qquad 2p < M.$$

The distance of γ_1 and Γ_0 is assumed to be positive: $\mathrm{dist}(\gamma_1, \Gamma_0) > 0$. The finite dimensional spaces $H_{\gamma_1 h}$ and $H_{\Gamma_0 h}$ may be constructed in the standard way, cf. e.g. [3], by means of basis functions denoted by w_j:

basis for $H_{\gamma_1 h}:\ \{w_j;\ j = p+1, ..., M\}$; $\qquad$ basis for $H_{\Gamma_0 h}:\ \{w_i;\ i = 1, ..., p, 2p+1, ..., M\}$.

The functions w_j are assumed to satisfy

$$w_j(P_i) \;=\; \delta_{ij} \quad \text{for} \quad i, j = 1, ..., M.$$

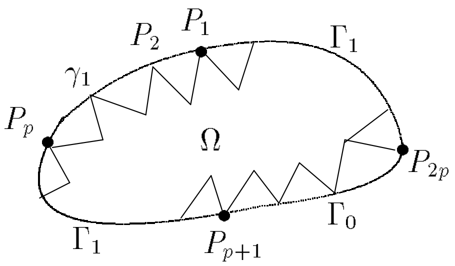

Fig. 2 The characterization of the nodes on the boundary parts γ_1 and Γ_0. A triangulation is indicated.

$u_h \in H_{\gamma_1 h}$ is written

$$u_h = \sum_{j=p+1}^{M} u_j w_j \tag{2.19}$$

where the coefficients u_j are solutions (if a solution exists) of the linear system

$$\sum_{j=p+1}^{M} a_{ij} u_j = f(w_j) \quad \text{for} \quad i = 1, ..., p, 2p+1, ..., M, \tag{2.20}$$

$a_{ij} = a(w_j, w_i)$. This system results from Problem 4. What is essential is the definition set of the indices i and j in eqs. (2.20). Defining a square matrix B and vectors U and b by

$$B = (a_{ij}) \quad \text{with} \quad i = 1, ..., p, 2p+1, ..., M; \quad j = p+1, ..., M,$$
$$U = (u_{p+1}, ..., u_M)^T,$$
$$b = (b_1, ..., b_p, b_{2p+1}, ..., b_M)^T, \qquad b_i = f(w_i),$$

eqs. (2.20) can be written shortly

$$BU = b. \tag{2.21}$$

Now it may be that the matrix B is singular. In general there are the following

Observations:

1. The regularity of the coefficient matrix B depends on the triangulation. There are admissible triangulations giving a singular B independent of
   (a) the data of the problem (here: a_0, b_0),
   (b) the boundary Γ,
   (c) the numerical method to calculate the matrix elements a_{ij}.
2. Given an admissible triangulation and a set of standard FEM basis functions for the spaces $H_{\gamma_1 h}$ and $H_{\Gamma_0 h}$, the matrix B may be singular independent of 1.(a) – (c).
3. In general, the matrix B (if regular) is ill–conditioned. In typical examples the order of the condition number of B is

   $$\| B \| . \| B^{-1} \| \approx 10^4 ... 10^7$$

   for $p \in [8, 20]$ and $M \in [50, 200]$.

(For admissible triangulations see e.g. [3, p.38, p.51].)
On the other hand there are triangulations which generate a regular matrix B for the ill–posed elliptic Cauchy problem. The stated observations are in sharp contrast to "normal" well–posed elliptic boundary value problems where the matrix B is, in general, regular for admissible triangulations.
These observations suggest that it is suitable to distinguish between triangulations giving a regular or a singular matrix B. Therefore we introduce the

Definition: A triangulation $\mathcal{T}_h$ is called Cauchy regular if the matrix B corresponding to the (finite dimensional) Problem 4 is regular, otherwise $\mathcal{T}_h$ is called Cauchy singular (or not Cauchy regular).

In the next subsection some simple examples for Cauchy regular and Cauchy singular $\mathcal{T}_h$ are given.

2.3 Examples

The examples given below illustrate the possibilities which can appear when constructing the matrix B, eq. (2.21), by means of the eqs. (2.20). It is sufficient for our purposes to use standard finite elements of first order.

Example 1: Let $p = 3$, $M = 9$ and $\mathcal{T}_h$ be chosen as in Fig. 3.

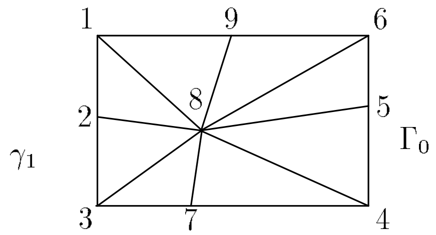

Fig. 3 Cauchy regular triangulation

This $\mathcal{T}_h$ is Cauchy regular as can be seen from eq. (2.21). In detail this system of equations reads

$$\sum_{j=4}^{9} a_{ij} u_j = b_i, \qquad i = 1,2,3,7,8,9,$$

and the matrix B takes the form

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & a_{18} & a_{19} \\ 0 & 0 & 0 & 0 & a_{28} & 0 \\ 0 & 0 & 0 & a_{37} & a_{38} & 0 \\ a_{74} & 0 & 0 & a_{77} & a_{78} & 0 \\ a_{84} & a_{85} & a_{86} & a_{87} & a_{88} & a_{89} \\ 0 & 0 & a_{96} & 0 & a_{98} & a_{99} \end{pmatrix}.$$

The elements a_{ij} shown are different from zero. It is easy to see that B is regular. Note that diagonal elements are different from zero only if the corresponding vertices are neither on γ_1 nor on Γ_0. This fact is one reason for the bad condition of the matrices B for elliptic Cauchy problems.

Example 2: This admissible triangulation with $p = 3$ and $M = 7$ is Cauchy singular.

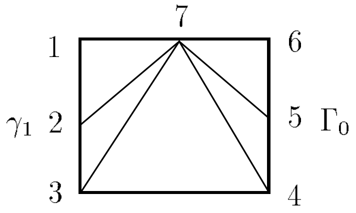

Fig. 4 Cauchy singular triangulation

The system of equations (2.20) is in detail

$$\sum_{j=4}^{7} a_{ij} u_j = b_i, \qquad i = 1, 2, 3, 7,$$

and the matrix B is

$$B = \begin{pmatrix} 0 & 0 & 0 & a_{17} \\ 0 & 0 & 0 & a_{27} \\ a_{34} & 0 & 0 & a_{37} \\ a_{74} & a_{75} & a_{76} & a_{77} \end{pmatrix}.$$

It is obviously singular.

Example 3: A further Cauchy singular triangulation is shown in Fig. 5.

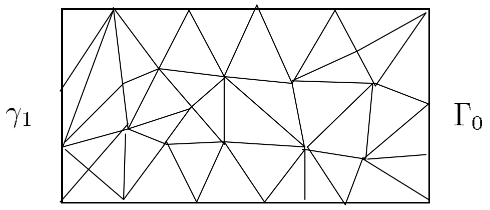

Fig. 5 Cauchy singular triangulation

It can easily be changed into a regular Cauchy triangulation.
These examples show that the triangulation for Cauchy problems must be done carefully.

2.4 Results for the finite element solution

We consider here a traditional finite element solution of Problem 4 (see subsection 2.2). As a consequence of Lemma 2.1 and the fact that the triangulation must be taken into account (subsections 2.2 and 2.3) we can state

Lemma 2.2: Let $H_{\gamma_1 h}$ and $H_{\Gamma_0 h}$ be finite dimensional spaces (of piecewise defined polynomials of given order). Furthermore let the triangulation be Cauchy regular and assume

(2.22) $$\dim(H_{\gamma_1 h}) = \dim(H_{\Gamma_0 h}).$$

Then there is a unique solution u_h of Problem 4.

This Lemma can easily be proved either with the help of Lemma 2.1 or with the representation of u_h in the form (2.19) and with the basis of $H_{\Gamma_0 h}$ mentioned in subsection 2.2.
If the condition (2.22) is not satisfied then Problem 4 has either no solution or the solution is not unique.

Based on the considerations in the foregoing subsections we can summarize some results concerning the finite dimensional solution of elliptic Cauchy problems by means of the variational Problems 3 resp. 4.

Results:

1. In contrast to well-posed boundary value problems of linear elliptic differential equations of second order, the regularity of the coefficient matrix B (stiffness matrix) depends on the triangulation.
2. Although the bilinear form $a(\cdot,\cdot)$ is symmetric and $H^1(\Omega)-$elliptic, the matrix B is, in general, neither symmetric nor definit.
3. The condition number of the matrix B is large for Cauchy problems. The system of equations (2.20) (or (2.21)) and the matrices B of the simple Examples 1 and 2 show the origin for the ill–conditioness. The Cauchy boundary conditions (1.2) and (1.3) produce in Problem 4 the fact that the diagonal elements of B (and the most neighbour elements) are different from zero only if the corresponding node is neither on γ_1 nor on Γ_0. In the finite element solution of "normal" well-posed boundary value problems this cannot happen.

3 Simultaneous determination of the flux on Γ_0

In applications sometimes the flux $-a_0\partial u/\partial n$ on the boundary part Γ_0 (where no boundary condition is prescribed) is also needed. We consider a possibility to calculate an approximation of it simultaneous with u_h. This is based on

Problem 5: Find $(u,\lambda) \in H^1(\Omega) \times H^{1/2}(\Gamma_0)$ such that

$$a(u,v) + <\lambda, v>_{\Gamma_0} = f(v) \quad \forall\, v \in H^1(\Omega) \tag{3.1}$$

under the side condition

$$<u,\mu>_{\gamma_1} = 0 \quad \forall\, \mu \in H^{1/2}(\Gamma). \tag{3.2}$$

$a(\cdot,\cdot)$ and $f(\cdot)$ are defined in eqs. (2.3) and (2.4) respectively. $H^{1/2}(\Gamma_0)$ denotes the linear space of functions in $H^{1/2}(\Gamma)$ restricted to Γ_0 :

$$H^{1/2}(\Gamma_0) := \{w \in H^{1/2}(\Gamma)\,; \quad w\,|_{\Gamma_0}\}.$$

Problem 5 reproduces the classical equations of Problem 1 (with $\varphi = 0$) and in addition

$$a_0 \frac{\partial u}{\partial n} + \lambda = 0 \quad \text{on} \quad \Gamma_0. \tag{3.3}$$

Thus λ is identical with the flux on Γ_0. The boundary condition $u = 0$ on γ_1 is satisfied by the constraint (3.2). For a finite dimensional approximation, eq. (3.1) is used below. For some theoretical considerations the following Problem 6 is defined. It is equivalent to Problem 5.

Problem 6: Find $(u, \lambda) \in H_{\gamma_1} \times H^{1/2}(\Gamma_0)$ such that

$$a(u, v) + < \lambda, v >_{\Gamma_0} = f(v) \quad \forall\, v \in H^1(\Omega) \tag{3.4}$$

The following Lemma states the relation between the solutions of the Problems 3 and 6.

Lemma 3.1: Let $\overline{u} \in H_{\gamma_1}$ be the (unique) solution of Problem 3 and $(u, \lambda) \in H_{\gamma_1} \times H^{1/2}(\Gamma_0)$ be a solution of Problem 6. Then

1. $u = \overline{u}$.
2. The solution (u, λ) is unique.

Note that for given $\lambda = \lambda_0$, $\overline{u}$ is (in general) not a solution component of Problem 6.

Proof: Since $H_{\Gamma_0} \subset H^1(\Omega)$ the difference of eqs. (2.6) and (3.4) yields $a(\overline{u} - u, v) - < \lambda, v >_{\Gamma_0} = 0 \quad \forall v \in H_{\Gamma_0}$ where $< .,. >_{\Gamma_0}$ is the surface integral on Γ_0. But this surface integral is zero for $v \in H_{\Gamma_0}$. Hence $a(\overline{u} - u, v) = 0\ \forall\, v \in H_{\Gamma_0}$ where $\overline{u} - u \in H_{\gamma_1}$. This variational equation is a special case of Problem 3. Thus we can use Lemma 2.1 in conjunction with eq. (2.11) which can be applied here. This proves $u = \overline{u}$ and the uniqueness of u.
To show that λ is unique too, we suppose there are two different solutions λ_1 and λ_2 (for $u = \overline{u}$). Subtracting the two corresponding equations (3.4) one from another we get $< \lambda_1 - \lambda_2, v >_{\Gamma_0} = 0 \quad \forall\, v \in H^1(\Omega)$. From this the uniqueness of λ follows. This proves the Lemma. $\square$

3.1 Finite dimensional solution with collocation

We use the notation and the general assumptions of subsection 2.2 and construct a finite element solution which is based on a triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ and the obvious conform finite dimensional version of Problem 5. We define the finite dimensional spaces $H_h^1(\Omega) \subset H^1(\Omega)$ and $H_h^{1/2}(\Gamma_0) \subset H^{1/2}(\Gamma_0)$ with traditional basis functions w_j and $\overline{w}_j$, respectively, as follows:

$$H_h^1(\Omega): \quad \{w_j\}_1^M, \quad u_h = \sum_1^M u_j w_j, \qquad \text{and} \qquad H_h^{1/2}(\Gamma_0): \quad \{\overline{w}_j\}_{p+1}^{2p}, \quad \lambda_h = \sum_{p+1}^{2p} \lambda_j \overline{w}_j.$$

Inserting this into the finite dimensional form of eq. (3.1), we get

$$\sum_{j=1}^{M} a_{ij}u_j \;=\; b_i - \sum_{j=p+1}^{2p} s_{ij}\lambda_j, \quad i = 1, 2, ..., M, \tag{3.5}$$

where the elements a_{ij} and b_i are given as in subsection 2.2. s_{ij} is defined by

$$s_{ij} = \begin{cases} < \overline{w}_j, w_i >_{\Gamma_0} & : \; i, j \in \{p+1, ..., 2p\} \\ 0 & : \; \text{otherwise.} \end{cases} \tag{3.6}$$

The side condition (3.2) can be written in several finite dimensional representations. The simplest one is a collocation method which can be generated by choosing

$$< u_h(x), \delta(x - \overline{x}_i) >= 0, \quad i = 1, ..., p, \quad \text{i. e.} \quad \overline{x}_i \in \gamma_1, \tag{3.7}$$

where δ is the Dirac generalized function.

It is convenient to identify the collocation points $\overline{x}_i$ with the p knots of $\mathcal{T}_h$ on γ_1. This gives immediately $u_1 = u_2 = ... = u_p = 0$ (i. e. $u_h = 0$ on γ_1). Therefore, the index j in the sum on the left of eq. (3.5) runs from $p+1$ until M only. Defining the vector of the unknowns $X := (\lambda_{p+1}, ..., \lambda_{2p}, u_{p+1}, ..., u_M)^T$ and the right hand vector $\overline{b} = (b_1, ..., b_M)^T$, eq. (3.5) can be rewritten in matrtix form

$$\overline{B}X \;=\; \overline{b} \quad \text{with} \quad \overline{B} = \begin{pmatrix} O_{11} & \overline{B}_{12} \\ S & \overline{B}_{22} \\ O_{31} & \overline{B}_{32} \end{pmatrix}. \tag{3.8}$$

O_{11} and O_{31} are $(p,p)-$ and $(M-2p,p)-$matrices with zero elements. The remaining matrices are of the following structure:
$S:\ (p,p), \qquad \overline{B}_{12}:\ (p, M-p), \qquad \overline{B}_{22}:\ (p, M-p), \qquad \overline{B}_{32}:\ (M-2p, M-p).$
The elements of S and $\overline{B}$ are s_{ij} and a_{ij}, respectively, where the definition set of the indices must be taken from eq. (3.5).
The $(M,M)-$matrix $\overline{B}$ has the same qualitative characteristics concerning the regularity and the condition as the matrix B in eq. (2.21). This follows from the structure of $\overline{B}$. For example, the determinant of $\overline{B}$ is given by

$$\det(\overline{B}) = \det(S)\det\begin{pmatrix} \overline{B}_{12} \\ \overline{B}_{32} \end{pmatrix} \tag{3.9}$$

where the matrix

$$\begin{pmatrix} \overline{B}_{12} \\ \overline{B}_{32} \end{pmatrix}$$

is identical with B, eq. (2.21). S is (in general) a regular matrix (of moderate condition), see eq. (3.6). Thus if $\mathcal{T}_h$ is Cauchy regular then both B and $\overline{B}$ are regular (and, of course, ill–conditioned).
The influence of the triangulation on the regularity of the coefficient matrix $\overline{B}$ is illustrated

by the following example.

Example 4: Suppose the triangulation is the same as in Example 2 (subsection 2.3). Then the linear system (3.5) takes the form

$$\sum_{j=4}^{7} a_{ij}u_j + \sum_{j=4}^{6} s_{ij}\lambda_j \;=\; b_i, \qquad i = 1, ..., 7,$$

and the matrix $\overline{B}$ is

$$\overline{B} \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & a_{17} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{27} \\ 0 & 0 & 0 & a_{34} & 0 & 0 & a_{37} \\ s_{44} & s_{45} & s_{46} & a_{44} & a_{45} & 0 & a_{47} \\ s_{54} & s_{55} & s_{56} & a_{54} & a_{55} & a_{56} & a_{57} \\ s_{64} & s_{65} & s_{66} & 0 & a_{65} & a_{66} & a_{67} \\ 0 & 0 & 0 & a_{74} & a_{75} & a_{76} & a_{77} \end{pmatrix}$$

which has the structure indicated in (3.8). If the rows 4, 5, 6 and the columns 1, 2, 3 are eliminated from $\overline{B}$ then we obtain the matrix B given in Example 2. This is in accordance with eq. (3.9). $\overline{B}$ is singular for this $\mathcal{T}_h$, as it should be.

The results of this section can be summarized by the following

Lemma 3.2: Assume

1. The triangulation $\mathcal{T}_h$ is Cauchy regular.
2. The number of nodes on γ_1 is the same as on Γ_0.

Then there is a unique finite element collocation solution $(u_h, \lambda_h) \in H_h^1(\Omega) \times H_h^{1/2}(\Gamma_0)$ of the finite dimensional version of Problem 5.
(u_h, λ_h) is determined by eq. (3.8). The coefficient matrix $\overline{B}$ is (if regular) ill–conditioned.

4 Convergence and Error Estimates

In this section we consider error estimates for $\| u - u_h \|$ where $\| \cdot \|$ is the $H^1(\Omega)$ norm. The general assumptions are:

A1. The bilinear form $a(\cdot,\cdot)$ is H^1-elliptic and continuous in $H^1(\Omega)\times H^1(\Omega)$ with constants α_0 and α_1, respectively (ineqs. (2.12), (2.13)).

A2. u and u_h are the unique solutions of Problem 3 and Problem 4, respectively.

A3. The notation and the assumptions for the finite element solution u_h (cf. subsection 2.2) are valid.

For a first consideration the numerical solution $u_h \in H_{\gamma_1 h}$, eq. (2.19), is decomposed into two parts:

$$u_h = w_h + \sum_{2p+1}^{M} u_j w_j \qquad \text{with} \qquad w_h = w_h[u_h] := \sum_{p+1}^{2p} u_j w_j. \tag{4.1}$$

The second term on the right of u_h is zero on $\gamma_1 \cup \Gamma_0$. w_h is zero on all inner nodes of $\mathcal{T}_h$ and, of course, $w_h \in H_{\gamma_1 h}$.
The first result is a Lemma (one could it call "generalized Cea-Lemma") which shows the special features of the elliptic Cauchy problem:

Lemma 4.1: Under the assumptions A1 – A3 the estimate

$$\| u - u_h \| \le \frac{\alpha_1}{\alpha_0} \inf_{v_h \in H_{\Gamma_0 h}} \| u - (w_h[u_h] + v_h) \| \tag{4.2}$$

is valid.
Proof: For $v_h \in H_{\Gamma_0 h}$ we subtract eq. (2.7) from eq. (2.6) to get $a(u - u_h, v_h) = 0$. Since $u_h - w_h \in H_{\Gamma_0 h}$ it is also $a(u - u_h, u_h - w_h) = 0$. Therefore $a(u - u_h, u - (u_h - w_h)) = a(u - u_h, u - v_h) \forall v_h \in H_{\Gamma_0 h}$. This can be written $a(u - u_h, u - u_h) = a(u - u_h, u - (w_h + v_h))$. Using now assumption A1 and taking the inf in $H_{\Gamma_0 h}$, ineq. (4.2) follows. $\square$

In this formulation there are two basic problems (which are in contrast to well-posed elliptic boundary value problems of second order) for error estimates of elliptic Cauchy problems. Firstly, the right hand side of ineq. (4.2) depends explicitly on u_h (because of w_h). Secondly, the inf must be taken in $H_{\Gamma_0 h}$. However, the element $u - w_h \in H_{\gamma_1}$ is, in general, not an element of H_{Γ_0}. (In special cases, $u - w_h \in H_{\gamma_1} \cap H_{\Gamma_0}$ may be.) This is a difficulty for error estimates of elliptic Cauchy problems.
The Lemma can be reformulated. For this a few assumptions are needed:

A4. Let $w[u] \in H^r(\Omega)$ with $w[u] \mid_{\Gamma_0} = u \mid_{\Gamma_0}$, $w[u] \neq u$, $u - w[u] \in H_{\Gamma_0} \cap H_{\gamma_1} \cap H^r(\Omega)$, $r > 1$, $k \le r$. As suggested in Fig. 6, $w[u]$ may be chosen such that it is different from zero only near Γ_0.

A5. The space $H_{\Gamma_0 h}$ is of approximation class $S_{k,r}$, $k \le r$, see e.g. [6, Section 1.3]. This means there is for any $u \in H_{\Gamma_0} \subset H^r(\Omega)$ an element $\chi \in H_{\Gamma_0 h} \subset H^k(\Omega)$ with

$$\| u - \chi \|_{H^k(\Omega)} \le c_1 h^{r-k} \| u \|_{H^r(\Omega)}, \; c_1 > 0, \tag{4.3}$$

c_1 independent of h.

Lemma 4.2: Under the assumptions A1 – A5 the estimate

$$\| u - u_h \| \le \frac{\alpha_1}{\alpha_0} \left[c_1 h^{r-1} \| u - w[u] \|_{H^r(\Omega)} + \inf_{v_h \in H_{\Gamma_0 h}} \| w[u] - w_h[u_h] - v_h \| \right] \tag{4.4}$$

holds.

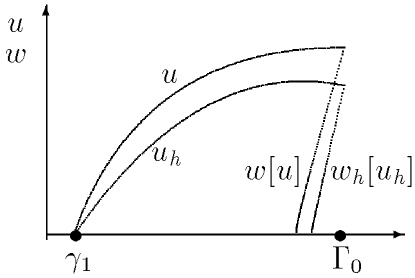

Fig. 6 Illustration for the proof of Lemma 4.2. Behavior of several functions along a cut from a point on γ_1 towards a point on Γ_0.

Proof: We use the proof of Lemma 4.1 to get

$$\frac{\alpha_0}{\alpha_1} \parallel u - u_h \parallel \leq \parallel u - (w_h + v_{h1} + v_{h2}) \parallel =$$

$$= \parallel u - w[u] - (w_h[u_h] - w[u]) - v_{h1} - v_{h2} \parallel \leq \parallel u - w[u] - v_{h1} \parallel + \parallel w[u] - w_h[u_h] - v_{h2} \parallel$$

where v_{h1} and v_{h2} are arbitrary elements in $H_{\Gamma_0 h}$. Since $k = 1$ in the context considered here and $u - w[u] \in H_{\Gamma_0} \cap H^r(\Omega)$, the first term on the right can be estimated by using assumption A5. Taking in the second term the infimum with respect to $H_{\Gamma_0 h}$ we obtain the assertion. □

In this Lemma, only $r > 1$ is of interest.

While Lemma 4.2 gives $\parallel u - u_h \parallel$ in terms of the error on Γ_0 (as expressed by the second term on the right of ineq. (4.4)), the following Lemma shows that for sufficiently small h, $\parallel u - u_h \parallel$ becomes zero.
To show this the adjoint problem for $W \in H_{\Gamma_0} \cap H^r(\Omega)$, $r > 1$,

$$a(\tilde{v}, W) = (u - u_h, \tilde{v})_{H^1} \quad \forall \tilde{v} \in H_{\gamma_1} \tag{4.5}$$

is considered. $(\cdot,\cdot)_{H^1}$ means the scalar product in $H^1(\Omega)$. Comparing this problem with Problem 3 it is seen that, in principle, the role of H_{Γ_0} and H_{γ_1} has been interchanged. We assume that there is an unique solution W of this problem (cf. Lemma 2.1).

Lemma 4.3: Let

(1) the assumptions A1–A3 be fulfilled,

(2) $W \in H_{\Gamma_0} \cap H^r(\Omega)$, $r > 1$, be the unique solution of the adjoint problem,

(3) $H_{\Gamma_0 h}$ be of approximation class $S_{1,r}$, $r > 1$ (cf. assumption A5).

Then there is a finite constant $C > 0$ such that

$$\parallel u - u_h \parallel^2 \leq C h^{r-1} \parallel u - u_h \parallel^2 . \tag{4.6}$$

Proof: Choosing in eq. (4.5) $\tilde{v} = u - u_h \in H_{\gamma_1}$ we obtain

$$\begin{aligned} \parallel u - u_h \parallel^2 &= (u - u_h, u - u_h)_{H^1} = \mid a(u - u_h, W) \mid \\ &= \mid a(u - u_h, W - v_h) \mid \leq \alpha_1 \parallel u - u_h \parallel \cdot \parallel W - v_h \parallel \end{aligned} \tag{4.7}$$

for $v_h \in H_{\Gamma_0 h}$. Here we used the fact that $a(u-u_h, v_h) = 0 \,\forall\, v_h \in \Gamma_{0h}$.
Since $W \in H_{\Gamma_0} \cap H^r(\Omega)$, assumption (3) implies

$$\| W - v_h \| \leq C_1 h^{r-1} \| W \|_{H^r(\Omega)} .$$

On the other hand the adjoint problem (4.5) also gives $\| W \|_{H^r(\Omega)} \leq C_2 \| u - u_h \|$ (cf. ineq. (2.11)). Hence it is

$$\| W - v_h \| \leq C_1 C_2 h^{r-1} \| u - u_h \| .$$

Inserting this into the right of ineq. (4.7) we get the Lemma. $\square$

As already pointed out, this Lemma is of interest in the case $h \to 0$. Using an argument similar to that described e.g. in [5] the relation (4.6) shows (under the assumptions stated) that $u_h \to u$ for $h \to 0$.

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