Two properties of sequencing functions and a class of combinatorical structures

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Abstract

In this paper, two properties of sequencing functions will be considered, which we will call (P1) and (P2). (P2) is a modification of the strong series network property, which is important in the theory of single-machine sequencing.

The aim is to characterize those hereditary languages for which the greedy algorithm works optimally for sequencing functions satisfying (P1) and (P2).

Keywords : Combinatorical Optimization, greedy algorithm, sequencing functions, combinatorical structures.

1 Introduction

In Combinatorical Optimization relationships between greedy algorithms and combinatorical structures are often considered. Many authors have presented "greedy algorithm characterizations" of matroid and greedoid structures, for example cf. Korte, Lovász, Schrader [4, p. 157, Theorem 1.4] or Dress, Wenzel [1, p. 2, Theorem 1].

Roughly speaking, a greedy algorithm characterization of a matroid or greedoid is a theorem of the following kind : A hereditary set system or language is a matroid or greedoid if and only if the greedy algorithm works optimally for a certain class of maps or sequencing functions. [1]

Discussing the connection between algorithms and combinatorical structures it is important to describe which type of map or sequencing function is considered. So, in algorithmic characterizations of matroids and greedoids a special class of functions is always referred to for which the optimality of the greedy algorithm is guaranteed. These classes, for example, may be linear functions if the underlying structure is a matroid (cf. Jungnickel [3, pp. 94, 95]) or bottleneck functions if the underlying structure is a greedoid (cf. Goecke, Korte, Lovász [2, pp. 46–48]). The type of function beeing considered can also be described by properties instead of giving an explicit definition. For example, Korte, Lovász and Schrader in [4, pp. 154–155] show that the greedy algorithm produces optimal solutions if it is applied to so-called $\mathcal{L}$-admissible functions.

In this paper two properties of sequencing functions – (P1) and (P2) – are defined and it is given a class of sequencing functions which fullfil these properties. Next the greedy algorithm is applied to sequencing functions which fullfil (P1) and (P2).

Last for a given sequencing function f satisfying (P1) and (P2) a class of hereditary languages is defined for which the greedy algorithm works optimally.

[1] For a more detailed explanation of the concept "greedy algorithmic characterization" cf. Dress and Wenzel [1, p. 1].

2 Definitions and notations

Let us consider the optimization problem

$$min\,\{w(\alpha): \quad \alpha \text{ is a basic word in } \mathcal{S}\} \tag{1}$$

which is related to a simple hereditary language $(J, \mathcal{S})$ and a sequencing function $f: \quad \mathcal{S} \longrightarrow \mathbb{R}$. The elements of the finite set J are called letters and the elements of $\mathcal{S}$ are simple words constructed from J. "Simple" means that each letter may appear only once in a word. A basic word of $\mathcal{S}$ is a word for which does not exist a feasible augmention by a letter of J. The set of all basic words of a hereditary languages is called $\mathcal{S}_B$. For a more detailed explanation of these concepts see Korte, Lovász, Schrader [4, p. 5].

In further considerations the symbol $\{\alpha\}$ denotes the set of elements of the word α, J^* is used for denoting the set of all simple words which can be performed by the letters of J and $\Gamma(\alpha)$ denotes the set of all letters x which are feasible augmentions for the word α (i.e. αx is an element of $\mathcal{S}$).

Now we give the greedy procedure for the optimization problem 1 as it is described in [4, p. 154]:

<u>Greedy Algorithm</u>

(1) Set $\alpha = 0$.
(2) If $\Gamma(\alpha) = 0$, stop.
(3) Choose $x \in \Gamma(\alpha)$ such that $w(\alpha x) \leq w(\alpha y)$ for all $y \in \Gamma(\alpha)$.
(4) Set $\alpha = \alpha x$ and goto 2.

Looking at step (3) it becomes clear that the result of the greedy algorithm is not necessarily unique. Several bases of $(J, \mathcal{S})$ may be greedy solutions. Here an example for a hereditary language and a sequencing function f for which the greedy algorithm yields two bases is given. This example shows that different greedy bases also may have different f-values.

Example 1 $J := \{a, b\}; \mathcal{S} := \{\emptyset, a, ab, b, ba\}; f(\emptyset) := 0, f(a) := 1, f(b) := 1, f(ab) := 1, f(ba) := 2$

Since $f(a) = f(b)$, the basis ab as well as the basis ba are greedy solutions with respect to the optimzation problem 1. Due to the definition of f we have $f(ab) \neq f(ba)$.

The set of all greedy solutions for (1) related to a hereditary language $(J, \mathcal{S})$ and a sequencing function f in the following is denoted by $\Lambda(\mathcal{S}, f)$.

3 Two properties of sequencing functions and interchanges of letters in greedy solutions

Definition 1 (P1) *Consider the inequalities*

$$f(\eta^1 x \eta^2) \leq f(\eta^1 y \eta^2) \tag{2}$$
$$f(x\vartheta y) \leq f(y \vartheta x), \tag{3}$$

where f is a sequencing function. Then the function f possesses the property (P1) if the following conditions hold :

(a) *(2) for $\eta^1, \eta^2 \in J^*$ and for $x, y \in J$ such that $\eta^1 x \eta^2, \eta^1 y \eta^2 \in J^*$ implies (3) for every $\vartheta \in J^*$ with $x\vartheta y, y\vartheta x \in J^*$.*

(b) *Conversely, (3) for $x, y \in J$ and for $\vartheta \in J^*$ such that $x\vartheta y, y\vartheta x \in J^*$ implies (2) for every pair $\eta^1, \eta^2 \in J^*$ with $\eta^1 x \eta^2, \eta^1 y \eta^2 \in J^*$.*

Remark 1 *We can get a modification (P1') of property (P1) by substituting "$<$" for "$\leq$" in (1) as well as in (2). Then (P1) implies (P1').*

Definition 2 (P2) *A sequencing function possesses the property (P1) if*

$$f(\chi) \leq f(\psi) \quad \Longrightarrow \quad f(\delta^1 \chi \delta^2) \leq f(\delta^1 \psi \delta^2), \tag{4}$$

where $\delta^1, \delta^2 \in J^*$ are arbitrary sequences and χ and ψ are permutations of one and the same set, i.e. $\{\chi\} = \{\psi\}$.

(P2) is similar to the property strong SND, which we quote from Momna, Sidney [5, p. 24]):

Definition 3 (strong SND) *A sequencing function f possesses the strong series network decomposition (strong SND) property if the following conditions hold for all permutations χ and ψ of the same set (i.e., $\{\chi\} = \{\psi\}$) : For all sequences δ^1 and δ^2 of the same set, $f(\chi) \leq f(\psi)$ if and only if $f(\delta^1 \chi \delta^2) \leq f(\delta^1 \psi \delta^2)$.*

Now a class of sequencing functions which satisfy both (P1) and (P2) is given. Therefore we need :

Definition 4 (weighted-position sequencing function) *Let $\omega = (w_1, w_2 \dots w_n)$ be a n–tupel of real numbers called weight coefficients and t a function from J into $\mathbb{R}$, i.e.*

$$\begin{array}{rccc} t: & J & \longrightarrow & \mathbb{R} \\ & b_i & \longrightarrow & t(b_i). \end{array}$$

Then each function

$$\begin{array}{rcl} g: \quad J^* & \longrightarrow & \mathbb{R} \\ g(x_1 x_2 \dots x_k) & = & \displaystyle\sum_{j=1}^{k} w_i t(x_i) \end{array} \tag{5}$$

is called weighted-position sequencing function.

It is easy to show that we will get a sequencing function with properties (P1) and (P2) if we choose $w_i := w^i$ where $w \in (0, 1)$ is arbitrary.

It might be also interesting to ask for all sequencing functions with weighted positions which satisfy (P1) and (P2) or (P1) and (strong SND) respectively. A first result concerning this question is :

Theorem 1 (Monotonicity) *Let f be a weighted-position sequencing function (cf. Definition 4) where t is non-constant over J* [2]*. Assume (P1) and (strong SND) hold.*

Then the sequence of the coefficients w_i is decreasing, i.e.

$$w_1 > w_2 > \ldots > w_n. \tag{6}$$

Proof. We have to show that $w_i \geq w_{i+1}$ for all $i \in \{1, \ldots, n-1\}$. Without loss of generality, let x_i, x_{i+1} be two letters of J with $t(x_i) < t(x_{i+1})$. Furthermore, from $J \setminus \{x_i, x_{i+1}\}$ number $i-1$ arbitrary letters are chosen, which are denoted by $x_1, x_2, \ldots, x_{i-1}$.

Then the following conclusions hold :

$$\begin{array}{lrcl}
 & t(x_i) & < & t(x_{i+1}) \\
\Longrightarrow & g(x_1 x_2 \ldots x_{i-1} x_i) & < & g(x_1 x_2 \ldots x_{i-1} x_{i+1}) \\
\stackrel{(SND)}{\Longrightarrow} & g(x_i) & < & g(x_{i+1}) \\
\stackrel{(P1')}{\Longrightarrow} & g(x_i x_{i+1}) & < & g(x_{i+1} x_i) \\
\stackrel{(SND)}{\Longrightarrow} & g(x_1 \ldots x_{i-1} x_i x_{i+1}) & < & g(x_1 \ldots x_{i-1} x_{i+1} x_i) \\
\Longrightarrow & w_i t(x_i) + w_{i+1} t(x_{i+1}) & < & w_i t(x_{i+1}) + w_{i+1} t(x_i) \\
\Longrightarrow & (w_{i+1} - w_i) \underbrace{(t(x_{i+1}) - t(x_i))}_{>0} & < & 0 \\
\Longrightarrow & w_{i+1} & < & w_i.
\end{array}$$

\\ \\ \\

Concerning sequencing functions satisfying (P1) and (P2), it was accidently noticed by the author that an exchange of two letters of an greedy solution does not yield a decrease of the sequencing function. This fact is now formulated as a lemma.

Lemma 2 (interchange lemma) *Let $(J, \mathcal{S})$ be a simple hereditary language and f a sequencing function over $\mathcal{S}$. Assume (P1) and (P2) hold.*

Furthermore let

$$\lambda = x_1 x_2 \ldots x_{i-1} x_i x_{i+1} \ldots x_{j-1} x_j x_{j+1} \ldots x_k$$

a greedy solution with respect to f and $(J, \mathcal{S})$ and

$$\mu = x_1 x_2 \ldots x_{i-1} x_j x_{i+1} \ldots x_{j-1} x_i x_{j+1} \ldots x_k$$

a sequence of $\mathcal{S}$ which results from one interchange of two elements of the basis λ. [3]

Then

$$f(\lambda) \leq f(\mu). \tag{7}$$

[2] I.e., there are at least two letters $b_1, b_2 \in J$ with $t(b_1) \neq t(b_2)$.

[3] We assume that at least one sequence which results from such an interchange of two jobs is feasible in $\mathcal{S}$.

Proof. In order to obtain a contradiction, we assume

$$f(x_1x_2\dots x_{i-1}x_jx_{i+1}\dots x_{j-1}x_ix_{j+1}\dots x_k) < \\ f(x_1x_2\dots x_{i-1}x_ix_{i+1}\dots x_{j-1}x_jx_{j+1}\dots x_k). \tag{8}$$

(8) implies

$$f(x_jx_{i+1}\dots x_{j-1}x_i) < f(x_ix_{i+1}\dots x_{j-1}x_j), \tag{9}$$

since otherwise we have contradiction to (8) by (P2).

Applying (P1') (see Remark 1) to (9) yields

$$f(x_1\dots x_{i-1}x_j) < f(x_1\dots x_{i-1}x_i). \tag{10}$$

Since $(J,\mathcal{S})$ is a hereditary language, we have $x_1\dots x_{i-1}x_j \in \mathcal{S}$. Thus, (10) is a contradiction to step (3) of the greedy algorithm. \\\\

4 Interchange stable hereditary languages

In this section we are going to use Lemma 2 for defining a combinatorical structure. Therefore we need the following definitions.

Definition 5 ((1)-interchange set of a sequence) *Given a sequence $\sigma = z_1z_2\dots z_m$ the set*

$$\hat{\mathcal{V}}(\sigma) := \bigcup_{i=1}^{m-1}\bigcup_{j=i}^{m}\begin{pmatrix} 1 & \dots & i & \dots & j & \dots & n \\ z_1 & \dots & z_j & \dots & z_i & \dots & z_m \end{pmatrix} \tag{11}$$

is called (1)-interchange set of the sequence σ.

Remark 2 *Following this defintion the set $\hat{\mathcal{V}}(\sigma)$ consists of all sequences which result from one interchange of two letters of σ. Since j=i is possible in (11) the sequence σ itself is an element $\hat{\mathcal{V}}(\sigma)$.*

Definition 6 ((1)-interchange set of a system of sequences) *Given a system of sequences $\mathcal{M}$ (i.e. $\sigma \in J^*$ for each $\sigma \in \mathcal{M}$) the set*

$$\hat{\mathcal{V}}(\mathcal{M}) := \bigcup_{\sigma\in\mathcal{M}} \hat{\mathcal{V}}(\sigma)$$

is called (1)-interchange set of the system $\mathcal{M}$.

Definition 7 (hereditary closure of a sequence) *Given a sequence $\sigma \in J^*$ the set*

$$\hat{\mathcal{H}}(\sigma) := \{\tilde{\sigma} : \quad \tilde{\sigma} \subseteq \sigma\}$$

is called hereditary closure of the sequence σ.

Definition 8 (hereditary closure of a system of sequences) *Given a system of sequences $\mathcal{M}$ the set*

$$\hat{\mathcal{H}}(\mathcal{M}) := \bigcup_{\sigma\in\mathcal{M}} \hat{\mathcal{H}}(\sigma)$$

is called hereditary closure of the system $\mathcal{M}$.

Definition 9 ((1)-interchange stable language) *Let $(J, \mathcal{S})$ be a simple hereditary language and f a sequencing function over $\mathcal{S}$. Then $(J, \mathcal{S})$ is called (1)-interchange stable with respect to the set of greedy bases $\Lambda(\mathcal{S}, f)$ if and only if*

$$\mathcal{S} \subseteq \hat{\mathcal{H}}\left(\hat{\mathcal{V}}\left(\Lambda\left(\mathcal{S}, f\right)\right)\right). \tag{12}$$

Remark 3 *The property "(1)-interchange stability" of an hereditary language $(J, \mathcal{S})$ is always dependent on the function f which is defined in $\mathcal{S}$. It is not possible to declare a language is (1)-interchange stable without referring to a sequencing function f.*

Now we are ready for :

Theorem 3 ***(Optimality of the greedy algorithm for (1)-interchange stable hereditary languages)*** *Let $(J, \mathcal{S})$ be a (1)-interchange stable hereditary language with respect to f. Furthermore, assume that all basic words of $\mathcal{S}$ have the same cardinality.*

Then at least on element of the set $\Lambda(\mathcal{S}, f)$ of all greedy bases is an optimal solution of the optimization problem (1).

Proof. By (12) and by the fact that all bases of $\mathcal{S}$ have the same cardinality we can conclude that for each basis $\beta = x_1 x_2 \ldots x_{i-1} x_i x_{i+1} \ldots x_{j-1} x_j x_{j+1} \ldots x_m$ of $(J, \mathcal{S})$ there exists a greedy solution $\lambda = x_1 x_2 \ldots x_{i-1} x_j x_{i+1} \ldots x_{j-1} x_i x_{j+1} \ldots x_m$ in which two letters of β are exchanged. By Lemma 2 we have

$$f(\beta) \geq f(\lambda).$$

Therefore the optimal solution of (1) must be an element of the set $\Lambda(\mathcal{S}, f)$. \ \ \

Remark 4 *If alle bases of $(J, \mathcal{S})$ have the same cardinality (12) in Definition 9 is equivalent to*

$$\mathcal{S}_B \subseteq \hat{\mathcal{V}}\left(\Lambda\left(\mathcal{S}, f\right)\right).$$

We complete this section by giving an example of a (1)-interchange stable hereditary language $(J, \mathcal{S})$ with respect to a sequencing function f.

Example 2

Letters of J : $J := \{a, b, c, d\}$

The set of feasible words in $(J, \mathcal{S})$:

abcd	*abdc*	*acbd*	*adcb*	*bacd*	*cbad*	*dbca*
abc	*abd*	*acb*	*adc*	*bac*	*cba*	*dbc*
ab		*ac*	*ad*	*ba*	*cb*	*db*
a				*b*	*c*	*d*
$\emptyset$						

The sequencing function f :

f is defined by

$$\begin{array}{rcl} f: \quad \mathcal{S} & \longrightarrow & \mathbb{R} \\ f(x_1 x_2 \ldots x_k) & := & \sum_{i=1}^{k} w_i [t(x_i)]^{k-i+1} \end{array}$$

where the weight coefficients are given by $w(1) := 1, w(2) := 2, w(3) := 7$ and $w(4) := 6$ and the t-values by $t(a) := 1; t(b) := 2; t(c) := 3$ and $t(d) := 4$.

The set of greedy bases $\Lambda(\mathcal{S}, f)$:

The greedy algorithm applied to f over $(J, \mathcal{S})$ yields $\Lambda(\mathcal{S}, f) = \{abcd\}$, i.e. the basis $\lambda = abcd$ is the unique greedy solution. For $\lambda = abcd$ we have $f(\lambda) = 1^4 + 2 \cdot 2^3 + 7 \cdot 3^2 + 6 \cdot 4 = 104$. $\lambda = abcd$ is an optimal basis of $(J, \mathcal{S})$ with respect to the optimization problem 1.

In Example 2, the language $(J, \mathcal{S})$ is (1)-interchange stable with respect to $\Lambda(\mathcal{S}, f)$ because we can get all bases of $(J, \mathcal{S})$ by only one exchange of two letters of the greedy solution $\lambda = abcd$. It might be regarded as an interesting fact that $(J, \mathcal{S})$ is not a greedoid. To verify this, for example consider the sequences $\alpha = abd$ and $\beta = ad$ and notice that the Steinitz exchange property (cf. Korte, Lovász, Schrader [4, Seite 45, Axiom 1.3]) is not satisfied for this pair of sequences. [4]

5 Concluding Remarks

Weighted-position sequencing functions were introduced in Definition 4 in order to show that the set of sequencing functions satisfy the properties (P1) and (P2) is not empty.

Even if the greedy algorithm works optimally for this kind of sequencing functions (cf. Theorem 3), weighted-position sequencing functions supposely will be not suitable for giving an algorithmic characterization of (1)-interchange stable hereditary languages. In order to give arguments for this statement imagine that a weighted-position sequencing function is defined in the set of feasible words of Example 2. Than it will be observed that $f(abcd)$ is minimal in the set of f-values of all permutations of the sequence $abcd$. [5]

The sequencing function which was chosen in Example 2 is not a weighted-position sequencing function because powers of the t-values were used for its definition. Contrary to the observation above, permutations of the sequence $abcd$ which are not bases of $(J, \mathcal{S})$ can be found for which the f-value is smaller than $f(abcd) = 104$, for example $f(bcad) = 2^4 + 2 \cdot 3^3 + 7 \cdot 1^2 + 6 \cdot 4 = 101$.

[4] The Steinitz exchange property is not satisfied because there is no letter $x \in \{\alpha\}$ such that $\beta x \in \mathcal{S}$.

[5] Remember that a simple hereditary language was constructed in order to present a hereditary language for which the greedy algorithm yields the minimum of f-values in the set of all bases of $(J, \mathcal{S})$.

A next task should be try to find sequencing functions which are typical for (1)-interchange stable hereditary languages. That means : It would be interesting to find a class of sequencing functions which characterize (1)-interchange stable hereditary languages exactly as those hereditary languages for which the greedy algorithm works optimally.

The general idea of this paper can be formulated as a method in the following way :

Step 1. Notice that the value of a sequencing function f does not decrease if an arbitrary member of a set of permutations – this set of permuations in the following is called $\hat{\mathcal{P}}$ – is applied to an arbitrary element of the set of greedy-solutions $\Lambda(\mathcal{S}, f)$.

Step 2. Define a combinatorical structure to a given sequencing function by

$$\mathcal{S}_B \subseteq \hat{\mathcal{P}}(\Lambda(\mathcal{S}, f))$$

in the case that all bases of $(J, \mathcal{S})$ have the same cardinality.

The set $\hat{\mathcal{P}}$ may be not necessarily restricted to the set of interchanges of two letters (cf. the introduction of the operator $\hat{\mathcal{V}}$ in Definition 5 and Definition 6) in further considerations as we have done it in this paper.

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