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REMARKS ON THE INTERNAL EXPONENTIAL STABILIZATION TO A NONSTATIONARY SOLUTION FOR 1D BURGERS EQUATIONS*

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Abstract. The feedback stabilization of the Burgers system to a nonstationary solution using finite-dimensional internal controls is considered. Estimates for the dimension of the controller are derived. In the particular case of no constraint on the support of the control, a better estimate is derived and the possibility of getting an analogous estimate for the general case is discussed; some numerical examples are presented illustrating the stabilizing effect of the feedback control and suggesting that the existence of an estimate in the general case analogous to that in the particular one is plausible.

Key words. Burgers equations, exponential stabilization, feedback control, finite elements

AMS subject classifications. 93B52, 93C20, 93D15, 93C50

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1. Introduction.

Let $L > 0$ be a positive real number. We consider the controlled Burgers equations in the interval $\Omega = (0,\, L) \subset \mathbb{R}$:

$$\partial_t u + u\partial_x u - \nu\partial_{xx} u + h + \zeta = 0, \qquad u|_\Gamma = 0. \tag{1.1}$$

Here, u stands for the unknown velocity of the fluid, $\nu > 0$ is the viscosity, h is a fixed function, $\Gamma = \partial\Omega$ stands for the boundary $\{0,\, L\}$ of Ω, and ζ is a control taking values in the space of square-integrable functions in Ω, whose support, in x, is contained in a given open subset $\omega \subset \Omega$.

Let us be given a positive constant $\lambda > 0$, a continuous Lipschitz function $\chi \in W^{1,\infty}(\Omega, \mathbb{R})$ with nonempty support, and a solution $\hat{u} \in \mathcal{W}$ of (1.1) with $\zeta = 0$, in a suitable Banach space $\mathcal{W}$. Then, following the procedure presented in [7], we can prove that there exists an integer M, a function $\eta = \eta(t,\, x)$, defined for $t > 0$, $x \in \Omega$, such that the solution $u = u(t,\, x)$ of problem (1.1) with $\zeta = \chi P_M \eta$, and supplemented with the initial condition

$$u(0, x) = u_0(x) \tag{1.2}$$

is defined on $[0,\, +\infty)$ and satisfies the relation $|u(t) - \hat{u}(t)|^2_{L^2(\Omega,\, \mathbb{R})} \le C\mathrm{e}^{-\lambda t}|u(0) - \hat{u}(0)|^2_{L^2(\Omega,\, \mathbb{R})}$, provided $|u(0) - \hat{u}(0)|_{L^2(\Omega,\, \mathbb{R})} < \epsilon$ for small enough ϵ. Here, M, C, and ϵ can be taken depending only on $(|\hat{u}|_{\mathcal{W}}\,,\, \lambda)$, and P_M is the orthogonal projection in $L^2(\Omega,\, \mathbb{R})$ onto the subspace $L^2_M(\Omega,\, \mathbb{R}) := \mathrm{span}\{\sin(\frac{i\pi x}{L}) \mid i \in \mathbb{N},\, 1 \le i \le M\}$. That is, the internal control $\zeta = \chi P_M \eta$ stabilizes exponentially, with rate $\frac{\lambda}{2}$, the Burgers system to the reference trajectory $\hat{u}$.

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Notice that the support of the control ζ is necessarily contained in that of χ and that the control is finite-dimensional. Furthermore, we also know that the control can be taken in feedback form, $\zeta(t) = \mathrm{e}^{-\lambda t}\chi P_M \chi Q_{\hat{u}}^{t,\,\lambda}(u(t) - \hat{u}(t))$, for a suitable family of linear continuous operators $Q_{\hat{u}}^{t,\,\lambda} \colon L^2(\Omega,\, \mathbb{R}) \to L^2(\Omega,\, \mathbb{R})$, $t \geq 0$ (cf. [7, section 3.2]).

We can see that the dimension M of the range of the controller depends on the norm $|\hat{u}|_{\mathcal{W}}$ of $\hat{u}$ but, up to now no precise estimate has been known. In the case $\hat{u}$ is independent of time, it is possible to give, for the case of the Navier–Stokes equations, a rather sharp description of its dimension M, though the range of the controller depends on $\hat{u}$; see, for example, [2, 5, 6, 8, 39] (cf. [7, Remark 3.11(c)]). The procedure uses the spectral properties of the Oseen–Stokes system and cannot be (at least not straightforwardly) used in the time-dependent case.

The aim of this paper is to establish some first results concerning the dimension M of the range of the internal stabilizing controller, in the case of a reference time-dependent trajectory $\hat{u}$. Notice that this case is not less important for applications because often we are confronted with external forces h that depend on time.

In [7], the proof of the existence of an M-dimensional stabilizing control uses a contradition argument (cf. [7, proofs of Lemma A.4 and Proposition A.3]) which makes it difficult to find an estimate for M. Here, we prove the existence of a stabilizing control by a more constructive procedure.

In the case we impose no restriction on the support of the control, more precisely, if we take $\chi(x) = 1$ for all $x \in \Omega$, we obtain that is it enough to take

$$M \geq \tfrac{L}{\pi}\left(\tfrac{3\mathrm{e}}{2}\right)^{\frac{1}{2}}\left(\nu^{-2}|\hat{u}|_{\mathcal{W}}^2 + \nu^{-1}\lambda\right)^{\frac{1}{2}}, \tag{1.3}$$

where e is the Napier's constant. In the case our control is supported in a small subset $\omega = \mathrm{supp}(\chi)$, we can also derive that it is "enough" to take

$$M \geq C_1 \mathrm{e}^{C_2\left(1 + (\nu^{-1}\lambda)^{\frac{1}{2}} + (\nu^{-1}\lambda)^{\frac{2}{3}} + \nu^{-1}|\hat{u}|_{\mathcal{W}} + \nu^{-2}|\hat{u}|_{\mathcal{W}}^2\right)}, \tag{1.4}$$

where C_1 and C_2 are constants depending on χ and Ω. Estimates (1.3) and (1.4) are the main results of this paper. We easily see that the estimate in the case of the support constraint is much less reasonable, if we think about an application. The reason for the gap is that the idea used to derive (1.3) cannot be (at least not straightforwardly) used for general $\chi(x)$. So one question arises: can we improve (1.4)? To derive (1.4), we depart from an exact null controllability result, carrying the cost associated with the respective control. For stabilization, with a given (finite) positive rate $\frac{\lambda}{2} > 0$, we do not need to reach zero; that is why we believe the estimate can be improved, if we can avoid using the exact controllability result.

We have performed some numerical simulations whose results suggest that the possibility of getting, also in the general case, an estimate analogous to (1.3) is plausible. We focus on the one-dimensional (1D) Burgers equations because the simulations are much simpler to perform in this setting. However, we believe that the difficulties to find an estimate for M will be analogous for the two-dimensional (2D) and three-dimensional (3D) Burgers and Navier–Stokes systems and for a suitable class of parabolic systems.

The rest of the paper is organized as follows. In section 2 we recall some well-known results and set up our problem; in particular, we recall that the problem can be reduced to the stabilization to zero of the Oseen–Burgers system. In section 3, for the linearized Oseen–Burgers system, we present the first estimates for a lower bound for the suitable dimension M of the controller; section 3.1 deals with the particular case where we impose no restriction on the support of the control, and section 3.2

deals with the general case. In section 4 we consider the full nonlinear Oseen–Burgers system. The discretization of our problem is presented in section 5, and in sections 6 and 7 we present the results of some simulations we have performed. Finally, in section 8 we give a few more comments on the results.

Notation. We write $\mathbb{R}$ and $\mathbb{N}$ for the sets of real numbers and nonnegative integers, respectively, and we define $\mathbb{R}_r \coloneqq (r, +\infty)$, for $r \in \mathbb{R}$, and $\mathbb{N}_0 \coloneqq \mathbb{N} \setminus \{0\}$. We denote by $\Omega \subset \mathbb{R}$ a bounded interval. Given a vector function $u\colon (t, x) \mapsto u(t, x) \in \mathbb{R}$, defined in an open subset of $\mathbb{R} \times \Omega$, its partial time derivative $\frac{\partial u}{\partial t}$ will be denoted by $\partial_t u$. The partial spatial derivative $\frac{\partial u}{\partial x}$ will be denoted by $\partial_x u$, and $\frac{\partial}{\partial x}\frac{\partial}{\partial x}$ by ∂_{xx}.

Given a Banach space X and an open subset $O \subset \mathbb{R}^n$, let us denote by $L^p(O, X)$, with either $p \in [1, +\infty)$ or $p = \infty$, the Bochner space of measurable functions $f\colon O \to X$, and such that $|f|_X^p$ is integrable over O for $p \in [1, +\infty)$, and such that $\operatorname{ess\,sup}_{x \in O} |f(x)|_X < +\infty$ for $p = \infty$. In the case $X = \mathbb{R}$, we recover the usual Lebesgue spaces. By $W^{s,p}(O, \mathbb{R})$ for $s \in \mathbb{R}$, denote the usual Sobolev space of order s. In the case $p = 2$, as usual, we denote $H^s(O, \mathbb{R}) \coloneqq W^{s,2}(O, \mathbb{R})$. Recall that $H^0(O, \mathbb{R}) = L^2(O, \mathbb{R})$. For each $s > 0$, we recall also that $H^{-s}(O, \mathbb{R})$ stands for the dual space of $H_0^s(O, \mathbb{R}) =$ closure of $\{f \in C^\infty(O, \mathbb{R}) \mid \operatorname{supp} f \subset O\}$ in $H^s(O, \mathbb{R})$. Notice that $H^{-s}(O, \mathbb{R})$ is a space of distributions.

For a normed space X, we denote by $|\cdot|_X$ the corresponding norm, by X' its dual, and by $\langle \cdot, \cdot \rangle_{X',X}$ the duality between X' and X. The dual space is endowed with the usual dual norm: $|f|_{X'} \coloneqq \sup\{\langle f, x\rangle_{X',X} \mid x \in X \text{ and } |x|_X = 1\}$. In the case that X is a Hilbert space, we denote the inner product by $(\cdot, \cdot)_X$.

Given an open interval $I \subseteq \mathbb{R}$ and two Banach spaces X and Y, we write $W(I, X, Y) \coloneqq \{f \in L^2(I, X) \mid \partial_t f \in L^2(I, Y)\}$, where the derivative $\partial_t f$ is taken in the sense of distributions. This space is endowed with the natural norm $|f|_{W(I, X, Y)} \coloneqq (|f|^2_{L^2(I, X)} + |\partial_t f|^2_{L^2(I, Y)})^{\frac{1}{2}}$. In the case $X = Y$, we write $H^1(I, X) \coloneqq W(I, X, X)$. Again, if X and Y are endowed with a scalar product, then $W(I, X, Y)$ is also. The space of continuous linear mappings from X into Y will be denoted by $\mathcal{L}(X \to Y)$.

If $\bar{I} \subset \mathbb{R}$ is a closed bounded interval, $C(\bar{I}, X)$ stands for the space of continuous functions $f\colon \bar{I} \to X$ with the norm $|f|_{C(\bar{I}, X)} = \max_{t \in \bar{I}} |f(t)|_X$.

$\overline{C}_{[a_1,\ldots,a_k]}$ denotes a nonnegative function of nonnegative variables a_j that increases in each of its arguments.

C, C_i, $i = 1, 2, \ldots$, stand for unessential positive constants.

2. Preliminaries.

2.1. Reduction to local null stabilization. We will denote $V \coloneqq H_0^1(\Omega, \mathbb{R})$, $H \coloneqq L^2(\Omega, \mathbb{R})$, $\mathrm{D}(\partial_{xx}) \coloneqq V \cap H^2(\Omega, \mathbb{R})$, and $V' \coloneqq H^{-1}(\Omega, \mathbb{R})$. The space H is supposed to be endowed with the usual $L^2(\Omega, \mathbb{R})$-scalar product, and the space V with the scalar product $(u, v)_V \coloneqq (\partial_x u, \partial_x v)_H$. The space H is taken as the pivot space, and V' is the dual of V. The inclusions $V \subset H \subset V'$ are dense, continuous, and compact. The space $\mathrm{D}(\partial_{xx})$ is endowed with the scalar product $(u, v)_{\mathrm{D}(\partial_{xx})} \coloneqq (\partial_{xx} u, \partial_{xx} v)_H$.

Let us denote

$$\mathcal{W} \coloneqq L^\infty(\mathbb{R}_0, L^\infty(\Omega, \mathbb{R})) \tag{2.1}$$

and, for given Banach spaces X and Y,

$$L^2_{\mathrm{loc}}(\mathbb{R}_0, X) \coloneqq \{f \mid f|_{(0,T)} \in L^2((0, T), X) \text{ for all } T > 0\},$$
$$W_{\mathrm{loc}}(\mathbb{R}_0, X, Y) \coloneqq \{f \mid f|_{(0,T)} \in W((0, T), X, Y) \text{ for all } T > 0\}.$$

Fix a function $h \in L^2_{\mathrm{loc}}(\mathbb{R}_0, V')$ and suppose that $\hat{u} \in \mathcal{W} \cap W_{\mathrm{loc}}(\mathbb{R}_0, V, V')$ solves the Burgers system (1.1) with $\zeta = 0$ and initial condition $\hat{u}_0 \coloneqq \hat{u}(0) \in H$.

Let us be given a Lipschitz continuous function $\chi \in W^{1,\infty}(\Omega, \mathbb{R})$ with nonempty support, an open interval $\mathcal{O} = (l_1, l_2)$ such that $\mathrm{supp}(\chi) \subseteq \overline{\mathcal{O}} \subseteq \overline{\Omega}$, a constant $\lambda > 0$, and another function u_0 such that $|u_0 - \hat{u}(0)|_H$ is small enough.

Our goal is to find an integer $M \in \mathbb{N}_0$ and a control $\eta \in L^2(\mathbb{R}_0, H)$ such that the solution of the problem (1.1)–(1.2) with $\zeta = \chi \mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}(\eta|_{\mathcal{O}})$ is defined for all $t > 0$ and converges exponentially to $\hat{u}$, that is, for some positive constant $C > 0$ independent of $u_0 - \hat{u}_0$,

$$|u(t) - \hat{u}(t)|_H^2 \leq C\, \mathrm{e}^{-\lambda t} |u_0 - \hat{u}_0|_H^2 \quad \text{for } t \geq 0. \tag{2.2}$$

Here, $P_M^{\mathcal{O}}$ stands for the orthogonal projection in $L^2(\mathcal{O}, \mathbb{R})$ onto the subspace spanned by the first M eigenfunctions $\underline{s}_n$ of the Dirichlet Laplacian in $\mathcal{O}$, that is, onto

$$L^2_M(\mathcal{O}, \mathbb{R}) \coloneqq \mathrm{span}\{\underline{s}_n \mid n \in \mathbb{N}_0,\, n \leq M\},$$

where $\mathbb{E}_0^{\mathcal{O}} \colon L^2(\mathcal{O}, \mathbb{R}) \to H$ is the extension by zero outside $\mathcal{O}$, defined by

$$\mathbb{E}_0^{\mathcal{O}} f(x) \coloneqq \begin{cases} f(x) & \text{if } x \in \mathcal{O}, \\ 0 & \text{if } x \in \Omega \setminus \overline{\mathcal{O}}. \end{cases}$$

Recall that it is well known that the complete system of (normalized) Dirichlet eigenfunctions $\{\underline{s}_n \mid n \in \mathbb{N}_0\} \subset \mathrm{D}(\partial_{xx})$ and the corresponding system of eigenvalues $\{\alpha_n \mid n \in \mathbb{N}_0\}$ are given explicitly by

$$\underline{s}_n(x) \coloneqq \left(\tfrac{2}{l}\right)^{\frac{1}{2}} \sin\left(\tfrac{n\pi(x-l_1)}{l}\right), \quad \alpha_n = \left(\tfrac{\pi}{l}\right)^2 n^2, \quad -\partial_{xx}\underline{s}_n = \alpha_n \underline{s}_n, \quad x \in \mathcal{O}, \tag{2.3}$$

where $l = l_2 - l_1$ stands for the length of $\mathcal{O}$.

Let us notice that, seeking the control η and considering the corresponding solution u, we find that $v = u - \hat{u}$ will solve the Oseen–Burgers system

$$\partial_t v - \nu \partial_{xx} v + v \partial_x v + \partial_x(\hat{u} v) + \zeta = 0, \qquad v|_\Gamma = 0, \qquad v(0) = v_0, \tag{2.4}$$

with $\zeta = \chi \mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}(\eta|_{\mathcal{O}})$ and $v_0 = u(0) - \hat{u}(0)$. It is now clear that to achieve (2.2), it suffices to consider the problem of local exponential stabilization to zero for solutions of (2.4), where "local" means that the property is to hold "provided $|v_0|_H$ is small enough."

2.2. Weak solutions. The existence and uniqueness of weak solutions for system (2.4) can be proved by classical arguments, where weak solutions are understood in the classical sense as in [37, Chapter 1, sections 6.1 and 6.4], [42, sections 2.4 and 3.2], [43, Chapter 3, section 3].

THEOREM 2.1. *Given $\hat{u} \in \mathcal{W}$, $\zeta \in L^2((0, T), V')$, and $v_0 \in H$, there exists a weak solution $v \in W((0, T), V, V')$ for system (2.4) in $(0, T) \times \Omega$. Moreover, v is unique and depends continuously on the given data (v_0, η):*

$$|v|^2_{W((0,T),V,V')} \leq \overline{C}_{[T,\, |\hat{u}|_{\mathcal{W}}]} \left(|v_0|_H^2 + |\zeta|^2_{L^2((0,T),V')} \right). \tag{2.5}$$

Notice that the proof of the existence and uniqueness of a weak solution can be done following the argument in [43, Chapter 3, section 3.2] by using the estimate

$$|\partial_x(wv)|^2_{V'} \leq C |w|^2_{L^\infty(\Omega,\mathbb{R})} |v|^2_{L^2(\Omega,\mathbb{R})} \leq C_1 |w|^2_{H^1(\Omega,\mathbb{R})} |v|^2_{L^2(\Omega,\mathbb{R})} \leq C_2 |w|_V^2 |v|_H^2.$$

DEFINITION 2.2. *We say that* $v \in W_{\mathrm{loc}}(\mathbb{R}_0, V, V')$ *is a global weak solution for system* (2.4) *in* $\mathbb{R}_0 \times \Omega$ *if* $v|_{(0,\,T)} \in W((0, T), V, V')$ *is a weak solution for the same system in* $(0, T) \times \Omega$ *for all* $T > 0$.

COROLLARY 2.3. *Given* $\hat{u} \in \mathcal{W}$, $\zeta \in L^2_{\mathrm{loc}}(\mathbb{R}_0, V')$, *and* $v_0 \in H$, *there exists a weak solution* $v \in W_{\mathrm{loc}}(\mathbb{R}_0, V, V')$ *for system* (2.4) *in* $\mathbb{R}_0 \times \Omega$ *which is unique and there holds estimate* (2.5).

Finally, notice that system (1.1)–(1.2) is a particular case of (2.4) (with $\hat{u} = 0$), and hence Theorem 2.1 and Corollary 2.3 also hold for (1.1)–(1.2) (with $h + \zeta$ in the role of ζ).

3. The Oseen–Burgers system. The dimension of the controller. Here, we look for a control in the form $\zeta = \chi \mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}(\eta|_{\mathcal{O}})$ with $\eta \in L^2(\mathbb{R}_0, H)$ that stabilizes exponentially the linearized Oseen–Burgers system

$$\partial_t v - \nu \partial_{xx} v + \partial_x(\hat{u} v) + \zeta = 0, \qquad v|_\Gamma = 0, \qquad v(0) = v_0, \tag{3.1}$$

to zero with a desired exponential rate $\frac{\lambda}{2} > 0$. We also provide some first estimates, concerning a lower bound for the integer M, depending on the triple $(\lambda, |\hat{u}|_{\mathcal{W}}, \nu)$. Later, the results will follow for system (2.4), provided $|v_0|_H$ is small enough, by a fixed point argument.

Remark 3.1. Theorem 2.1 and Corollary 2.3 also hold for system (3.1) in the role of system (2.4).

It is well known that controllability properties for system 3.1 are closely related to observability properties for the "time-backward" adjoint Oseen–Burgers

$$-\partial_t q - \nu \partial_{xx} q - \hat{u} \partial_x q + f = 0, \qquad q|_\Gamma = 0, \qquad q(T) = q_1, \tag{3.2}$$

for $q_1 \in H$ and $f \in L^2((0, T), V')$; below, in section 3.2, we will use some suitable observability inequalities for this adjoint system.

3.1. The particular case $\boldsymbol{\chi = 1_\Omega}$. We consider the case $\mathcal{O} = \Omega$ and $\chi = 1_\Omega$ with $1_\Omega(x) \coloneqq 1$ for all $x \in \Omega$. In particular, there is no constraint in the support of the controller.

THEOREM 3.2. *For given* $\hat{u} \in \mathcal{W}$ *and* $\lambda > 0$, *set*

$$M \geq \tfrac{L}{\pi} \left(\tfrac{3\mathrm{e}}{2}\right)^{\frac{1}{2}} \left(\tfrac{1}{\nu^2} |\hat{u}|^2_{\mathcal{W}} + \tfrac{1}{\nu}\lambda\right)^{\frac{1}{2}}, \tag{3.3}$$

where e *is the Napier's constant. Then for any given* $v_0 \in H$, *there is a control* $\eta^{\lambda,\hat{u},\nu}(v_0) \in L^2(\mathbb{R}_0, H)$ *such that the corresponding solution* v *of system* (3.1) *with* $\zeta = \chi \mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}(\eta^{\lambda,\hat{u},\nu}|_{\mathcal{O}})$ *satisfies the inequality*

$$|v(t)|^2_H \leq (1 + \mathrm{e}^{\frac{1}{2}}) \mathrm{e}^{-\lambda t} |v_0|^2_H, \quad t \geq 0. \tag{3.4}$$

Moreover, the mapping $v_0 \mapsto \eta^{\lambda,\hat{u},\nu}(v_0)$ *is well defined, is linear, and satisfies*

$$\left| \mathrm{e}^{(\hat{\lambda}/2)t} \eta^{\lambda,\hat{u},\nu}(v_0) \right|^2_{L^2(\mathbb{R}_0, H)} \leq \tfrac{4\mathrm{e}^{\frac{1}{2}}}{\lambda - \hat{\lambda}} \left(\tfrac{1}{\nu} |\hat{u}|^2_{\mathcal{W}} + \lambda \right) |v_0|^2_H \quad \textit{for} \ \ 0 \leq \hat{\lambda} < \lambda.$$

Proof. Let w solve

$$\partial_t w = \nu \partial_{xx} w - \partial_x(\hat{u} w) + \tfrac{\lambda}{2} w, \quad w|_\Gamma = 0, \quad w(0) = v_0. \tag{3.5}$$

By standard arguments, we can find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|w|_H^2 &\le -2\nu|\partial_x w|_H^2 + 2|\hat{u}|_{L^\infty(\Omega,\,\mathbb{R})}|w|_H|\partial_x w|_H + \lambda|w|_H^2\\ &\le \frac{1}{2\nu}|\hat{u}|^2_{L^\infty(\Omega,\,\mathbb{R})}|w|_H^2 + \lambda|w|_H^2,\end{aligned}$$

from which we can derive that

$$|w|^2_{L^\infty((0,\,T),\,H)} \quad\le\quad \mathrm{e}^{\left(\frac{1}{2\nu}|\hat{u}|^2_{\mathcal{W}}+\lambda\right)T}\,|v_0|_H^2\,. \tag{3.6}$$

Now let $\varphi(t) \coloneqq 1 - \frac{t}{T} \in C^1([0,T],\mathbb{R})$, and set $\delta \coloneqq \varphi w$. Notice that δ solves

$$\partial_t\delta = \nu\partial_{xx}\delta - \partial_x(\hat{u}\delta) + \tfrac{\lambda}{2}\delta + (\partial_t\varphi)w, \quad \delta|_\Gamma = 0, \quad \delta(0) = v_0,$$

with $\delta(T) = 0$. Now let $M \in \mathbb{N}_0$ be a positive integer and consider the solution δ_M for the system

$$\partial_t\delta_M = \nu\partial_{xx}\delta_M - \partial_x(\hat{u}\delta_M) + \tfrac{\lambda}{2}\delta_M + (\partial_t\varphi)P_M^\Omega w, \quad \delta_M|_\Gamma = 0, \quad \delta_M(0) = v_0.$$

The difference $d \coloneqq \delta - \delta_M$ solves

$$\partial_t d = \nu\partial_{xx}d - \partial_x(\hat{u}d) + \tfrac{\lambda}{2}d + (\partial_t\varphi)(1-P_M^\Omega)w, \quad d|_\Gamma = 0, \quad d(0) = 0,$$

from which we can also derive

$$\begin{aligned}|d|^2_{L^\infty((0,\,T),\,H)} &\le \mathrm{e}^{\left(\frac{3}{2\nu}|\hat{u}|^2_{\mathcal{W}} + \lambda\right)T}\left(|d(0)|_H^2 + \tfrac{3}{4\nu}\left|(\partial_t\varphi)(1-P_M^\Omega)w\right|^2_{L^2((0,\,T),\,V')}\right)\\ &\le T^{-2}\mathrm{e}^{\left(\frac{3}{2\nu}|\hat{u}|^2_{\mathcal{W}} + \lambda\right)T}\tfrac{3}{4\nu}\alpha_M^{-1}\,|w|^2_{L^2((0,\,T),\,H)}\end{aligned}$$

and, from $|w|^2_{L^2((0,\,T),\,H)} \le T|w|^2_{L^\infty((0,\,T),\,H)}$ and (3.6), we can arrive at

$$|d|^2_{L^\infty((0,\,T),\,H)} \le T^{-1}\mathrm{e}^{2\left(\nu^{-1}|\hat{u}|^2_{\mathcal{W}}+\lambda\right)T}\tfrac{3}{4\nu}\alpha_M^{-1}\,|v_0|_H^2\,.$$

Since we are interested in the stabilization of the system, we can see T as a parameter at our disposal. Minimizing the right-hand side over $T > 0$, we can see that the minimizer T_* is defined by $T_*^{-1} \coloneqq 2(\nu^{-1}|\hat{u}|^2_{\mathcal{W}} + \lambda)$; then, setting $T = T_*$, we have that

$$|d|^2_{L^\infty((0,\,T_*),\,H)} \le 2(\nu^{-1}|\hat{u}|^2_{\mathcal{W}} + \lambda)\mathrm{e}^1\tfrac{3}{4\nu}\alpha_M^{-1}\,|v_0|_H^2\,. \tag{3.7}$$

Now, from $\alpha_M = (\frac{M\pi}{L})^2$ (cf. (2.3) with $\mathcal{O} = \Omega$), setting M satisfying (3.3), and recalling that $\delta_M(0) = v_0$ and $\delta_M(T_*) = -d(T_*)$, we find

$$\alpha_M \ge (\nu^{-1}|\hat{u}|^2_{\mathcal{W}} + \lambda)\tfrac{3\mathrm{e}^1}{2\nu} \tag{3.8}$$

and $|\delta_M(T_*)|_H^2 \le |\delta_M(0)|_H^2$.

Further, from (3.6) and (3.7), we find $|\delta_M|^2_{L^\infty((0,\,T_*),\,H)} = |\delta - d|^2_{L^\infty((0,\,T_*),\,H)} \le C_M^\delta\,|\delta_M(0)|_H^2$ with

$$\begin{aligned}C_M^\delta &\coloneqq \mathrm{e}^{\left(\frac{1}{2\nu}|\hat{u}|^2_{\mathcal{W}}+\lambda\right)T^*} + 2(\nu^{-1}|\hat{u}|^2_{\mathcal{W}} + \lambda)\mathrm{e}^1\tfrac{3}{4\nu}\alpha_M^{-1}\\ &\le \mathrm{e}^{\frac{1}{2}} + 2(\nu^{-1}|\hat{u}|^2_{\mathcal{W}} + \lambda)\mathrm{e}^1\tfrac{3}{4\nu}\alpha_M^{-1} \le \mathrm{e}^{\frac{1}{2}} + 1 \eqqcolon \Upsilon_\delta.\end{aligned}$$

Now, notice that we can consider system (3.5) in $(T_*, +\infty) \times \Omega$ with $w(T_*) = \delta_M(T_*)$ and repeat the arguments. Recursively, we conclude that in each interval $J_*^i \coloneqq (iT_*, (i+1)T_*)$, $i \in \mathbb{N}_0$, we have $|\delta_M((i+1)T_*)|_H^2 \le |\delta_M(iT_*)|_H^2$ and $|\delta_M|^2_{L^\infty(J_*^i,\, H)} \le \Upsilon_\delta\, |\delta_M(iT_*)|_H^2$. Hence, we conclude that $|\delta_M|^2_{L^\infty(\mathbb{R}_0,\, H)} \le \Upsilon_\delta\, |v_0|_H^2$.

Next, we notice that $v \coloneqq \mathrm{e}^{-\frac{\lambda}{2}t}\delta_M$ solves (3.1), in $\mathbb{R}_0 \times \Omega$, with the concatenated control $\zeta = \chi P_M^\Omega(\mathrm{e}^{-\frac{\lambda}{2}t}(-T_*^{-1})w) = -T_*^{-1}\mathrm{e}^{-\frac{\lambda}{2}t}\chi P_M^\Omega w$, where $w|_{J_*^i} =: w_i$ solves (3.5), in $J_*^i \times \Omega$ with $w_i(iT_*) = w(iT_*) = \delta_M(iT_*)$; from (3.6) and from the boundedness of $\{|\delta_M(iT_*)|_H \mid i \in \mathbb{N}\}$, we can conclude that the family $\{|w|_{L^2(J_*^i,\, H)} \mid i \in \mathbb{N}\}$ is bounded, so we have that $\mathrm{e}^{\frac{\hat\lambda}{2}t}\zeta \in L^2(\mathbb{R}_0, H)$ for all $\hat\lambda < \lambda$. Finally, we observe that $|v(t)|_H^2 \le \mathrm{e}^{-\lambda t}\, |\delta_M|^2_{L^\infty(\mathbb{R}_0,\, H)} \le \Upsilon_\delta \mathrm{e}^{-\lambda t}\, |v_0|_H^2$ and that for $\eta^{\lambda,\hat u,\nu} \coloneqq \mathrm{e}^{-\frac{\lambda}{2}t}(-T_*^{-1})w$,

$$\begin{aligned}\left|\mathrm{e}^{\frac{\hat\lambda}{2}t}\eta^{\lambda,\hat u,\nu}\right|^2_{L^2(\mathbb{R}_0,\, H)} &= \int_{\mathbb{R}_0} \mathrm{e}^{(\hat\lambda-\lambda)s} T_*^{-2}\, |w(s)|_H^2 \,\mathrm{d}s \\ &\le \tfrac{1}{\lambda-\hat\lambda} T_*^{-2} \mathrm{e}^{\left(\frac{1}{2\nu}|\hat u|^2_{\mathcal{W}}+\lambda\right)T^*}\, |v_0|_H^2 \le \tfrac{1}{\lambda-\hat\lambda}(2(\nu^{-1}|\hat u|^2_{\mathcal{W}}+\lambda))^2 \mathrm{e}^{\frac12}\, |v_0|_H^2 .\end{aligned}$$

That is, $|\mathrm{e}^{\frac{\hat\lambda}{2}t}\eta^{\lambda,\hat u,\nu}|^2_{L^2(\mathbb{R}_0,\, H)} \le \frac{4\mathrm{e}^{\frac12}}{\lambda-\hat\lambda}(\nu^{-1}|\hat u|^2_{\mathcal{W}}+\lambda)^2\, |v_0|_H^2$. $\square$

3.2. The general case. Let w solve the system

$$\partial_t w = \nu\partial_{xx}w - \partial_x(\hat u w) + \tfrac{\lambda}{2}w + \chi\tilde\eta, \quad w|_\Gamma = 0, \quad w(0) = v_0. \tag{3.9}$$

To simplify the exposition, we rescale time as $t = \frac{\tau}{\nu}$. Then $\breve w(\tau) \coloneqq w(\frac{\tau}{\nu})$ solves

$$\partial_\tau \breve w = \partial_{xx}\breve w - \partial_x(\breve u\breve w) + \tfrac{\breve\lambda}{2}\breve w + \chi\breve\eta, \quad \breve w|_\Gamma = 0, \quad \breve w(0) = v_0, \tag{3.10}$$

with $(\breve u, \breve\lambda, \breve\eta) = \nu^{-1}(\hat u, \lambda, \tilde\eta)$. Next, consider the adjoint system

$$-\partial_\tau q = \partial_{xx}q + \breve u\partial_x q + \tfrac{\breve\lambda}{2}q, \quad q|_\Gamma = 0, \quad q(T) = q_T \tag{3.11}$$

with $q_T \in H$ (here with no external force; cf. system (3.2)). From, for example, [18, Theorem 2.1] and [17, Theorem 2.3] (e.g., reversing time in system (3.11)), we have that given an open set $\omega \subseteq \Omega$, there exists a constant $C_{\omega,\Omega} > 0$, depending on ω and Ω, such that for any time $T > 0$, the weak solution q for (3.11) satisfies

$$|q(0)|_H^2 \le \mathrm{e}^{C_{\omega,\Omega}\left(1+\frac{1}{T}+T\breve\lambda+\breve\lambda^{\frac23}+(1+T)|\breve u|^2_{\mathcal{W}}\right)}\, |q|^2_{L^2((0,T),\, L^2(\omega,\mathbb{R}))} . \tag{3.12}$$

Proposition 3.3. *For every $v_0 \in H$, we can find a control $\breve\eta = \bar\eta(v_0) \in L^2((0,T),H)$, driving system (3.10) to $\breve w(T) = 0$ at time $t = T > 0$. Moreover, the mapping $\bar\eta\colon v_0 \mapsto \bar\eta(v_0)$ is linear and continuous: $\bar\eta \in \mathcal{L}(H \to L^2((0,T),H))$ and there is a constant $C_{\chi,\Omega}$ such that*

$$|\bar\eta(v_0)|^2_{L^2((0,T),\, H)} \le \mathrm{e}^{C_{\chi,\Omega}\left(1+\frac{1}{T}+T\breve\lambda+\breve\lambda^{\frac23}+(1+T)|\breve u|^2_{\mathcal{W}}\right)}\, |v_0|_H^2 . \tag{3.13}$$

Sketch of the proof. The proof can be done following the arguments in [7]. First, from (3.12) we can derive an observability of the form

$$|q(0)|_H^2 \le \mathrm{e}^{C_{\chi,\Omega}\left(1+\frac{1}{T}+T\breve\lambda+\breve\lambda^{\frac23}+(1+T)|\breve u|^2_{\mathcal{W}}\right)}\, |\chi q|^2_{L^2((0,T),\, H)} \tag{3.14}$$

for the solution q of system (3.11) (cf. [7, equation (A.8)]). Then we can prove the null controllability considering the following minimization problem (cf. [7, Problem 3.3])

$$J_\epsilon(\breve{w},\breve{\eta}) = |\breve{\eta}|^2_{L^2} + \frac{1}{\epsilon}|\breve{w}(T)|^2_H \;\to \min \qquad \text{with } (\breve{w},\breve{\eta}) \text{ solving (3.10).}$$

Next, we can consider the minimization problem

$$J_\infty(\breve{w},\breve{\eta}) = |\breve{\eta}|^2_{L^2} \;\to \min \qquad \text{with } (\breve{w},\breve{\eta}) \text{ solving (3.10) and } \breve{w}(T)=0$$

whose unique minimizer $(\overline{w},\overline{\eta})(v_0)$ depends linearly on v_0 (cf. [7, Problem 3.4]). ☐

Considering the null controllability of linear parabolic equations, we also refer to [23, section 2] and [46, section 5.2.2] and references therein.

THEOREM 3.4. *For given $\hat{u}\in\mathcal{W}$ and $\lambda>0$, set*

$$M \ge \tfrac{l}{\pi}C^0_{\chi,\,\Omega}\mathrm{e}^{\frac{3}{2}(1+C_{\chi,\Omega})\left(1+(\frac{\lambda}{\nu})^{\frac12}+(\frac{\lambda}{\nu})^{\frac23}+\frac{1}{\nu}|\hat{u}|_{\mathcal{W}}+\frac{1}{\nu^2}|\hat{u}|^2_{\mathcal{W}}\right)}, \tag{3.15}$$

where $C^0_{\chi,\,\Omega} = (2+2(\frac{L}{\pi})^2)^{\frac12}\,|\chi|_{W^{1,\,\infty}(\Omega,\,\mathbb{R})}$, l is the length of $\mathcal{O}$, and $C_{\chi,\Omega}$ is the constant from (3.13). Then for any given $v_0\in H$, there is a control $\eta^{\lambda,\hat{u},\nu}(v_0)\in L^2(\mathbb{R}_0,\,H)$ such that, taking $\zeta=\chi\mathbb{E}^{\mathcal{O}}_0P^{\mathcal{O}}_M(\eta^{\lambda,\hat{u},\nu}|_{\mathcal{O}})$, the corresponding solution v of system (3.1) satisfies, for $t\ge 0$, the inequality

$$|v(t)|^2_H \le K_{\chi,\Omega}\mathrm{e}^{-\lambda t}|v_0|^2_H \tag{3.16}$$

with $K_{\chi,\Omega} := (1+\mathrm{e}^{(\frac{\lambda}{\nu}+\frac{1}{\nu^2}|u|^2_{\mathcal{W}})^{\frac12}} + \frac{(C^0_{\chi,\,\Omega})^2}{\alpha_1}\mathrm{e}^{3(C_{\chi,\Omega}+1)(1+(\frac{\lambda}{\nu})^{\frac12}+(\frac{\lambda}{\nu})^{\frac23}+\frac{1}{\nu}|\hat{u}|_{\mathcal{W}}+\frac{1}{\nu^2}|\hat{u}|^2_{\mathcal{W}})})$. Moreover, the mapping $v_0\mapsto\eta^{\lambda,\hat{u},\nu}(v_0)$ is well defined, is linear, and satisfies for $0\le\hat{\lambda}<\lambda$ the inequality

$$\left|\mathrm{e}^{\frac{\hat{\lambda}}{2}t}\eta^{\hat{u},\lambda}(v_0)\right|^2_{L^2(\mathbb{R}_0,H)} \le \frac{\nu\mathrm{e}^{C_{\chi,\Omega}\left(1+3(\frac{\lambda}{\nu})^{\frac12}+(\frac{\lambda}{\nu})^{\frac23}+3\frac{1}{\nu}|\hat{u}|_{\mathcal{W}}+\frac{1}{\nu^2}|\hat{u}|^2_{\mathcal{W}}\right)}}{1-\mathrm{e}^{(\hat{\lambda}-\lambda)(2(\nu\lambda+|\hat{u}|^2_{\mathcal{W}}))^{-\frac12}}}\,|v_0|^2_H\,.$$

Proof. Let $\breve{w}$ solve (3.10) for $t\in(0,T)$ with the control $\breve{\eta}=\bar{\eta}(v_0)$ given by Proposition 3.3, and let $\breve{w}_M$ be the solution of

$$\partial_\tau\breve{w}_M = \partial_{xx}\breve{w}_M - \partial_x(\breve{u}\breve{w}_M) + \tfrac{\breve{\lambda}}{2}\breve{w}_M + \chi\mathbb{E}^{\mathcal{O}}_0P^{\mathcal{O}}_M(\bar{\eta}(v_0)|_{\mathcal{O}}),\;\; \breve{w}_M|_\Gamma=0,\;\; \breve{w}_M(0)=v_0.$$

Then, the difference $d := \breve{w}-\breve{w}_M$ solves

$$\partial_\tau d = \partial_{xx}d - \partial_x(\breve{u}d) + \tfrac{\breve{\lambda}}{2}d + \chi\mathbb{E}^{\mathcal{O}}_0(1-P^{\mathcal{O}}_M)(\bar{\eta}(v_0)|_{\mathcal{O}}), \quad d|_\Gamma=0, \quad d(0)=0,$$

and taking the scalar product with d, in H, we can arrive at

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\,|d|^2_H &\le -2\,|\partial_x d|^2_H + 2\,|\breve{u}|_{L^\infty(\Omega,\,\mathbb{R})}\,|d|_H\,|\partial_x d|_H + \breve{\lambda}\,|d|^2_H \\ &\quad + 2\langle\chi\mathbb{E}^{\mathcal{O}}_0(1-P^{\mathcal{O}}_M)(\bar{\eta}(v_0)|_{\mathcal{O}}),\,d\rangle_{V',V}.\end{aligned} \tag{3.17}$$

For the last term, we find

$$\begin{aligned}\langle\chi\mathbb{E}^{\mathcal{O}}_0(1-P^{\mathcal{O}}_M)(\bar{\eta}(v_0)|_{\mathcal{O}}),\,d\rangle_{V',V} &= (\chi\mathbb{E}^{\mathcal{O}}_0(1-P^{\mathcal{O}}_M)(\bar{\eta}(v_0)|_{\mathcal{O}}),\,d)_H \\ &\le |\bar{\eta}(v_0)|_{\mathcal{O}}|_{L^2(\mathcal{O},\,\mathbb{R})}\left|(1-P^{\mathcal{O}}_M)(\chi d|_{\mathcal{O}})\right|_{L^2(\mathcal{O},\,\mathbb{R})},\end{aligned}$$

and from

$$\left|(1-P_M^{\mathcal{O}})(\chi d|_{\mathcal{O}})\right|^2_{L^2(\mathcal{O},\mathbb{R})}\le \alpha_M^{-1}\left|(1-P_M^{\mathcal{O}})(\chi d|_{\mathcal{O}})\right|^2_V\le 2\alpha_M^{-1}\left|\chi\right|^2_{W^{1,\infty}(\Omega,\mathbb{R})}\left|d\right|^2_{H_0^1(\Omega,\mathbb{R})}$$

(which makes sense for a.e. $t\in(0,T)$, since from $2\langle\chi\mathbb{E}_0^{\mathcal{O}}(1-P_M^{\mathcal{O}})(\bar\eta(v_0)|_{\mathcal{O}}),\,d\rangle_{V',V}\le 2|\chi\mathbb{E}_0^{\mathcal{O}}(1-P_M^{\mathcal{O}})(\bar\eta(v_0)|_{\mathcal{O}})|_H\,|d|_H$ and (3.17), by standard arguments it follows that $d\in L^\infty((0,T),H)\cap L^2((0,T),V))$ and $|\partial_x d|^2_{L^2(\Omega,\mathbb{R})}\ge\frac{\alpha_1^\Omega}{1+\alpha_1^\Omega}|d|^2_{H_0^1(\Omega,\mathbb{R})}$, where $\alpha_1^\Omega=\frac{\pi^2}{L^2}$, we find

$$\langle\chi\mathbb{E}_0^{\mathcal{O}}(1-P_M^{\mathcal{O}})(\bar\eta(v_0)|_{\mathcal{O}}),\,d\rangle_{V',V}\le\alpha_M^{-\frac12}D_{\chi,\Omega}\left|\partial_x d\right|_{L^2(\Omega,\mathbb{R})}\left|\bar\eta(v_0)|_{\mathcal{O}}\right|_{L^2(\mathcal{O},\mathbb{R})}$$

with $D_{\chi,\Omega}=(2\frac{1+\alpha_1^\Omega}{\alpha_1^\Omega})^{\frac12}\left|\chi\right|_{W^{1,\infty}(\Omega,\mathbb{R})}=(2+2(\frac{L}{\pi})^2)^{\frac12}\left|\chi\right|_{W^{1,\infty}(\Omega,\mathbb{R})}$. Then, from (3.17),

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left|d\right|^2_H\le\left|\breve u\right|^2_{L^\infty(\Omega,\mathbb{R})}\left|d\right|^2_H+\breve\lambda\left|d\right|^2_H+\alpha_M^{-1}D^2_{\chi,\Omega}\left|\bar\eta(v_0)|_{\mathcal{O}}\right|^2_{L^2(\mathcal{O},\mathbb{R})}$$

and, using (3.13), we obtain

$$\left|d\right|^2_{L^\infty((0,T),H)}\le\alpha_M^{-1}D^2_{\chi,\Omega}\mathrm{e}^{C_{\chi,\Omega}\left(1+\breve\lambda^{\frac23}+|\breve u|^2_{\mathcal{W}}\right)}\mathrm{e}^{C^1_{\chi,\Omega}\left(\frac1T+2(\breve\lambda+|\breve u|^2_{\mathcal{W}})T\right)}\left|v_0\right|^2_H \tag{3.18}$$

with $C^1_{\chi,\Omega}=\max\{1,\,C_{\chi,\Omega}\}$. Now the function $E(T)=\mathrm{e}^{C^1_{\chi,\Omega}(\frac1T+2(\breve\lambda+|\breve u|^2_{\mathcal{W}})T)}$ takes its minimum when $T=T_*$ with T_* defined by $\frac{1}{T_*^2}=2(\breve\lambda+|\breve u|^2_{\mathcal{W}})$. Then, choosing $T=T_*$ and recalling that $\breve w_M(T)=-d(T)$ and $\breve w_M(0)=v_0$, we arrive at

$$\left|\breve w_M(T_*)\right|^2_H\le\alpha_M^{-1}D^2_{\chi,\Omega}\mathrm{e}^{C_{\chi,\Omega}\left(1+\breve\lambda^{\frac23}+|\breve u|^2_{\mathcal{W}}\right)}\mathrm{e}^{2^{\frac32}C^1_{\chi,\Omega}\left(\breve\lambda+|\breve u|^2_{\mathcal{W}}\right)^{\frac12}}\left|\breve w_M(0)\right|^2_H.$$

Thus, choosing $M\in\mathbb{N}_0$ satisfying (3.15) and recalling that $\alpha_M=(\frac{M\pi}{l})^2$, we have

$$\alpha_M\ge D^2_{\chi,\Omega}\mathrm{e}^{3(C_{\chi,\Omega}+1)\left(1+\breve\lambda^{\frac23}+|\breve u|^2_{\mathcal{W}}+\breve\lambda^{\frac12}+|\breve u|_{\mathcal{W}}\right)} \tag{3.19}$$

and $|\breve w_M(T_*)|^2_H\le|\breve w_M(0)|^2_H$. Moreover, we can deduce from (3.13) and (3.18) that $|\breve w_M|^2_{L^\infty((0,T_*),H)}=|\breve w-d|^2_{L^\infty((0,T_*),H)}\le C^d_M\,|\breve w_M(0)|^2_H$ with

$$\begin{aligned}C^d_M&\coloneqq\mathrm{e}^{(\breve\lambda+|\breve u|^2_{\mathcal{W}})T_*}+(\alpha_1^{-1}+\alpha_M^{-1})D^2_{\chi,\Omega}\mathrm{e}^{C_{\chi,\Omega}\left(1+\breve\lambda^{\frac23}+|\breve u|^2_{\mathcal{W}}\right)}\mathrm{e}^{C^1_{\chi,\Omega}\left(\frac{1}{T_*}+2(\breve\lambda+|\breve u|^2_{\mathcal{W}})T_*\right)}\\&\le\mathrm{e}^{(\breve\lambda+|\breve u|^2_{\mathcal{W}})^{\frac12}}+\alpha_1^{-1}D^2_{\chi,\Omega}\mathrm{e}^{3(C_{\chi,\Omega}+1)\left(1+\breve\lambda^{\frac23}+|\breve u|^2_{\mathcal{W}}+\breve\lambda^{\frac12}+|\breve u|_{\mathcal{W}}\right)}+1\eqqcolon\Upsilon_d.\end{aligned}$$

Recursively, repeating the argument in the time interval $(iT_*,+\infty)$ with $\breve w(iT_*)=\breve w_M(iT_*)$ in (3.10), we can conclude that the solution $\breve w_M$ will remain bounded for all time $\tau\ge0$. That is, $|\breve w_M|^2_{L^\infty(\mathbb{R}_0,H)}\le\Upsilon_d\,|v_0|^2_H$.

Next, we notice that $v(t)\coloneqq\mathrm{e}^{-\frac{\lambda}{2}t}\breve w_M(\nu t)$ solves (3.1) in $\mathbb{R}_0\times\Omega$ with the concatenated control $\zeta=\chi\mathbb{E}_0^{\mathcal{O}}P_M^{\mathcal{O}}(\nu\mathrm{e}^{-\frac{\lambda}{2}t}\bar\eta(\breve w_M(iT_*))(\nu t)|_{\mathcal{O}})$, where $\bar\eta(\breve w_M(iT_*))$, $i\in\mathbb{N}$, is the control given in Proposition 3.3 when we consider system (3.10) in $J_*^i\times\Omega$ with $J_*^i\coloneqq(iT_*,\,(i+1)T_*)$, $i\in\mathbb{N}_0$, and $\breve w(iT_*)=\breve w_M(iT_*)$; in particular, $\bar\eta(\breve w_M(iT_*))(\nu t)$ is defined for $t\in(i\nu^{-1}T_*,\,(i+1)\nu^{-1}T_*)$. We can also conclude from (3.13) and from the boundedness of $\{|\breve{w}_M(iT_*)|_H \mid i \in \mathbb{N}\}$ that the family $\{|\bar{\eta}(\breve{w}_M(iT_*))|_{L^2(J_*^i,\, H)} \mid i \in \mathbb{N}\}$ is bounded, so $\mathrm{e}^{\frac{\hat{\lambda}}{2}t}\zeta \in L^2(\mathbb{R}_0,\, H)$ for all $\hat{\lambda} < \lambda$. Finally, we observe that $|v(t)|_H^2 \le \mathrm{e}^{-\lambda t}\,|\breve{w}_M(\nu t)|_H^2 \le \Upsilon_d \mathrm{e}^{-\lambda t}\,|v_0|_H^2$, and for $\eta^{\lambda,\hat{u},\nu}(t) \coloneqq \nu \mathrm{e}^{-\frac{\lambda}{2}t}\bar{\eta}(\breve{w}_M(iT_*))(\nu t)$, $t \in J_*^i$, it follows that

$$\begin{aligned}\left|\mathrm{e}^{\frac{\hat{\lambda}}{2}t}\eta^{\lambda,\hat{u},\nu}\right|^2_{L^2(\mathbb{R}_0,\, H)} &= \lim_{j\to+\infty}\sum_{i=0}^{j}\int_{\frac{iT_*}{\nu}}^{\frac{(i+1)T_*}{\nu}} \mathrm{e}^{(\hat{\lambda}-\lambda)s}\nu^2\,|\bar{\eta}(\breve{w}_M(iT_*))(\nu s)|_H^2\,\mathrm{d}s\\
&\le \lim_{j\to+\infty}\nu\sum_{i=0}^{j}\mathrm{e}^{(\hat{\lambda}-\lambda)\frac{iT_*}{\nu}}\int_{iT_*}^{(i+1)T_*}|\bar{\eta}(\breve{w}_M(iT_*))(\tau)|_H^2\,\mathrm{d}\tau\\
&\le \frac{\nu}{1-\mathrm{e}^{(\hat{\lambda}-\lambda)\frac{T_*}{\nu}}}\mathrm{e}^{C_{\chi,\Omega}\left(1+\frac{1}{T_*}+T_*\breve{\lambda}+\breve{\lambda}^{\frac{2}{3}}+(1+T_*)|\breve{u}|^2_{\mathcal{W}}\right)}\,|v_0|_H^2\,,\end{aligned}$$

from which, using the equality $T^* = (2(\breve{\lambda} + |\breve{u}|^2_{\mathcal{W}}))^{-\frac{1}{2}}$, we can arrive at the estimate $|\mathrm{e}^{\frac{\hat{\lambda}}{2}t}\eta^{\lambda,\hat{u},\nu}|^2_{L^2(\mathbb{R}_0,\, H)} \le \frac{\nu \mathrm{e}^{C_{\chi,\Omega}(1+3(\frac{\lambda}{\nu})^{\frac{1}{2}}+(\frac{\lambda}{\nu})^{\frac{2}{3}}+3\frac{1}{\nu}|\hat{u}|_{\mathcal{W}}+\frac{1}{\nu^2}|\hat{u}|^2_{\mathcal{W}})}}{1-\mathrm{e}^{(\hat{\lambda}-\lambda)(2(\nu\lambda+|\hat{u}|^2_{\mathcal{W}}))^{-\frac{1}{2}}}}|v_0|_H^2$. ∎

Remark 3.5. Notice that when we shrink the support of χ, the constant $C_{\chi,\,\Omega}$ in (3.13) is expected to increase; we cannot expect the right-hand side of (3.15) to go to 0 as the length l of $\mathcal{O}$ does.

3.3. The gap between (3.3) and (3.15). Comparing estimates (3.3) and (3.15), we see that there is a big gap; the former is proportional to $(\frac{1}{\nu^2}|\hat{u}|^2_{\mathcal{W}} + \frac{1}{\nu}\lambda)^{\frac{1}{2}}$, while the latter depends exponentially on both $\frac{1}{\nu}|\hat{u}|_{\mathcal{W}}$ and $(\frac{\lambda}{\nu})^{\frac{1}{2}}$. For application purposes, the latter is much less convenient, so one question arises naturally: can we improve (3.15)?

It seems that the idea used to derive (3.3) cannot (at least straightforwardly) be applied in the general case. On the other side, to derive (3.15) we start from an exact null controllability result and carry the cost of the respective control. This means that to improve (3.15), we will probably need a different idea.

In section 6, in order to understand if it is possible to improve (3.15), say, that we also have an estimate like (3.3) in the general case, we present results of some numerical simulations comparing the number of controls $M = M_{\text{need}}$ that we need to stabilize the system (3.1) to zero with the following reference real numbers

$$M_{\text{ref}} \coloneqq \tfrac{L}{\pi}(\nu^{-2}\,|\hat{u}|^2_{\mathcal{W}} + \nu^{-1}\lambda)^{\frac{1}{2}}; \quad M_{\text{exp}} \coloneqq \tfrac{L}{\pi}\mathrm{e}^{M_{\text{ref}}}. \tag{3.20}$$

The value M_{ref} is motivated by (3.3), and the value M_{exp} by (3.15). Notice that $\frac{l}{\pi}\mathrm{e}^{M_{\text{ref}}}$ is a lower bound for the right-hand side of (3.15); we take $\frac{L}{\pi}$ instead of $\frac{l}{\pi}$ in front of $\mathrm{e}^{M_{\text{ref}}}$ in order to avoid giving the wrong idea that (3.15) goes to 0 with l (cf. Remark 3.5).

Notice that in the case $\hat{u} = 0$, we can see that the unstable modes of system (3.5) are those defined by the inequality $\nu\alpha_i < \frac{\lambda}{2}$, that is, $i < \frac{L}{\pi}\nu^{-\frac{1}{2}}(\frac{\lambda}{2})^{\frac{1}{2}} = 2^{-\frac{1}{2}}M_{\text{ref}} < M_{\text{ref}}$. Thus, in this case and with $\chi = 1_\Omega$, it is enough (and necessary) to take the $M = \lfloor 2^{-\frac{1}{2}}M_{\text{ref}}\rfloor$ controls in $\{(\frac{2}{L})^{\frac{1}{2}}\sin(\frac{i\pi x}{L}) \mid i \in \{1,\, 2,\, \ldots,\, M\}\}$ (taking the family of controls considered in section 3.1). Here, $\lfloor y\rfloor \in \mathbb{N}$ stands for the biggest integer that is strictly smaller than $y > 0$.

3.4. Feedback control and Riccati equation. By the dynamic programing principle, for example, following the arguments in [7, section 3.2], considering the family of minimization problems

(3.21, s)

$$\begin{aligned}&\text{Minimize}\quad \mathcal{J}^s(v,\,\eta) \coloneqq \textstyle\int_{\mathbb{R}_s} \mathrm{e}^{\lambda t}(\nu\,|v(t)|_V^2 + |\eta(t)|_H^2)\,\mathrm{d}t \quad \text{on the space } \mathcal{X}\\ &\coloneqq \left\{ (v,\,\eta) \;\middle|\; \begin{array}{l} \mathrm{e}^{\frac{\lambda}{2}t}(v,\,\eta) \in W(\mathbb{R}_s,\,V,\,V') \times L^2(\mathbb{R}_s,\,H) \text{ and, for } t \in \mathbb{R}_s,\\ (v,\,\zeta) \text{ solves (3.1) with } v(s) = w \in H \text{ and } \zeta = \chi\mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}(\eta|_{\mathcal{O}}) \end{array}\right\},\end{aligned}$$

where s runs over $[0,\,+\infty)$ and $\mathbb{R}_s = (s,\,+\infty)$, we can derive the following result.

THEOREM 3.6. *The controls ζ given in Theorems* 3.2 *and* 3.4 *can be taken in feedback form*

$$\zeta = \mathrm{e}^{-\lambda t}\chi\mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}((\chi Q_{\hat{u}}^{t,\,\lambda} v)|_{\mathcal{O}}) \tag{3.22}$$

for a suitable family of operators $Q_{\hat{u}}^{t,\,\lambda}\colon H \to H$, $t \geq 0$, with $|Q_{\hat{u}}^{t,\,\lambda}|_{\mathcal{L}(H\to H)} \leq \overline{C}_{[\lambda,\,\hat{u},\,\frac{1}{\nu}]}\mathrm{e}^{\lambda t}$. Furthermore, the family $\{Q_{\hat{u}}^{t,\,\lambda} \mid t \geq 0\}$ is continuous in the weak operator topology, and $Q \coloneqq Q_{\hat{u}}^{t,\,\lambda}$ satisfies the differential Riccati equation

$$\dot{Q} - Q(-\nu\partial_{xx} + \mathcal{B}(\hat{u})) - (-\nu\partial_{xx} + \mathcal{B}(\hat{u}))^* Q, -QB_M^{\mathcal{O}}{B_M^{\mathcal{O}}}^* Q - \mathrm{e}^{\lambda t}\nu\partial_{xx} = 0 \tag{3.23}$$

where $\mathcal{B}(\hat{u})v \coloneqq \partial_x(\hat{u}v)$, and $B_M^{\mathcal{O}}\colon H \to H$ and its adjoint ${B_M^{\mathcal{O}}}^*\colon H \to H$ given by

$$B_M^{\mathcal{O}}\eta \coloneqq \mathrm{e}^{-\frac{\lambda}{2}t}\chi\mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}(\eta|_{\mathcal{O}}), \qquad {B_M^{\mathcal{O}}}^*\xi = \mathrm{e}^{-\frac{\lambda}{2}t}\mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}((\chi\xi)|_{\mathcal{O}}), \tag{3.24}$$

and $(Q_{\hat{u}}^{s,\,\lambda} v_*^0(s),\, v_*^0(s))_H = \mathcal{J}^s(v_*^0|_{\mathbb{R}_s},\, \eta_*^0|_{\mathbb{R}_s})$, where $(v_*^0,\, \eta_*^0)$ is the unique minimizer of problem $(3.21, 0)$. *Further, $(Q_{\hat{u}}^{s,\,\lambda} w,\, w)_H = \mathcal{J}^s(v_*^s,\, \eta_*^s)$, where $(v_*^s,\, \eta_*^s)$ is the unique minimizer of problem* $(3.21, s)$.

Remark 3.7. Equation (3.23) can be seen as an evolutionary equation, and $\dot{Q} \coloneqq \frac{\mathrm{d}}{\mathrm{d}t}Q$. A solution for (3.23) is understood in the sense that

$$\begin{aligned}(\dot{Q}w^1,\, w^2)_H &= (Q\mathbb{A}w_1,\, w^2)_H + (\mathbb{A}^* Q w_1,\, w^2)_H\\ &\quad + (QB_M^{\mathcal{O}}{B_M^{\mathcal{O}}}^* Q w_1,\, w^2)_H + (\mathrm{e}^{\lambda t}\nu\partial_{xx} w_1,\, w^2)_H\end{aligned}$$

holds for all $(w^1,\, w^2) \in \mathrm{D}(\partial_{xx}) \times \mathrm{D}(\partial_{xx})$ with $\mathbb{A} = \mathbb{A}(t) \coloneqq -\nu\partial_{xx} + \mathcal{B}(\hat{u}(t))$. See, for example, [15, section 5.4], [35, Chapter 1, Theorem 1.4.6.4 and Corollary 1.5.3.3]. Observe also that $(Q_{\hat{u}}^{s,\,\lambda} w,\, w)_H = \mathcal{J}^s(v_*^s,\, \eta_*^s) > 0$ for all $w \neq 0$, that is, $Q_{\hat{u}}^{s,\,\lambda}$ is definite positive.

Remark 3.8. Notice that $(\partial_{xx})^* = \partial_{xx}$ and $\mathcal{B}(\hat{u})^* = -\hat{u}\partial_x$. Notice also that from Theorems 3.2 and 3.4 (taking, e.g., $(2\lambda,\, \lambda)$ in place of $(\lambda,\, \hat{\lambda})$), we have that the space $\mathcal{X}$ in problem (3.21) is nonempty.

Remark 3.9. For any $T > 0$ and $w \in H$, the function $q \coloneqq Qv_*^0$ solves the system (3.2) with $f = -\mathrm{e}^{\lambda t}\nu\partial_{xx}v_*^0$ and $q(T) = Q_{\hat{u}}^{T,\,\lambda} v_*^0(T)$, where $(v_*^0,\, \eta_*^0) = (v_*^0,\, \eta_*^0)(w)$ is the minimizer of problem $(3.21, 0)$.

We already know that Q satisfies (3.23), and we can also show that it is unique in the class of operators $\mathrm{e}^{\lambda t}\mathcal{C}$ with

$$\mathcal{C} \coloneqq \left\{ \widehat{R} \in L^\infty(\mathbb{R}_0, \mathcal{L}(H \to H)) \;\middle|\; \begin{array}{l} \widehat{R}(t) \text{ is self-adjoint positive definite for all } t \geq 0,\\ \text{the family } \{\widehat{R}(t) \mid t \geq 0\} \text{ is continuous in the}\\ \text{weak operator topology} \end{array}\right\}.$$

LEMMA 3.10. *If R satisfies* (3.23) *and $\widehat{R} \coloneqq \mathrm{e}^{-\lambda t}R \in \mathcal{C}$, then the feedback control*

$$\zeta = \mathrm{e}^{-\lambda t}\chi\mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}((\chi R v)|_{\mathcal{O}}) = B_M^{\mathcal{O}}{B_M^{\mathcal{O}}}^* R v$$

exponentially stabilizes system (3.1) *to zero with rate* $\frac{\lambda}{2}$.

Proof. We find that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(Rv,\, v)_H &= (\dot{R}v,\, v)_H + (R\partial_t v,\, v)_H + (Rv,\, \partial_t v)_H\\ &= ((R\mathbb{A} + \mathbb{A}^*R + RB_M^{\mathcal{O}}{B_M^{\mathcal{O}}}^*R + \mathrm{e}^{\lambda t}\nu\partial_{xx})v,\, v)_H\\ &\quad + (-R(\mathbb{A} + B_M^{\mathcal{O}}{B_M^{\mathcal{O}}}^*R)v,\, v)_H + (Rv,\, -(\mathbb{A} + B_M^{\mathcal{O}}{B_M^{\mathcal{O}}}^*R)v)_H\\ &= -\mathrm{e}^{\lambda t}(\nu|v|_V^2 + |\mathrm{e}^{-\frac{\lambda}{2}t}{B_M^{\mathcal{O}}}^*Rv|_H^2).\end{aligned}$$

Notice that $\chi\mathbb{E}_0^{\mathcal{O}}P_M^{\mathcal{O}}((\chi\mathrm{e}^{-\frac{\lambda}{2}t}{B_M^{\mathcal{O}}}^*Rv)|_{\mathcal{O}}) = B_M^{\mathcal{O}}{B_M^{\mathcal{O}}}^*Rv$. Thus, we have that $(Rv,\, v)_H$ is decreasing and, after integration, that $\mathcal{J}^s(v,\, \mathrm{e}^{-\frac{\lambda}{2}t}{B_M^{\mathcal{O}}}^*Rv) = (R(s)v(s),\, v(s))_H - \lim_{T\to+\infty}(R(T)v(T),\, v(T))_H \le (R(s)v(s),\, v(s))_H \le |\widehat{R}|_{L^\infty(\mathbb{R}_0,\, \mathcal{L}(H\to H))}\mathrm{e}^{\lambda s}\,|v(s)|_H^2$. This inequality, together with $\partial_t(\mathrm{e}^{\frac{\lambda}{2}t}v) = \frac{\lambda}{2}\mathrm{e}^{\frac{\lambda}{2}t}v + \mathrm{e}^{\frac{\lambda}{2}t}(\nu\partial_{xx}v - \partial_x(\hat{u}v) + B_M^{\mathcal{O}}{B_M^{\mathcal{O}}}^*Rv)$ and $\int_{\mathbb{R}_s}|\mathrm{e}^{\frac{\lambda}{2}t}B_M^{\mathcal{O}}{B_M^{\mathcal{O}}}^*Rv|_H^2\,\mathrm{d}t \le C_0\int_{\mathbb{R}_s}|{B_M^{\mathcal{O}}}^*Rv|_H^2\,\mathrm{d}t \le C_0\mathcal{J}^s(v,\, \mathrm{e}^{-\frac{\lambda}{2}t}{B_M^{\mathcal{O}}}^*Rv)$, imply that $\partial_t(\mathrm{e}^{\frac{\lambda}{2}t}v)$ is in $L^2(\mathbb{R}_s,\, V')$ with $|\partial_t(\mathrm{e}^{\frac{\lambda}{2}t}v)|_{L^2(\mathbb{R}_s,\, V')} \le C\mathrm{e}^{\lambda s}\,|v(s)|_H$. Hence, it follows that $|\mathrm{e}^{\frac{\lambda}{2}t}v|_{C([s,\,+\infty),\, H)} \le C_1\mathrm{e}^{\frac{\lambda}{2}s}\,|v(s)|_H$ for suitable positive constants C_0, C, and C_1. That is, the feedback control $\zeta = B_M^{\mathcal{O}}{B_M^{\mathcal{O}}}^*Rv$ stabilizes system (3.1) to zero with rate $\frac{\lambda}{2}$, $|v(t)|_H \le C_1\mathrm{e}^{-\frac{\lambda}{2}(t-s)}\,|v(s)|_H$. ∎

The uniqueness of Q will follow from the uniqueness of $Q_1 \coloneqq \mathrm{e}^{-\lambda t}Q \in \mathcal{C}$ satisfying

$$\dot{Q}_1 - Q_1\mathbb{A} - \mathbb{A}^*Q_1 - Q_1\mathbb{B}\mathbb{B}^*Q_1 - \nu\partial_{xx} + \lambda Q_1 = 0 \tag{3.25}$$

with $\mathbb{B} = \mathbb{B}(t) \coloneqq \mathrm{e}^{\frac{\lambda}{2}t}B_M^{\mathcal{O}}$. From the exponential stability, with rate $\frac{\lambda}{2}$, of system (3.1) with ζ given by (3.22), it follows that

$$\partial_t z - \nu\partial_{xx}z + \partial_x(\hat{u}z) - \tfrac{\lambda}{2}z + \mathbb{B}\mathbb{B}^*Q_1 z = 0, \qquad z|_\Gamma = 0, \qquad z(0) = z_0, \tag{3.26}$$

is stable, that is, there is a constant $C > 0$ independent of z_0 such that $|z(t)|_H \le C\,|z_0|_H$ for all $t \in \mathbb{R}_0$. Actually, we can prove that it is uniformly exponentially stable, that is, there are $\alpha > 0$ and $K > 0$ such that

$$|z(t)|_H \le K\mathrm{e}^{-\alpha(t-t_0)}\,|z(t_0)|_H \text{ for all } 0 \le t_0 \le t. \tag{3.27}$$

Indeed, notice that we can consider the system (3.1) in the interval of time $\mathbb{R}_{t_0} = (t_0,\, +\infty)$ instead of $\mathbb{R}_0$, and we obtain the analogues to Theorems 3.2 and 3.4, replacing the initial time $t = 0$ by $t = t_0$. This means that if we denote by $\mathcal{S}(t,\, t_0)w$ the solution of system (3.26) for $t \in \mathbb{R}_{t_0}$ with initial condition $z(t_0) = z_{t_0}$, we will have that $|\mathcal{S}(t,\, t_0)w|_{L^2(\mathbb{R}_0,\, H)}^2 \le C\,|z_{t_0}|_H^2$, where C is given in Theorems 3.2 and 3.4 and can be taken independent of t_0. The uniform exponential stability follows then by [14, Theorem 1]; see also [47, Chapter 3, Theorem 3.1].

Remark 3.11. The operator (or family of operators) $\mathcal{S}(t,\, t_0)$ is sometimes called an "evolutionary process" as in [14, section 1], "Green operator" as in [36, Chapter IV, section 3], or "evolution operator" as in [13, section 2].

THEOREM 3.12. *The solution of* (3.25) *is unique in* $\mathcal{C}$.

Proof. We follow ideas from [12, 13, 45]; see also [35, Chapter 1]. Let $Q_2 \in \mathcal{C}$ solve (3.25). Then with $\mathcal{A}_1 = \mathbb{A} + \mathbb{B}\mathbb{B}^*Q_1 - \frac{\lambda}{2}I$ and $\mathcal{A}_2 = \mathbb{A} + \mathbb{B}\mathbb{B}^*Q_2 - \frac{\lambda}{2}I$, where I is the identity operator, the difference $D \coloneqq Q_2 - Q_1$ solves

$$\dot{D} = D\mathcal{A}_1 + \mathcal{A}_1^*D + D\mathbb{B}\mathbb{B}^*D, \tag{3.28a}$$

$$\dot{D} = D\mathcal{A}_2 + \mathcal{A}_2^*D - D\mathbb{B}\mathbb{B}^*D. \tag{3.28b}$$

Let $w \in H$ and let $\mathcal{S}_i(t,\, s)w$ stand for the solution of system (3.26) in the interval of time $t \in \mathbb{R}_s$ with $z(s) = w$ and with Q_i in the place of Q_1, $i \in \{1,\, 2\}$. Then we have that $\partial_t \mathcal{S}_i(t,\, s)w = -\mathcal{A}_i(t)\mathcal{S}_i(t,\, s)w$, and we find $\partial_t \mathcal{S}_i(t,\, s) = -\mathcal{A}_i(t)\mathcal{S}_i(t,\, s)$ and $\partial_t \mathcal{S}_i(t,\, s)^* = -\mathcal{S}_i(t,\, s)^*\mathcal{A}_i(t)^*$. For $t \in (s,\, T)$, we also have $0 = \partial_t \mathcal{S}_i(T,\, s)w = \partial_t(\mathcal{S}_i(T,\, t)\mathcal{S}_i(t,\, s)w)$, which gives us $0 = (\partial_t \mathcal{S}_i(T,\, t))\mathcal{S}_i(t,\, s) + \mathcal{S}_i(T,\, t)\partial_t \mathcal{S}_i(t,\, s)$, that is, $\partial_t \mathcal{S}_i(T,\, t) = \mathcal{S}_i(T,\, t)\mathcal{A}_i(t)$ and $\partial_t \mathcal{S}_i(T,\, t)^* = \mathcal{A}_i(t)^*\mathcal{S}_i(T,\, t)^*$.

Now we fix $s \geq 0$ and, for $t > s$, set $G_i(t) \coloneqq \mathcal{S}_i(t,\, s)^* D(t)\mathcal{S}_i(t,\, s)$. Using (3.28), it follows that

$$\dot{G}_1 = \mathcal{S}_1(t,\, s)^*(D\mathbb{B}\mathbb{B}^*D)(t)\mathcal{S}_1(t,\, s) \quad \text{and} \quad \dot{G}_2 = -\mathcal{S}_2(t,\, s)^*(D\mathbb{B}\mathbb{B}^*D)(t)\mathcal{S}_2(t,\, s).$$

Then we obtain $(G_1(t)w,\, w)_H - (G_1(s)w,\, w)_H = \int_s^t |(\mathbb{B}^*D)(r)\mathcal{S}_1(r,\, s)w|_H^2 \,\mathrm{d}r$ and $(G_2(t)w,\, w)_H - (G_2(s)w,\, w)_H = -\int_s^t |(\mathbb{B}^*D)(r)\mathcal{S}_2(r,\, s)w|_H^2 \,\mathrm{d}r$. Then, from $G_1(s) = D(s) = G_2(s)$, we arrive to

$$(G_2(t)w,\, w)_H \leq (D(s)w,\, w)_H \leq (G_1(t)w,\, w)_H. \tag{3.29}$$

Notice that from Lemma 3.10 and (3.27) we have that

$$|\mathcal{S}_i(t,\, t_0)w|_H \leq K_i \mathrm{e}^{-\alpha_i(t-t_0)}\, |w|_H \;\text{ for all } 0 \leq t_0 \leq t$$

for suitable positive constants K_i and α_i. Thus, from (3.29) it follows that

$$-\widehat{D}K_2^2 \mathrm{e}^{-2\alpha_2(t-s)}|w|_H^2 \leq (D(s)w,\, w)_H \leq \widehat{D}K_1^2 \mathrm{e}^{-2\alpha_1(t-s)}|w|_H^2 \tag{3.30}$$

with $\widehat{D} \coloneqq |D|_{L^\infty(\mathbb{R}_0,\, \mathcal{L}(H\to H))}$. Letting t go to $+\infty$, we obtain $(D(s)w,\, w)_H = 0$. Hence, since w can be taken arbitrary and $D(s)$ is self-adjoint, it follows that $0 = (D(s)(w^1 + w^2),\, w^1 + w^2)_H = 2(D(s)w^1,\, w^2)_H$ for any $(w^1,\, w^2) \in H \times H$; necessarily, $D(s) = 0$ and $D = 0$ because s can be taken arbitrary. ∎

We know (cf. Remark 3.7) that $Q_1(t) = \mathrm{e}^{-\lambda t}Q(t)$ is self-adjoint and positive definite for all $t \geq 0$. From (3.25) we can also conclude that if, at some $T > 0$, we impose a final condition $Q_1(T) = Q_1^T$ with Q_1^T self-adjoint and positive definite, and then $Q_1(t)$ remains self-adjoint and positive definite for all $t \in [0,\, T]$. Indeed, from

$$\dot{Q}_1 = Q_1\mathcal{A}_1 + \mathcal{A}_1^* Q_1 - Q_1\mathbb{B}\mathbb{B}^*Q_1 + \nu\partial_{xx} \tag{3.31}$$

we can see that Q_1 can be written as

$$Q_1(t) = \mathcal{S}_1(T,\, t)^* Q_1^T \mathcal{S}_1(T,\, t) + \int_t^T \mathcal{S}_1(s,\, t)^*(Q_1\mathbb{B}\mathbb{B}^*Q_1 - \nu\partial_{xx})(s)\mathcal{S}_1(s,\, t)\,\mathrm{d}s \tag{3.32}$$

and, for $u \neq 0$, we have

$$\begin{aligned}(Q_1(t)u,\, u)_H &= (Q_1^T\mathcal{S}_1(T,\, t)u,\, \mathcal{S}_1(T,\, t)u)_H \\ &\quad + \int_t^T |\mathbb{B}^*Q_1\mathcal{S}_1(s,\, t)u|_H^2 + \nu\, |\partial_x \mathcal{S}_1(s,\, t)u|_H^2 \,\mathrm{d}s > 0.\end{aligned} \tag{3.33}$$

Further, if Q_2 also solves (3.25) with $Q_2(T) = Q_1^T$, then necessarily $D(s) \coloneqq Q_2(s) - Q_1(s) = 0$ for all $s \in (0,\, T)$, because from (3.29) if $D(T) = 0$, we can derive that $0 = (G_2(T)w,\, w)_H \leq (D(s)u,\, u)_H \leq (G_1(T)w,\, w)_H = 0$.

Remark 3.13. Denoting $F_1 \coloneqq Q_1\mathbb{B}\mathbb{B}^*Q_1 - \nu\partial_{xx}$ and differentiating (3.32), we find

$$\begin{aligned}\dot{Q}_1 = \partial_t Q_1 &= \mathcal{A}_1(t)^*\mathcal{S}_1(T,t)^*Q_1^T\mathcal{S}_1(T,t) + \mathcal{S}_1(T,t)^*Q_1^T\mathcal{S}_1(T,t)\mathcal{A}_1(t) - F_1(t)\\ &\quad + \int_t^T \mathcal{A}_1(t)^*\mathcal{S}_1(s,t)^*F_1(s)\mathcal{S}_1(s,t) + \mathcal{S}_1(s,t)^*F_1(s)\mathcal{S}_1(s,t)\mathcal{A}_1(t)\,\mathrm{d}s\\ &= Q_1(t)\mathcal{A}_1(t) + \mathcal{A}_1(t)^*Q_1(t) - F_1(t),\end{aligned}$$

that is, we recover (3.31).

Remark 3.14. From (3.32), we see that $Q_1(t)$ is obtained from $Q_1^T = Q_1(T)$. Thus, the Riccati equations (3.31), (3.23), and (3.25) must be solved backward in time.

4. The nonlinear system. The next result is a corollary of Theorem 3.6. It will follow by a fixed point argument.

THEOREM 4.1. *Let M be the integer in Theorem* 3.6 *(i.e., as in either* (1.3) *or* (1.4)*). Then there are positive constants Θ and $\epsilon = \epsilon(\Theta)$ depending only on λ, $|\hat{u}|_{\mathcal{W}}$, and ν such that for $|v_0|_H \le \epsilon$, the solution v of system* (2.4) *with ζ as in* (3.22) *is well defined for all $t \ge 0$ and satisfies the inequality*

$$|v(t)|_H^2 \le \Theta \mathrm{e}^{-\lambda t}|v_0|_H^2 \quad \textit{for} \;\; t \ge 0. \tag{4.1}$$

Notice that the feedback rule is found to globally stabilize to zero the linear Oseen–Burgers system (3.1). Then, Theorem 4.1 says that the same feedback rule also locally stabilizes to zero the bilinear system (2.4).

The proof of Theorem 4.1 will be done following the arguments in [7, section 4]. The nonlinear system (2.4) with ζ as in (3.22) reads

$$\partial_t v - \nu\partial_{xx}v + v\partial_x v + \partial_x(\hat{u}v) + \mathcal{K}^{t,\lambda}v = 0, \quad v|_\Gamma = 0, \quad v(0) = v_0 \tag{4.2}$$

with $\mathcal{K}^{t,\lambda}v \coloneqq \mathrm{e}^{-\lambda t}\chi\mathbb{E}_0^{\mathcal{O}}P_M^{\mathcal{O}}((\chi Q_{\hat{u}}^{t,\lambda}v)|_{\mathcal{O}})$. Given $\lambda > 0$, we denote by $\mathcal{Z}^\lambda$ the space of functions $z \in C([0,+\infty), H) \cap L^2(\mathbb{R}_0, V)$ such that

$$|z|_{\mathcal{Z}^\lambda} \coloneqq \left(\left| e^{\frac{\lambda}{2}\cdot}z(\cdot)\right|^2_{L^\infty(\mathbb{R}_0,H)} + \left| e^{\frac{\lambda}{2}\cdot}z(\cdot)\right|^2_{L^2(\mathbb{R}_0,V)} \right)^{\frac{1}{2}} < \infty.$$

Let us fix a constant $\Theta > 0$ and a function $v_0 \in H$ and introduce the following subset of $\mathcal{Z}^\lambda$:

$$\mathcal{Z}^\lambda_\Theta \coloneqq \{ z \in \mathcal{Z}^\lambda \mid z(0) = v_0, |z|^2_{\mathcal{Z}^\lambda} \le \Theta|v_0|^2_H \}.$$

We define a mapping $\Xi : \mathcal{Z}^\lambda \to C([0,+\infty), H) \cap L^2_{\mathrm{loc}}(\mathbb{R}_0, V)$ that takes a function $a \in \mathcal{Z}^\lambda$ to the solution b of the problem

$$\partial_t b - \nu\partial_{xx}b + \partial_x(\hat{u}b) + \mathcal{K}^t b + a\partial_x a = 0, \quad b|_\Gamma = 0, \quad b(0) = v_0. \tag{4.3}$$

Recall that $\left|\mathcal{K}^{\cdot,\lambda}\right|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H\to H))} \le C_1$ and $|a\partial_x a|_{V'} \le C_2\,|a|_H\,|a|_V$.

LEMMA 4.2. *Let M be the integer in Theorem* 3.6. *Then, there exists $\Theta = \Theta(\lambda, |\hat{u}|_{\mathcal{W}}, \nu) > 0$ such that the following property holds: for any $\gamma \in (0,1)$, one can find a constant $\epsilon = \epsilon_{\Theta,\gamma} > 0$ such that for any $v_0 \in H$ with $|v_0|_H \le \epsilon$ the mapping Ξ takes the set Z^λ_Θ into itself and satisfies the inequality*

$$|\Xi(a_1) - \Xi(a_2)|_{\mathcal{Z}^\lambda} \le \gamma|a_1 - a_2|_{\mathcal{Z}^\lambda} \quad \textit{for all} \quad a_1, a_2 \in \mathcal{Z}^\lambda_\Theta. \tag{4.4}$$

Proof. Step 1. *For suitable* Θ *and* $\epsilon = \epsilon_\Theta$, Ξ *maps* $\mathcal{Z}^\lambda_\Theta$ *into itself.* By the Duhamel formula, we can write b as

$$b(t) = \mathcal{S}(t,\, 0)b(0) + \int_0^t \mathcal{S}(t,\, s)(a\partial_x a)(s)\,\mathrm{d}s, \tag{4.5}$$

where $\mathcal{S}(t,\, s)w$ denotes the solution of the system (4.3), for time $t \ge s$ with initial condition $b(s) = w$, and $a = 0$. Then, we derive

$$\begin{aligned}|b(t)|_H^2 &\le 2|\mathcal{S}(t,\, 0)b(0)|_H^2 + 2\left(\int_0^t |\mathcal{S}(t,\, s)(a\partial_x a)(s)|_H\,\mathrm{d}s\right)^2\\ &\le 2C_3 e^{-(\lambda+\beta)t}|b(0)|_H^2 + 2C_3 e^{-(\lambda+\beta)t}\left(\int_0^t e^{\frac{\lambda+\beta}{2}s}|(a\partial_x a)(s)|_H\,\mathrm{d}s\right)^2,\end{aligned}$$

where $\beta = \min\{\alpha,\, \lambda\} > 0$, and α is as in (3.27). From $|(a\partial_x a)(s)|_H \le C_4\,|a|_V^2$, it follows that

$$e^{(\lambda+\beta)t}|b(t)|_H^2 \le C_5\left(|b(0)|_H^2 + |a|_{\mathcal{Z}^\lambda}^4\right). \tag{4.6}$$

Now, multiplying (4.3) by b and following standard arguments, we also have that

$$\begin{aligned}\nu\int_s^{s+1} |b(\tau)|_V^2\,\mathrm{d}\tau &\le |b(s)|_H^2 + C_6\left(|\hat{u}|_{\mathcal{W}}^2 + \left|\mathcal{K}^{\cdot,\,\lambda}\right|^2_{L^\infty(\mathbb{R}_0,\,\mathcal{L}(H\to H))}\right)|b|^2_{L^2((s,\,s+1),\,H)}\\ &\quad + C_7\int_s^{s+1} |a\partial_x a(\tau)|_{V'}^2\,\mathrm{d}\tau,\end{aligned}$$

from which, using (4.6), it follows that

$$\begin{aligned}\nu\int_s^{s+1} \mathrm{e}^{\lambda\tau}\,|b(\tau)|_V^2\,\mathrm{d}\tau &\le \mathrm{e}^{\lambda(s+1)}\,|b(s)|_H^2 + \mathrm{e}^{\lambda}C_7\int_s^{s+1}\mathrm{e}^{\lambda\tau}\,|a\partial_x a(\tau)|_{V'}^2\,\mathrm{d}\tau\\ &\quad + C_6\left(|\hat{u}|_{\mathcal{W}}^2 + \left|\mathcal{K}^{\cdot,\,\lambda}\right|^2_{L^\infty(\mathbb{R}_0,\,\mathcal{L}(H\to H))}\right)\mathrm{e}^{\lambda(s+1)}\,|b|^2_{L^\infty((s,\,s+1),\,H)}\\ &\le \mathrm{e}^{-\beta s}C_8\left(|b(0)|_H^2 + |a|_{\mathcal{Z}^\lambda}^4\right) + C_9\int_s^{s+1}\mathrm{e}^{\lambda\tau}\,|a(\tau)|_H^2\,|a(\tau)|_V^2\,\mathrm{d}\tau.\end{aligned}$$

Thus, since $\mathrm{e}^{\lambda\tau} \le \mathrm{e}^{-\lambda s}\mathrm{e}^{2\lambda\tau} \le \mathrm{e}^{-\beta s}\mathrm{e}^{2\lambda\tau}$ for $\tau \in (s,\, s+1)$, summing up we obtain $\int_{\mathbb{R}_0}\mathrm{e}^{\lambda\tau}|b(\tau)|_V^2\,\mathrm{d}\tau \le C_{10}(|b(0)|_H^2 + |a|_{\mathcal{Z}^\lambda}^4)\sum_{j=1}^{+\infty}\mathrm{e}^{-\beta j}$. Then, from (4.6) we arrive at $|b|_{\mathcal{Z}^\lambda} \le C_{11}(|b(0)|_H^2 + |a|_{\mathcal{Z}^\lambda}^4)$, which implies that for $a \in \mathcal{Z}^\lambda_\Theta$, we have

$$|\Xi(a)|_{\mathcal{Z}^\lambda}^2 \le C_{11}\left(1 + \Theta^2\,|v_0|_H^2\right)|v_0|_H^2\,.$$

Setting $\Theta = 2C_{11}$ and choosing $\epsilon_\Theta > 0$ so small that $\Theta\epsilon_\Theta \le 1$, we see that if $|v_0|_H \le \epsilon_\Theta$, then Ξ maps the set $\mathcal{Z}^\lambda_\Theta$ into itself.

Step 2. *Given* $\gamma \in (0,\, 1)$, Ξ *is a* γ*-contraction for smaller* $\epsilon = \epsilon_{\Theta,\,\gamma}$. Let us take two functions $a_1, a_2 \in \mathcal{Z}^\lambda_\Theta$ and set $a \coloneqq a_1 - a_2$ and $b \coloneqq \Xi(a_1) - \Xi(a_2)$. Then the function b satisfies (4.3) with $b(0) = 0$ and $a_1\partial_x a_1 - a_2\partial_x a_2$ in the place of $a\partial_x a$. From

$$\begin{aligned}|a_1\partial_x a_1 - a_2\partial_x a_2|_H = |a_1\partial_x a + a\partial_x a_2|_H &\le C_4(|a_1|_V + |a_2|_V)\,|a|_V\,;\\ |a_1\partial_x a_1 - a_2\partial_x a_2|_{V'} &\le C_{12}(|a_1|_H + |a_2|_H)\,|a|_V\,;\end{aligned}$$

and proceeding as above we can arrive at

$$|\Xi(a_1) - \Xi(a_2)|^2_{\mathcal{Z}^\lambda} \le C_{13}\left(|a_1|^2_{\mathcal{Z}^\lambda} + |a_2|^2_{\mathcal{Z}^\lambda}\right)|a|^2_{\mathcal{Z}^\lambda} \le 2C_{13}\Theta\, |v_0|^2_H\, |a_1 - a_2|^2_{\mathcal{Z}^\lambda}.$$

Choosing $\tilde\epsilon_{\Theta,\gamma} > 0$ so small that $2\Theta C_{13}\tilde\epsilon^2_{\Theta,\gamma} \le \gamma^2$, we see that if $|v_0|_H \le \tilde\epsilon_{\Theta,\gamma}$, then (4.4) holds. Therefore, the lemma holds with $\epsilon_{\Theta,\gamma} = \min\{\epsilon_\Theta, \tilde\epsilon_{\Theta,\gamma}\}$. ☐

Proof of Theorem 4.1. If $|v_0|_H \le \epsilon_{\Theta,\gamma}$, the contraction mapping principle implies that there is a unique fixed point $v \in \mathcal{Z}^\lambda_\Theta$ for Ξ. It follows from the definition of Ξ and Z^λ_Θ that v is a solution of problem (4.2) and satisfies (4.1). We claim that v is the unique solution of (4.2) in the space $C([0, +\infty), H) \cap L^2(\mathbb{R}_0, V)$. Indeed, if w is another solution, then the difference $z = v - w$ satisfies

$$z_t - \nu\partial_{xx} z + z\partial_x z + \partial_x(wz) + \partial_x(\hat u z) + \mathcal{K}^{t,\lambda} z = 0, \quad z(0) = 0.$$

Multiplying this equation by z, in H, and following a standard procedure, we arrive at $\frac{\mathrm{d}}{\mathrm{d}t}|z|^2_H + \nu|z|^2_V \le C_{14}(|w|^2_V + |\hat u|^2_{\mathcal{W}})|z|^2_H$, which implies $z(t) = 0$ for all $t \ge 0$. ☐

Remark 4.3. Though it would be possible to derive more precise estimates on the Θ and ϵ in Theorem 4.1, it would lead to a more cumbersome exposition, and these estimates are not the main focus of this work.

5. Discretization. To perform the simulations in order to check the stabilization of systems (1.1) and (3.1), to a reference trajectory $\hat u$ and to zero, respectively, we must discretize those systems with the feedback control ζ as in (3.22).

5.1. Discretization in space. We use a finite-element-based approach. We introduce a uniform mesh

$$\Omega_D := \left(\tfrac{L}{N_x}, \tfrac{2L}{N_x}, \ldots, \tfrac{(N_x-1)L}{N_x}\right) \tag{5.1}$$

consisting of the interior points of Ω that are multiples of the space step $h = \frac{L}{N_x}$ with $2 \le N_x \in \mathbb{N}$. As basis functions, we take the classical hat-functions $\phi_i \in V$ defined for $x \in \Omega$ and each $i \in \{1, 2, \ldots, N_x - 1\}$ by $\phi_i(x) := \begin{cases} 1-i+\frac{x}{h} & \text{if } x\in[(i-1)h,\, ih]\,; \\ 1+i-\frac{x}{h} & \text{if } x\in[ih,\, (i+1)h]\,; \\ 0 & \text{if } x\notin[(i-1)h,\, (i+1)h]\,. \end{cases}$

Next, any function $u \in V$ can be approximated by the values it takes on Ω_D. More precisely, we approximate u by the function $\tilde u$, defined as

$$\tilde u := \sum_{i=1}^{N_x-1} u(ih)\phi_i.$$

We define the evaluation vector $\overline{u} := [u(ih)]^\top := [u(1h), u(2h), \ldots, u((N_x-1)h)]^\top \in \mathcal{M}_{(N_x-1)\times 1}$, where $A^\top$ stands for the transpose matrix of A.

Remark 5.1. Notice that $\widetilde{u} := \sum_{i=1}^{N_x-1} \overline{u}_i\phi_i$ is a piecewise (affine) linear function that takes the same values as u at the points of the mesh Ω_D. Also notice that, since we are dealing with homogeneous Dirichlet boundary conditions, only the values at interior points are unknown for the solution of our system

The next step is the weak discretization matrix L_D of a given linear operator $L \in \mathcal{L}(V \to V')$. We define L_D by the formula

$$\overline{v}^\top L_D \overline{u} = \langle L\tilde u, \tilde v\rangle_{V',V} \text{ for all } u, v \in V. \tag{5.2}$$

Of key importance are the identity and Laplace operators. For the identity operator $Iu = u$, we find that $I_D = [(\phi_i, \phi_j)_H] =: \mathbf{M}$ is the so-called mass matrix, while for the Laplace operator we find that $(\partial_{xx})_D = -[(\partial_x\phi_i,\, \partial_x\phi_j)_H] =: -\mathbf{S}$, where $\mathbf{S}$ is the so-called stiffness matrix. Explicitly, we have the tridiagonal matrices

$$\mathbf{M} := \frac{h}{6}\begin{bmatrix} 4 & 1 & 0 & 0 & \dots & 0\\ 1 & 4 & 1 & 0 & \dots & 0\\ 0 & 1 & 4 & 1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & 0\\ 0 & \dots & 0 & 1 & 4 & 1\\ 0 & \dots & 0 & 0 & 1 & 4 \end{bmatrix} \quad\text{and}\quad \mathbf{S} := \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0\\ -1 & 2 & -1 & 0 & \dots & 0\\ 0 & -1 & 2 & -1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & 0\\ 0 & \dots & 0 & -1 & 2 & -1\\ 0 & \dots & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Next, we recall the reference solution $\hat{u}$ and discretize the operator $v \mapsto \mathcal{B}(\hat{u})v = \partial_x(\hat{u}v)$, $v \in V$. We start by noticing that, for an arbitrary $w \in V$, $(\partial_x(\hat{u}v),\, w)_H = -(\hat{u}v,\, \partial_x w)_H$, and then we consider the approximation $\widetilde{\hat{u}v} = \sum_{j=1}^{N_x-1} \overline{\hat{u}}_j \overline{v}_j \phi_j$ of $\hat{u}v$, and we find that $-(\widetilde{\hat{u}v},\, \partial_x \tilde{w})_H = \sum_{i=1}^{N_x-1}\sum_{j=1}^{N_x-1} -\overline{\hat{u}}_j\overline{v}_j\overline{w}_i(\phi_j,\, \partial_x\phi_i)_H$ and

$$(\partial_x(\hat{u}v),\, w)_H \approx \overline{w}^\top \mathbf{B}\mathcal{D}_{\overline{\hat{u}}}\overline{v}$$

with

$$\mathbf{B} := \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0\\ -1 & 0 & 1 & 0 & \dots & 0\\ 0 & -1 & 0 & 1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & 0\\ 0 & \dots & 0 & -1 & 0 & 1\\ 0 & \dots & 0 & 0 & -1 & 0 \end{bmatrix} \quad\text{and}\quad \mathcal{D}_{\overline{\hat{u}}} := \begin{bmatrix} \overline{\hat{u}}_1 & 0 & 0 & \dots & 0\\ 0 & \overline{\hat{u}}_2 & 0 & \dots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \dots & 0 & \overline{\hat{u}}_{N_x-2} & 0\\ 0 & \dots & 0 & 0 & \overline{\hat{u}}_{N_x-1} \end{bmatrix}.$$

Notice also that, rewriting $v\partial_x v$ as $\frac{1}{2}\mathcal{B}(v)v$, we can discretize $v\partial_x v$ as $\frac{1}{2}\mathbf{B}\mathcal{D}_{\overline{v}}\overline{v}$.

Remark 5.2. Notice that above we consider the operator $v \mapsto \partial_x(\hat{u}v)$ as a composition $\partial_x \circ m_{\hat{u}}$, where $m_{\hat{u}}$ denotes the pointwise multiplication by $\hat{u}$, and then we just take the product of the discretized factors. Of course, we can also discretize directly and, after some computations, we can find that $\mathcal{B}(\hat{u})_D$ is a tridiagonal matrix $\mathbf{B}^{\overline{\hat{u}}}$:

$$\mathbf{B}^{\overline{\hat{u}}}_{ii} = -\sum_{\substack{k\in\{i-1,\, i+1\}\\ 1\le k\le N_x-1}} \overline{\hat{u}}_k \int_\Omega (\phi_k\phi_i\partial_x\phi_i)dx = \frac{1}{6}\begin{cases} \overline{\hat{u}}_2 & \text{if } i = 1,\\ \overline{\hat{u}}_{i+1} - \overline{\hat{u}}_{i-1} & \text{if } i \in \{2,\, \dots,\, N_x-2\},\\ -\overline{\hat{u}}_{N_x-2} & \text{if } i = N_x - 1 \end{cases}$$

$$\mathbf{B}^{\overline{\hat{u}}}_{ij} = -\sum_{\substack{k\in\{i,\, j\}\\ 1\le k\le N_x-1}} \overline{\hat{u}}_k \int_\Omega (\phi_k\phi_j\partial_x\phi_i)dx = \frac{1}{6}\begin{cases} 2\overline{\hat{u}}_{i+1} - \overline{\hat{u}}_i & \text{if } j = i+1,\\ -2\overline{\hat{u}}_{i-1} + \overline{\hat{u}}_i & \text{if } j = i-1,\\ 0 & \text{if } |i-j|_{\mathbb{R}} \ge 2. \end{cases}$$

We see that the composition-based procedure leads to a simpler result. We have also performed some simulations with the direct discretization (for the nonlinear system) and, though we have noticed no substantial difference, we must say that the direct discretization could lead to better results under suitable data.

To discretize the operators in the feedback control rule in (3.22), we start by rewriting it, recalling (3.24), as

$$\mathcal{F}v := B_M^{\mathcal{O}} {B_M^{\mathcal{O}}}^* Q_{\hat{u}}^{t,\,\lambda} v, \tag{5.3}$$

and we notice that what we essentially need is an approximation $\overline{\mathcal{F}v}$ of $\mathcal{F}v$ when we only know the approximation $\overline{v}$ of v.

We will construct $\overline{\mathcal{F}v}$ in a few steps. For the multiplication operator $v \mapsto \chi v$, we can of course take $\mathcal{D}_{\overline{\chi}}\overline{v} = \overline{\chi v}$ as an approximation of χv. For the orthogonal projection $P_M^{\mathcal{O}}$, we start by noticing that

$$P_M^{\mathcal{O}}(v|_{\mathcal{O}}) = \sum_{n=1}^{M} (v|_{\mathcal{O}}\,,\, \underline{s}_n)_{L^2(\mathcal{O},\mathbb{R})}\underline{s}_n = \sum_{n=1}^{M} (v,\, \mathbb{E}_0^{\mathcal{O}}\underline{s}_n)_H \underline{s}_n,$$

and then we can take the approximation $P_M^{\mathcal{O}}(v|_{\mathcal{O}}) \approx \sum_{n=1}^{M} (\overline{\mathbb{E}_0^{\mathcal{O}}\underline{s}_n}^\top \mathbf{M}\overline{v})\underline{s}_n$, from which we set the discrete approximation

$$\mathbf{P}_M \overline{v} \approx \mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}(v|_{\mathcal{O}}) \text{ with } \mathbf{P}_M \coloneqq \mathcal{S}_M \mathbf{M}, \text{ and } \mathcal{S}_M \coloneqq \sum_{n=1}^{M} \overline{\mathbb{E}_0^{\mathcal{O}}\underline{s}_n}\,\overline{\mathbb{E}_0^{\mathcal{O}}\underline{s}_n}^\top.$$

Finally, the linear operator $Q_{\hat{u}}^{t,\,\lambda}$ is, at this moment, unknown and (an approximation) has to be found. Note that denoting by $Q_D = (Q_{\hat{u}}^{t,\,\lambda})_D$ the discretization of $Q_{\hat{u}}^{t,\,\lambda}$, we may take $\mathbf{M}^{-1}Q_D\overline{v} \approx Q_{\hat{u}}^{t,\,\lambda}v$ and discretize the feedback rule (5.3) as follows: first we take the approximation $B_M^{\mathcal{O}\,*}\xi \approx \mathrm{e}^{-\frac{\lambda}{2}t}\mathbf{P}_M\mathcal{D}_{\overline{\chi}}\overline{\xi}$, and then from (5.2) and $(\mathcal{F}v,\, w)_H = (B_M^{\mathcal{O}\,*}Q_{\hat{u}}^{t,\,\lambda}v,\, B_M^{\mathcal{O}\,*}w)_H$, for $(v,\, w) \in H \times H$ we find

$$\begin{aligned}
(\mathcal{F}v,\, w)_H &\approx \left(\widetilde{B_M^{\mathcal{O}\,*}Q_{\hat{u}}^{t,\,\lambda}v},\, \widetilde{B_M^{\mathcal{O}\,*}w}\right)_H \\
&= \overline{B_M^{\mathcal{O}\,*}w}^\top \mathbf{M}\overline{B_M^{\mathcal{O}\,*}Q_{\hat{u}}^{t,\,\lambda}v} \\
&\approx \mathrm{e}^{-\lambda t}\left(\mathbf{P}_M\mathcal{D}_{\overline{\chi}}\overline{w}\right)^\top \mathbf{M}\left(\mathbf{P}_M\mathcal{D}_{\overline{\chi}}\mathbf{M}^{-1}Q_D\overline{v}\right) \\
&= \overline{w}^\perp \mathbf{M}\mathbf{M}^{-1}\mathrm{e}^{-\lambda t}\left(\mathbf{P}_M\mathcal{D}_{\overline{\chi}}\right)^\top \mathbf{M}\left(\mathbf{P}_M\mathcal{D}_{\overline{\chi}}\mathbf{M}^{-1}Q_D\overline{v}\right) \\
&= \overline{w}^\perp \mathbf{M}\mathrm{e}^{-\lambda t}\mathbf{R}\mathbf{R}^\top Q_D\overline{v},
\end{aligned}$$

where

$$\mathbf{R} \coloneqq \left(\mathbf{M}_{\mathrm{c}}\mathbf{P}_M\mathcal{D}_{\overline{\chi}}\mathbf{M}^{-1}\right)^\top \tag{5.4}$$

and $\mathbf{M}_{\mathrm{c}}$, satisfying $\mathbf{M}_{\mathrm{c}}^\top\mathbf{M}_{\mathrm{c}} = \mathbf{M}$, is the Cholesky factor of $\mathbf{M}$. (Notice that $\mathbf{M}$ is positive definite.) Thus, we take $\overline{\mathcal{F}v} = \overline{\mathcal{F}}\overline{v}$ with

$$\overline{\mathcal{F}} \coloneqq \mathrm{e}^{-\lambda t}\mathbf{R}\mathbf{R}^\top Q_D. \tag{5.5}$$

Remark 5.3. Denoting $Q = Q_{\hat{u}}^{t,\,\lambda}$, the approximation $\mathbf{M}^{-1}Q_D\overline{v} \approx Qv$ can be understood in the following formal sense: we have that $\overline{w}^\perp Q_D\overline{v} \approx (Qv, w)_H \approx \overline{w}^\perp\mathbf{M}\overline{Qv}$, and thus we can write $\overline{w}^\perp\mathbf{M}\mathbf{M}^{-1}Q_D\overline{v} \approx \overline{w}^\perp\mathbf{M}\overline{Qv}$, that is, $(\mathbf{M}^{-1}Q_D\overline{v}, \widetilde{w})_H \approx (Qv, w)_H$, where in the last expression the vector $\rho = (\rho_1,\, \rho_2,\, \dots,\, \rho_{N_x-1}) = \mathbf{M}^{-1}Q_D\overline{v}$ is to be seen as an element in $\mathcal{Y} = \mathrm{span}\{\phi_i \mid i \in \{1,\, 2,\, \dots,\, N_x - 1\}\} \subset L^2(\Omega, \mathbb{R})$, that is, ρ is to be understood as $\sum_{i=1}^{N_x-1}\rho_i\phi_i$. Further, notice that in this way the operator $\mathbf{M}^{-1}Q_D$ is symmetric in $\mathcal{Y}$, because if $(v,\, w) \in \mathcal{Y} \times \mathcal{Y}$, we have $(v,\, w) = (\widetilde{v},\, \widetilde{w})$ and $(\mathbf{M}^{-1}Q_D\overline{v},\, w)_H = \overline{w}^\perp\mathbf{M}\mathbf{M}^{-1}Q_D\overline{v} = \overline{w}^\perp Q_D\overline{v} = \overline{v}^\perp Q_D\overline{w} = \overline{v}^\perp\mathbf{M}\mathbf{M}^{-1}Q_D\overline{w} = (\mathbf{M}^{-1}Q_D\overline{w},\, v)_H$.

5.2. Discretization in time. For discretization in time of system (3.1), considered in a time interval $[0,\, T]$, where T is a positive real number, we introduce a uniform mesh

$$[0,\, T]_D \coloneqq \left(0,\, \tfrac{T}{N_t},\, \tfrac{2T}{N_t},\, \dots,\, \tfrac{(N_t-1)T}{N_t},\, T\right) \tag{5.6}$$

consisting of the points in $[0,\,T]$ that are proportional to the time step $k \coloneqq \frac{T}{N_t}$ with $N_t \in \mathbb{N}_0$. Then, any function $u \in H^1((0,\,T),\,V)$ is approximated by the values it takes in $[0,\,T]_D \times \Omega_D$, that is, we essentially approximate $u = u(t,\,x)$ by a matrix $[u] \in \mathcal{M}_{(N_x-1)\times(N_t+1)}$ whose jth column is the vector $\overline{u(jk,\cdot)}$. That is, $[u]_{ij} = u(jk,\,ih)$, for $i \in \{1,\,2,\,\ldots,\,N_x-1\}$ and $j \in \{0,1,\,2,\,\ldots,\,N_t\}$

5.3. Computation of the discretized feedback rule. We recall the operator $Q = Q^{s,\,\lambda}_{\hat{u}}$ satisfying, for $t > 0$, the differential Riccati equation (3.23).

5.3.1. Discretization of the differential Riccati equation. To construct the approximation Q_D for the operator Q, we can look for Q_D solving

$$\partial_t Q_D - Q_D X - X^\top Q_D - \mathrm{e}^{-\lambda t} Q_D \mathbf{R}\mathbf{R}^\top Q_D + \mathrm{e}^{\lambda t}\nu\mathbf{S} = 0, \quad t > 0,$$

with $\mathbf{R}$ as in (5.4) and

$$X = X(t) = \mathbf{M}^{-1}\left(\nu\mathbf{S} + \mathbf{B}\mathcal{D}_{\overline{\hat{u}(t)}}\right). \tag{5.7}$$

Equivalently, we can look for $P = \mathrm{e}^{-\lambda t} Q_D$ solving

$$\partial_t P - PX - X^\top P - P\mathbf{R}\mathbf{R}^\top P + \nu\mathbf{S} + \lambda P = 0, \quad t > 0. \tag{5.8}$$

Remark 5.4. Notice that from the relation $X\overline{v} \approx -\nu\partial_{xx}v + \mathcal{B}(\hat{u})v$, we have that $(Q(-\nu\partial_{xx} + \mathcal{B}(\hat{u}))v, w)_H \approx \overline{w}^\top Q_D X\overline{v}$. Similarly, $((-\nu\partial_{xx}v + \mathcal{B}(\hat{u}))^* Qv, w)_H \approx \overline{w}^\top X^\top Q_D\overline{v}$ and $(Q\mathcal{F}v, w)_H \approx \overline{w}^\top Q_D\overline{\mathcal{F}}\overline{v}$.

5.3.2. Initialization of the differential Riccati equation. Since we need to solve (3.23) backward in time (cf. Remark 3.14), we will also solve system (5.8) backward in time; thus, the question is how to initialize the system. Roughly speaking, it seems that we would need to know $P(+\infty)$, and even if we know this (limit) value, it is not clear how we could use it.

Recall that our main goal is to approach the desired solution $\hat{u}(t)$ as time t increases, but in a real application we also want to have an effective controller that, for example, guarantees us that after some time $t = \widehat{T} > 0$ we are indeed closer than we were at initial time $t = 0$, say, e.g., $|v(\widehat{T})|^2_H \le \frac{1}{2}|v(0)|^2_H$. Also, in applications it is reasonable to think of a problem set for a possibly very long time range $t \in [0,\,T]$ but never for an infinite time range.

Thus, we suppose that we are interested in the evolution for time $t \in [0,\,T]$, and then we may suppose that for time $t > T$, our solution is stationary, that is, we may study the same problem but now we suppose that $\hat{u}(t) = \hat{u}(T)$ for all $t \ge T$. Notice, however, that this does not reduce the full problem to the stationary case, because in the interesting time range $t \in (0,\,T)$ the reference trajectory $\hat{u}(t)$ remains unchanged.

Now, we can find P_T solving the algebraic Riccati equation

$$P(-X(T) + \tfrac{\lambda}{2}I) + (-X(T) + \tfrac{\lambda}{2}I)^\top P - P\mathbf{R}\mathbf{R}^\top P + \nu\mathbf{S} = 0, \tag{5.9}$$

and we can see that P_T will solve the autonomous system (5.8) for $t \ge T$ (under the supposition $\hat{u}(t) = \hat{u}(T)$ for $t \ge T$); see also [47, sections 1.4 and 4.4].

Then, it remains to solve (5.8) for $t \in [0,\,T]$ with the final condition $P(T) = P_T$.

5.3.3. Solving the Riccati systems.

- *General procedure.* To solve the algebraic Riccati system (5.9), we use the software available from [9]; in this way, we find P_T.
  To solve (backward in time) the differential system (5.8) for $t \in [0, T]$ with the initial condition $P(T) = P_T$, we proceed as follows. Recall the mesh $[0, T]_D$ of the interval $[0, T]$, defined in (5.6). We have $P^{N_t} := P(N_t k) = P(T) = P_T$; next, recursively, we construct $P^j := P(jk)$ from P^{j+1} for $j \in \{0, 1, \dots, N_t - 1\}$ as follows: we start by rewriting (5.8) as

$$-\partial_t P = P(-X + \tfrac{\lambda}{2} I) + (-X + \tfrac{\lambda}{2} I)^\top P - P\mathbf{R}\mathbf{R}^\top P + \nu \mathbf{S} =: R_F(P)$$

  and we use the Crank–Nicolson inspired scheme

$$-\tfrac{2}{k}(P^{j+1} - P^j) = R_F(P^j) + R_F(P^{j+1}),$$

  from which we obtain $R_F(P^j) - \frac{2}{k} P^j + R_F(P^{j+1}) + \frac{2}{k} P^{j+1} = 0$, that is,

$$P^j(-X + \tfrac{\lambda}{2} I - \tfrac{1}{k} I) + (-X + \tfrac{\lambda}{2} I - \tfrac{1}{k} I)^\top P^j - P^j \mathbf{R}\mathbf{R}^\top P^j + Z^{j+1} = 0 \tag{5.10}$$

  with $Z^{j+1} = R_F(P^{j+1}) + \frac{2}{k} P^{j+1} + \nu \mathbf{S}$. Hence, P^j solves again an algebraic Riccati equation and we can still use the software in [9].
- *Initial guess.* The software in [9] (see also [10]) uses a Newton method to solve an algebraic Riccati equation like (5.9). We have to provide an initial starting guess Y_0 such that $-X(T) + \frac{\lambda}{2} I - \mathbf{R}\mathbf{R}^\top Y_0^T Y_0$ is stable. This is of course a nontrivial task (see, e.g., the discussion after (1.4) in [27]), and we look for the initial guess in three steps:
  (a) We set $M = +\infty$ and $\chi = 1_\Omega$. That is, we impose no constraints either on the dimension or on the support of the controller. In this case, we can see that $\mathbf{R} = \left(\mathbf{M}_c \mathbf{M}^{-1}\right)^\top$ and $\mathbf{R}\mathbf{R}^\top = \mathbf{M}^{-1}$. Then, from (5.7) we can expect that

$$-X(T) + \tfrac{\lambda}{2} I - \mathbf{M}^{-1} Y_0^T Y_0 = -\mathbf{M}^{-1}\left(\nu \mathbf{S} + \mathbf{B}\mathcal{D}_{\overline{\hat{u}(T)}}\right) + \tfrac{\lambda}{2} I - \mathbf{M}^{-1} Y_0^T Y_0$$

  will be stable for $Y_0 = \beta \mathbf{M}_c$ with $\sqrt{2}\beta \ge \beta_0 := (\nu^{-1} \left|\hat{u}(T)\right|^2_{L^\infty(\Omega,\,\mathbb{R})} + \lambda)^{\frac{1}{2}}$. Notice that, proceeding as in the beginning of section 3.1, we see that a weak solution w for $w_t = \nu \partial_{xx} w - \partial_x(\hat{u}(T) w) + \frac{\lambda}{2} w - \beta^2 w$ will satisfy the estimate $\frac{\mathrm{d}}{\mathrm{d}t} |w|_H^2 \le -\nu |\partial_x w|_H^2 + \nu^{-1} |\hat{u}(T)|^2_{L^\infty(\Omega,\,\mathbb{R})} |w|_H^2 + \lambda |w|_H^2 - 2\beta^2 |w|_H^2$. That is, the lower bound β_0 works for the continuous system. However, when taking β strictly bigger than β_0, we may expect that the stability is preserved for the discretized system if N_x and N_t are big enough.
  Hence, we set $\beta = (\nu^{-1} \left|\hat{u}(T)\right|^2_{L^\infty(\Omega,\,\mathbb{R})} + \lambda)^{\frac{1}{2}}$ and solve (5.9), i.e.,

$$P(-X(T) + \tfrac{\lambda}{2} I) + (-X(T) + \tfrac{\lambda}{2} I)^\top P - P\mathbf{M}^{-1} P + \nu \mathbf{S} = 0,$$

  providing the initial guess $Y_0 = \beta \mathbf{M}_c$. Let us denote the solution by $P_T^{[1]}$.
  (b) We set $M = +\infty$ and the true χ. That is, now we include the constraints on the support of the controller. In this case, $\mathbf{R} = (\mathbf{M}_c \mathcal{D}_{\overline{\chi}} \mathbf{M}^{-1})^\top$; see (5.4). In some cases, it may happen that $P_T^{[1]}$ is not a "good" initial guess. In some cases (as we have observed in some simulations), the step

from $(+\infty, 1_\Omega)$ to $(+\infty, \chi)$ seems to be too big, in other words, $P_T^{[1]}$ is too far from the solution corresponding to $\mathbf{R} = (\mathbf{M}_c \mathcal{D}_{\overline{\chi}} \mathbf{M}^{-1})^\top$. Having this in mind, we connect the operators I and $\mathcal{D}_{\overline{\chi}}$ by the homotopy $\mathcal{H}_\tau = (1-\tau)I + \tau\mathcal{D}_{\overline{\chi}}$, $\tau \in [0,1]$ and set $\mathbf{H}_\tau \coloneqq (\mathbf{M}_c \mathcal{H}_\tau \mathbf{M}^{-1})^\top$. Now let us fix $N_\mathcal{H} \in \mathbb{N}_0$ and set the homotopy step $\rho = \frac{1}{N_\mathcal{H}}$ and solve

$$P(-X(T) + \tfrac{\lambda}{2}I) + (-X(T) + \tfrac{\lambda}{2}I)^\top P - P\mathbf{H}_\rho \mathbf{H}_\rho^\top P + \nu\mathbf{S} = 0,$$

providing the initial guess $Y_0 = P_T^{[1]}$. Let us denote the solution by $P_T^{[1+\rho]}$. Recursively, we solve, for $l \in 2, \ldots, N_\mathcal{H}$,

$$P(-X(T) + \tfrac{\lambda}{2}I) + (-X(T) + \tfrac{\lambda}{2}I)^\top P - P\mathbf{H}_{l\rho} \mathbf{H}_{l\rho}^\top P + \nu\mathbf{S} = 0,$$

providing the initial guess $Y^0 = P_T^{[1+(l-1)\rho]}$, and denote the solution by $P_T^{[1+l\rho]}$. After $N_\mathcal{H}$ steps, we have found a solution $P_T^{[2]}$ for

$$P(-X(T) + \tfrac{\lambda}{2}I) + (-X(T) + \tfrac{\lambda}{2}I)^\top P - P\mathbf{H}_1 \mathbf{H}_1^\top P + \nu\mathbf{S} = 0$$

with $\mathbf{H}_1 = \left(\mathbf{M}_c \mathcal{D}_{\overline{\chi}} \mathbf{M}^{-1}\right)^\top$.

(c) We set the true M and the true χ. That is, finally, we include also the constraints on the dimension of the controller. In this case, $\mathbf{R}$ is given by (5.4). Analogously to step (2), we consider the homotopy $\mathcal{H}_\tau = (1-\tau)\mathcal{D}_{\overline{\chi}} + \tau\mathbf{P}_M \mathcal{D}_{\overline{\chi}}$, set $\mathbf{H}_\tau \coloneqq (\mathbf{M}_c \mathcal{H}_\tau \mathbf{M}^{-1})^\top$, and, starting with $P_T^{[2]}$, we find, recursively after $N_\mathcal{H}$ steps, a solution $P_T^{[3]}$ for

$$P(-X(T) + \tfrac{\lambda}{2}I) + (-X(T) + \tfrac{\lambda}{2}I)^\top P - P\mathbf{H}_1 \mathbf{H}_1^\top P + \nu\mathbf{S} = 0$$

with $\mathbf{H}_1 = (\mathbf{M}_c \mathbf{P}_M \mathcal{D}_{\overline{\chi}} \mathbf{M}^{-1})^\top = \mathbf{R}$. That is, $P_T^{[3]}$ solves (5.9).

Of course, the number of homotopy steps $N_\mathcal{H}$ may be taken different in Steps 2 and 3. To get the convergence of the Newton method used to solve the algebraic Riccati equations at each homotopy step, we may need, depending on the situation, to increase the number of homotopy steps $N_\mathcal{H}$.

Notice, however, that in Step 3 increasing $N_\mathcal{H}$ can be sufficient for convergence at each homotopy step only if M is big enough. Indeed, we can see that the algebraic Riccati equation will have a solution up to the homotopy step before the last, because from the observability inequality (3.14) we can also derive $|q(0)|_H^2 \le (1-\tau)^{-2}C\,|(1-\tau)\chi q|_{L^2((0,T),\,H)}^2$, and from

$$\begin{aligned}
&|(1-\tau)\chi q|_{L^2((0,T),\,H)}^2\\
&\quad= \left|(1-\tau)(1-P_M^\mathcal{O})(\chi q|_\mathcal{O})\right|_{L^2((0,T),\,L^2(\mathcal{O},\mathbb{R}))}^2 + \left|(1-\tau)P_M^\mathcal{O}(\chi q|_\mathcal{O})\right|_{L^2((0,T),\,L^2(\mathcal{O},\mathbb{R}))}^2\\
&\quad\le \left|(1-\tau)(1-P_M^\mathcal{O})(\chi q|_\mathcal{O})\right|_{L^2((0,T),\,L^2(\mathcal{O},\mathbb{R}))}^2 + \left|P_M^\mathcal{O}(\chi q|_\mathcal{O})\right|_{L^2((0,T),\,L^2(\mathcal{O},\mathbb{R}))}^2\\
&\quad= \left|(1-\tau)(\chi q|_\mathcal{O}) + (1-(1-\tau))P_M^\mathcal{O}(\chi q|_\mathcal{O})\right|_{L^2((0,T),\,L^2(\mathcal{O},\mathbb{R}))}^2,
\end{aligned}$$

we arrive at $|q(0)|_H^2 \le (1-\tau)^{-2}C|(1-\tau)\chi q + \tau\mathbb{E}_0^\mathcal{O} P_M^\mathcal{O}(\chi q|_\mathcal{O})|_{L^2((0,T),\,H)}^2$. Then, from this observability inequality it will follow that there exists a stabilizing control (for system (3.1)) of the form $\zeta(t) = \mathbb{F}_\tau\eta \coloneqq (1-\tau)\chi\eta(t) + \tau\chi\mathbb{E}_0^\mathcal{O} P_M^\mathcal{O}(\eta(t)|_\mathcal{O})$ for $\tau < 1$. Reasoning as in section 3.4, by the dynamic programing principle it will follow that the control can be taken in feedback form $\zeta(t) = \mathbb{F}_\tau \mathbb{F}_\tau^* Q_\tau^t v(t)$, where $\mathrm{e}^{\lambda t} Q_\tau^t$ solves the Ricatti equation (3.25) with $\mathbb{F}_\tau$ in the role of $\mathbb{B}$, which corresponds on the discrete level to the case in we take $\mathbf{R} = (\mathbf{M}_\mathrm{c}[(1-\tau)\mathcal{D}_{\overline{\chi}} + \tau \mathbf{P}_M \mathcal{D}_{\overline{\chi}}]\mathbf{M}^{-1})^\top$ in (5.8).

For the last homotopy step, that is, for $\tau = 1$, the observability will hold if M is big enough (following the arguments in [7]). Also, from Theorem 3.4, a stabilizing control exists if M is big enough, so we cannot guarantee the existence of a stabilizing control of the form $\zeta(t) = \chi \mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}(\eta(t)|_{\mathcal{O}})$ for arbitrary (small) M, and then we cannot guarantee the existence of a solution for the algebraic Riccati equation (3.25).

In Step 2, increasing $N_{\mathcal{H}}$ should be sufficient to get the convergence at each homotopy step, because reasoning as above we can conclude that there exists a stabilizing control of the form $\zeta(t) = (1-\tau)\eta(t) + \tau\chi\eta(t)$ for all $\tau \in [0, 1]$.

Therefore, if convergence is not reached at a homotopy step in (2) *or at a homotopy step before the last in Step* 3*, we probably need either more homotopy steps or to refine our mesh; if convergence is not reached only at the last homotopy step in* (3)*, then probably the number of controls is not enough.*

In the simulations we present here, we have taken no more than $N_{\mathcal{H}} = 20$ in the second step and no more than $N_h = 10$ in the third step. Notice, however, that increasing the number of homotopy steps does not mean that the computational time will be much bigger because the Newton method may converge faster at each homotopy step.

Finally, in the process of solving the differential Riccati equation, to find P^j solving (5.10) we provide the natural initial guess P^{j+1}. Again, we cannot guarantee that the solution will always exist. If this process fails at some j-step, we can try to refine the mesh (in particular, by increasing the number N_t of time steps in (5.6)); if that does not work, it probably means that the number of controls M is not sufficient.

5.4. Solving the discretized Oseen–Burgers system. Once we have constructed P, we can simulate the evolution of the system (3.1). We look for $\overline{v}(t) \coloneqq \overline{v(t,\cdot)}$ that solves the system

$$\partial_t \overline{v} + \nu \mathbf{M}^{-1}\mathbf{S}\overline{v} + \mathbf{M}^{-1}\mathbf{B}\mathcal{D}_{\overline{\hat{u}}}\overline{v} + \mathbf{R}^\top \mathbf{R} P(\overline{v}) = 0, \quad \overline{v}(0) = \overline{v_0}, \tag{5.11}$$

and expect $\overline{v}$ to go exponentially to 0 as time increases, with an a priori prescribed rate $\frac{\lambda}{2} > 0$ as time goes to infinity (cf. (3.4) and (3.16)); recall that P depends on λ (cf. (5.8)). Notice that from (5.5), (5.4), and $P = \mathrm{e}^{-\lambda t} Q_D$, it follows that $\overline{\mathcal{F}}\overline{v} = \mathbf{R}^T \mathbf{R} P \overline{v}$.

Again, we will approximate $\overline{v}(t) \approx [\overline{v}(jk)]$, $j \in \{0, 1, \ldots, N_t\}$, and we apply a Crank–Nicolson inspired algorithm to solve system (5.11). For simplicity, we denote $\overline{\mathcal{F}}^j \coloneqq \mathbf{R}^\top \mathbf{R} P^j$ for $j \in \{0, 1, \ldots, N_t\}$. Set $\overline{v}^0 \coloneqq \overline{v(0k)} = \overline{v_0}$; then, the idea is to construct, recursively, $\overline{v}^{j+1} \coloneqq \overline{v((j+1)k)}$ from $\overline{v}^j \coloneqq \overline{v(jk)}$ by the scheme

$$\begin{aligned} \tfrac{2}{k}(\overline{v}^{j+1} - \overline{v}^j) = &-\nu \mathbf{M}^{-1}\mathbf{S}(\overline{v}^j + \overline{v}^{j+1}) - \mathbf{M}^{-1}\left(\mathbf{B}\mathcal{D}_{\overline{\hat{u}}^j}\overline{v}^j + \mathbf{B}\mathcal{D}_{\overline{\hat{u}}^{j+1}}\overline{v}^{j+1}\right) \\ &- \left(\overline{\mathcal{F}}^j \overline{v}^j + \overline{\mathcal{F}}^{j+1}\overline{v}^{j+1}\right) \end{aligned}$$

with $\overline{\hat{u}}^j \coloneqq \overline{\hat{u}(jk)}$, $j \in \{0, 1, \ldots, N_t\}$. Then, working the above scheme a little, we can obtain

$$\begin{aligned} \overline{v}^{j+1} = &A_\oplus^{-1} A_\ominus \overline{v}^j - \tfrac{k}{2} A_\oplus^{-1}\left(\mathbf{B}\mathcal{D}_{\overline{\hat{u}}^j}\overline{v}^j + \mathbf{B}\mathcal{D}_{\overline{\hat{u}}^{j+1}}\overline{v}^{j+1}\right) \\ &- \tfrac{k}{2} A_\oplus^{-1}\mathbf{M}\left(\overline{\mathcal{F}}^j \overline{v}^j + \overline{\mathcal{F}}^{j+1}\overline{v}^{j+1}\right) \end{aligned} \tag{5.12}$$

with $A_\oplus := \mathbf{M} + \frac{k}{2}\nu\mathbf{S}$ and $A_\ominus := \mathbf{M} - \frac{k}{2}\nu\mathbf{S}$. Notice that the unknown $\overline{v}^{j+1}$ is still present on the right-hand side of (5.12). In the argument of the feedback operator, we will replace $\overline{v}^{j+1}$ by a preliminary guess $\overline{v}_G^{j+1}$ and approximate $\mathbf{B}\mathcal{D}_{\overline{\hat{u}}^{j+1}}\overline{v}^{j+1}$ by $\mathbf{B}\mathcal{D}_{\overline{\hat{u}}^{j}}\overline{v}^{j} + k(\mathbf{B}\mathcal{D}_{\overline{\hat{u}}^{j}}\overline{v}^{j} - \mathbf{B}\mathcal{D}_{\overline{\hat{u}}^{j-1}}\overline{v}^{j-1})$ (where we define $\overline{v}^{\,-1} := \overline{v}^0 = \overline{v}_0$). In this way, we arrive at the scheme

$$\begin{aligned}\overline{v}^{j+1} &= A_\oplus^{-1}A_\ominus\overline{v}^j - kA_\oplus^{-1}\left((1+\tfrac{k}{2})\mathbf{B}\mathcal{D}_{\overline{\hat{u}}^{j}}\overline{v}^j - \tfrac{k}{2}\mathbf{B}\mathcal{D}_{\overline{\hat{u}}^{j-1}}\overline{v}^{j-1}\right)\\ &\quad - \tfrac{k}{2}A_\oplus^{-1}\mathbf{M}\left(\overline{\mathcal{F}}^j\overline{v}^j + \overline{\mathcal{F}}^{j+1}\overline{v}_G^{j+1}\right).\end{aligned} \tag{5.13}$$

We set $\overline{v}_G^{j+1}$ as the "uncontrolled" output

$$\overline{v}_G^{j+1} := A_\oplus^{-1}A_\ominus\overline{v}^j - kA_\oplus^{-1}\left((1+\tfrac{k}{2})\mathbf{B}\mathcal{D}_{\overline{\hat{u}}^{j}}\overline{v}^j - \tfrac{k}{2}\mathbf{B}\mathcal{D}_{\overline{\hat{u}}^{j-1}}\overline{v}^{j-1}\right).$$

5.5. Solving the discretized Burgers system. Concerning the evolution of the system (1.1)–(1.2), we look for $\overline{u}(t) := \overline{u(t,\cdot)}$ that solves the system

$$\partial_t\overline{u} + \nu\mathbf{M}^{-1}\mathbf{S}\overline{u} + \tfrac{1}{2}\mathbf{M}^{-1}\mathbf{B}\mathcal{D}_{\overline{u}}\overline{u} + \overline{h} + \mathbf{R}^\top\mathbf{R}P(\overline{u} - \overline{\hat{u}}) = 0, \quad \overline{u}(0) = \overline{u_0}, \tag{5.14}$$

and expect $\overline{u}$ to go exponentially to $\overline{\hat{u}}$ with an a priori prescribed rate $\frac{\lambda}{2} > 0$, as time increases (with $\mathbf{R}$ as in (5.4)). However, this would be meaningful if $\overline{\hat{u}}$ were a solution for the uncontrolled discrete system, which is not true. The solution of the uncontrolled discrete system

$$\partial_t\overline{\hat{u}}_{\mathrm{S}} + \nu\mathbf{M}^{-1}\mathbf{S}\overline{\hat{u}}_{\mathrm{S}} + \tfrac{1}{2}\mathbf{M}^{-1}\mathbf{B}\mathcal{D}_{\overline{\hat{u}}_{\mathrm{S}}}\overline{\hat{u}}_{\mathrm{S}} + \overline{h} = 0, \quad \overline{\hat{u}}_{\mathrm{S}}(0) = \overline{\hat{u}_0}, \tag{5.15}$$

will be an approximation $\overline{\hat{u}}_{\mathrm{S}}$ of $\overline{\hat{u}}$. There is no reason to expect that $\mathrm{e}^{\frac{\lambda}{2}t}(\overline{u}(t) - \overline{\hat{u}}(t))$ will remain bounded for $t \in \mathbb{R}_0$.

Nevertheless, there is a way to check the rate of exponential stabilization λ. We will just have to compute the discrete (fictitious) external force $\overline{h}_{\mathrm{f}}$, that makes $\overline{\hat{u}}$ a solution of the discrete system that is,

$$\partial_t\overline{\hat{u}} + \nu\mathbf{M}^{-1}\mathbf{S}\overline{\hat{u}} + \tfrac{1}{2}\mathbf{M}^{-1}\mathbf{B}\mathcal{D}_{\overline{\hat{u}}}\overline{\hat{u}} + \overline{h}_{\mathrm{f}} = 0, \quad \overline{\hat{u}}(0) = \overline{\hat{u}_0}. \tag{5.16}$$

Before that we present the scheme that we apply. Suppose for the moment that we know $\overline{h}_{\mathrm{f}}$. Then, we follow the idea in section 5.4 and arrive at the scheme

$$\begin{aligned}\overline{u}^{j+1} &= A_\oplus^{-1}A_\ominus\overline{u}^j - \tfrac{k}{2}A_\oplus^{-1}\left((1+\tfrac{k}{2})\mathbf{B}\mathcal{D}_{\overline{u}^{j}}\overline{u}^j - \tfrac{k}{2}\mathbf{B}\mathcal{D}_{\overline{u}^{j-1}}\overline{u}^{j-1}\right)\\ &\quad - \tfrac{k}{2}A_\oplus^{-1}\mathbf{M}\left(\overline{h}_{\mathrm{f}}^j + \overline{h}_{\mathrm{f}}^{j+1} + \overline{\mathcal{F}}^j(\overline{u}^j - \overline{\hat{u}}^j) + \overline{\mathcal{F}}^{j+1}(\overline{u}_G^{j+1} - \overline{\hat{u}}^{j+1})\right)\end{aligned} \tag{5.17}$$

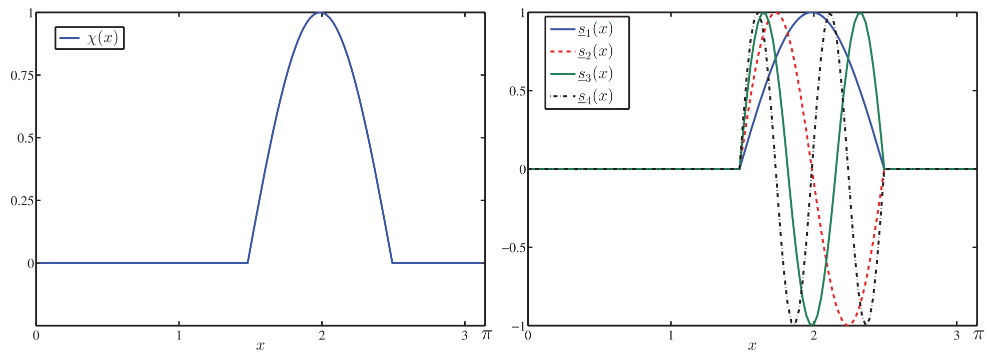
with the preliminary "uncontrolled" guess $\overline{u}_G^{j+1}$ given by

$$\begin{aligned}\overline{u}_G^{j+1} &:= A_\oplus^{-1}A_\ominus\overline{u}^j - \tfrac{k}{2}A_\oplus^{-1}\left((1+\tfrac{k}{2})\mathbf{B}\mathcal{D}_{\overline{u}^{j}}\overline{u}^j - \tfrac{k}{2}\mathbf{B}\mathcal{D}_{\overline{u}^{j-1}}\overline{u}^{j-1}\right)\\ &\quad - \tfrac{k}{2}A_\oplus^{-1}\mathbf{M}\left(\overline{h}_{\mathrm{f}}^j + \overline{h}_{\mathrm{f}}^{j+1}\right).\end{aligned}$$

It remains to explain how we construct the force $\overline{h}_{\mathrm{f}}$. Actually, from our scheme we can deduce that we only need to know the terms $\frac{k}{2}A_\oplus^{-1}\mathbf{M}(\overline{h}_{\mathrm{f}}^j + \overline{h}_{\mathrm{f}}^{j+1})$ for $j \in \{0, 1, \ldots, N_t - 1\}$ that we can easily compute as

$$\begin{aligned}&\tfrac{k}{2}A_\oplus^{-1}\mathbf{M}\left(\overline{h}_{\mathrm{f}}^j + \overline{h}_{\mathrm{f}}^{j+1}\right)\\ &= -\overline{\hat{u}}^{j+1} + A_\oplus^{-1}A_\ominus\overline{\hat{u}}^j - \tfrac{k}{2}A_\oplus^{-1}\left((1+\tfrac{k}{2})\mathbf{B}\mathcal{D}_{\overline{\hat{u}}^{j}}\overline{\hat{u}}^j - \tfrac{k}{2}\mathbf{B}\mathcal{D}_{\overline{\hat{u}}^{j-1}}\overline{\hat{u}}^{j-1}\right)\end{aligned}$$

(where we define $\overline{\hat{u}}^{\,-1} := \overline{\hat{u}}^0$).



(a) The function χ. (b) The first four sinus.

FIG. 1. *Basis for the control space* $\{\chi\mathbb{E}_0^{\mathcal{O}}\eta \mid \eta \in \operatorname{span}\{\underline{s}_i \mid i \in \{1, 2, 3, 4\}\}\}$.

6. Numerical examples: The linear Oseen–Burgers system. We present some results of the numerical simulations we have performed concerning the stabilization of system (3.1) to zero. Below, v_{u} stands for the solution of the uncontrolled (discretized) system (i.e., $\zeta = 0$), and v (or v_λ) stands for the solution of the (discretized) system under the action of a (discretized) feedback controller $\zeta = \chi\eta$ with $\eta = \mathrm{e}^{-\lambda t}\mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}((\chi Q_{\hat{u}}^{t,\,\lambda} v)|_{\mathcal{O}})$ as in (3.22). If nothing is said to the contrary, $\mathcal{O} = (\inf\{\Omega \cap \operatorname{supp}(\chi)\},\, \sup\{\Omega \cap \operatorname{supp}(\chi)\})$.

We follow a "trying and checking" procedure; we fix M and check the results of the simulations.

6.1. Testing with a family of reference trajectories. We set $\nu = \frac{1}{10}$, $\lambda = 2$, $\Omega = (0,\, \pi)$, $\mathcal{O} = (\frac{3}{2},\, \frac{5}{2})$, and

$$\chi(x) = \mathbb{E}_0^{\mathcal{O}}\left(\sin((x - \tfrac{3}{2})\pi)|_{\mathcal{O}}\right). \tag{6.1}$$

That is, $\chi = \mathbb{E}_0^{\mathcal{O}}\underline{s}_1$ (cf. section 2.1). Next, we set the family of reference trajectories

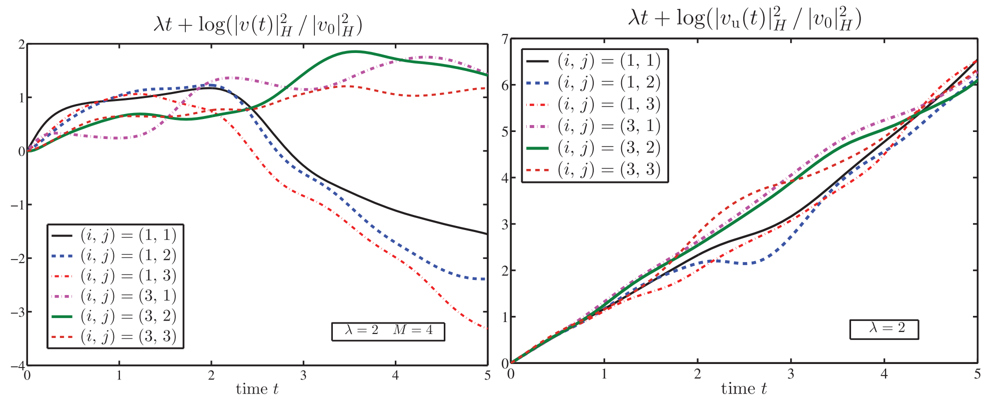
$$\hat{u} = \hat{u}^{(i,j)} = C_{\mathrm{nr}}\left(\sin(-t)\sin(ix) - \cos(3t)\sin(jx)\right), \tag{6.2}$$

where the constant C_{nr} is chosen so that $|\hat{u}|_{\mathcal{W}} = 1$. In this case, we have that $M_{\mathrm{ref}} = \sqrt{120} \approx 10.95$ and $M_{\mathrm{exp}} \approx 57208.12$, so our question is whether the number M of needed controls stays "close" to M_{ref} or to M_{exp} (cf. section 3.3). We will test with the smaller number $M = 4$, and $v_0(x) = \sin(2x)$. The function χ and the four controls are plotted in Figure 1.

In Figure 2 we can check that the feedback control is able to stabilize the system with the desired rate. Then, we change the initial condition to $v_0(x) = \sin(x) - \sin(6x)$ and test for some other reference trajectories (with higher frequencies) in the family (6.2); in Figure 3 we see that the feedback control is still able to stabilize the system with the desired rate; of course, the squared norm $|v|_H^2$ is to be understood as the discrete approximation $\overline{v}^\top\mathbf{M}\overline{v}$ (cf. section 5.1).

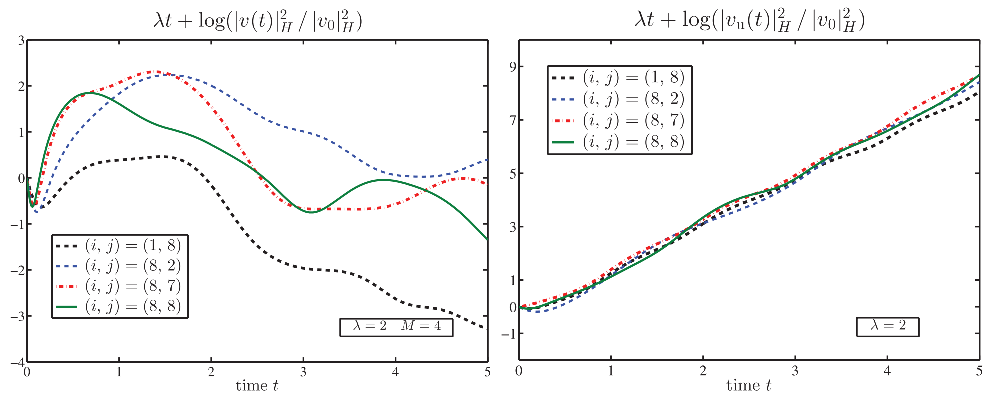
Remark 6.1. There is no particular reason to test with M far below M_{ref}; trivially, if M controls are enough to stabilize the system, then taking more controls we can also stabilize the system.

Initial data in $L^2(\Omega,\, \mathbb{R}) \setminus H^1(\Omega,\, \mathbb{R})$. We set $\lambda = 4$, $\nu = \frac{1}{10}$, and χ as in (6.1). But now we set $v_0(x) = x^{\frac{1}{2}}$ and the reference trajectory $\hat{u} = C_{\mathrm{nr}}1_{[0,\,\frac{\pi}{2}]}(\sin(-t)\sin(2x)-$



(a) With feedback control. (b) Without control.

FIG. 2. *Convergence rate is achieved with the feedback control.*



(a) With feedback control. (b) Without control.

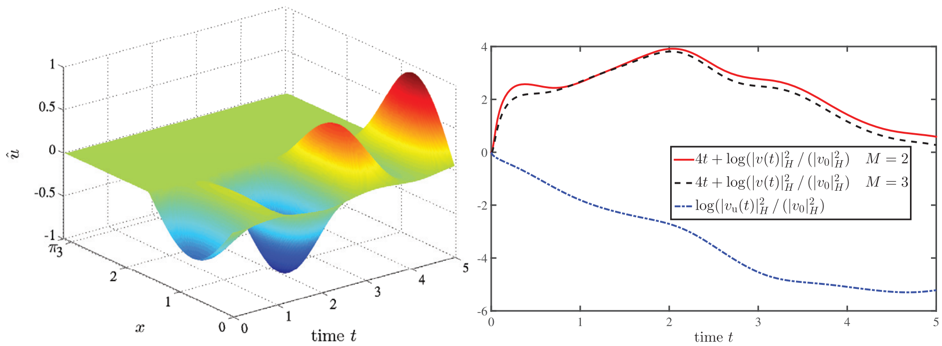
FIG. 3. *Convergence rate is achieved with the feedback control.*

$\cos(3t)\sin(2x))$, where

$$1_{[a,\,b]}(x) := \begin{cases} 1 & \text{if } x \in [a,\, b], \\ 0 & \text{if } x \in \Omega \setminus [a,\, b], \end{cases} \qquad a,\, b \in \mathbb{R}, \tag{6.3}$$

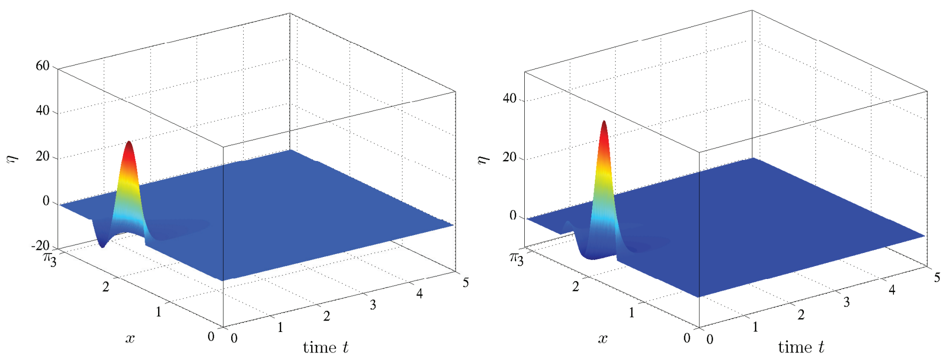
and C_{nr} is taken so that $|\hat{u}|_{\mathcal{W}} = 1$. We can see in Figure 4 that two controls stabilize the system (3.1) to zero with the desired rate. In Figure 5 we see the controls corresponding to the cases we take either two or three controls. Notice that in this case, the initial condition is in $H \setminus V$ and the support of the control is disjoint from that of $\hat{u}$.

6.2. Increasing number of needed controls. We set $\hat{u} = 0$, $\Omega = (0, \pi)$. In this example, we show that for any given $n \in \mathbb{N}_0$, we can construct χ supported in a subset $\omega \subset \overline{\omega} \subset \Omega$, $\lambda > 0$, and an initial condition v_0, such that the first $2n$ controls cannot stabilize the system (3.1) to zero with the rate λ. However, by increasing the number of controls, we can obtain the desired stabilization. Notice that here we look for $\chi \neq 1_\Omega$ (cf. the last paragraph in section 3.3).



(a) The trajectory $\hat{u}$. (b) Convergence rates of v_u and v.

FIG. 4. *Two controls stabilize the system to zero with desired rate.*



(a) The control η for $M = 2$. (b) The control η for $M = 3$.

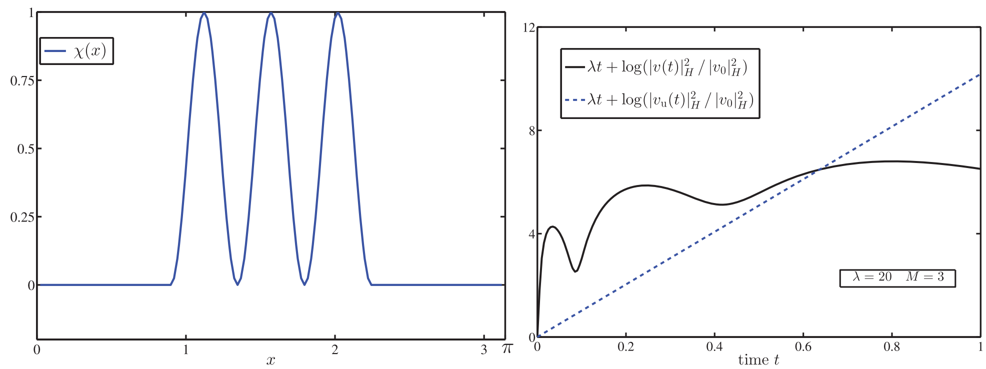
FIG. 5. *The controls.*

Let $n \in \mathbb{N}_0$. Set $v_0(x) = \sin((2n+5)x)$, $\mathcal{O} = (\frac{2\pi}{2n+5}, \frac{(2n+3)\pi}{2n+5})$, and $\chi = \mathbb{E}_0^{\mathcal{O}} v_0^2|_{\mathcal{O}}$. We claim that the controls $\chi P_{2n}^{\mathcal{O}}\eta$ cannot stabilize the equation with rate $\lambda > 2\nu(2n+5)^2$. Indeed, we can write $(v_0, \chi P_{2n}^{\mathcal{O}}\eta)_H = \sum_{i=1}^{2n} \eta_i \int_{\mathcal{O}} v_0^3 \underline{s}_i \, \mathrm{d}\mathcal{O} = \sum_{i=1}^{2n} \eta_i (\frac{l}{2})^{\frac{3}{2}} \int_{\mathcal{O}} \underline{s}_{2n+1}^3 \underline{s}_i \, \mathrm{d}\mathcal{O}$ and also $\int_{\mathcal{O}} \underline{s}_{2n+1}^3 \underline{s}_i \, \mathrm{d}\mathcal{O} = \frac{1}{4} \int_{\mathcal{O}} (1 - \underline{c}_{2(2n+1)})(\underline{c}_{2n+1-i} - \underline{c}_{2n+1+i}) \, \mathrm{d}\mathcal{O}$ with

$$\underline{c}_j(x) \coloneqq \left(\tfrac{2}{l}\right)^{\frac{1}{2}} \cos\left(\tfrac{j\pi(x-l_1)}{l}\right), \quad x \in \mathcal{O}, \quad l \coloneqq \operatorname{length}(\mathcal{O}) = \tfrac{(2n+1)\pi}{2n+5}, \quad l_1 = \tfrac{2\pi}{2n+5}$$

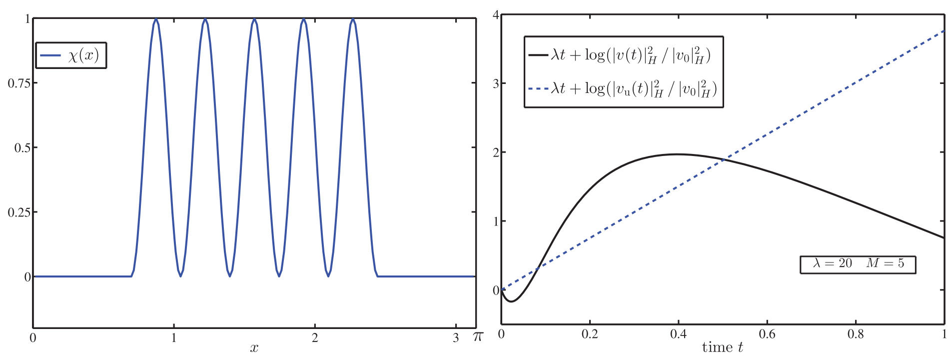
(cf. definition of the functions $\underline{s}_n$ in section 2.1). Now, notice that since $i \le 2n$, we have that $0 < 2n+1 \pm i < 2(2n+1)$, and thus we can conclude that $(v_0, \chi P_{2n}^{\mathcal{O}}\eta)_H = 0$. Therefore, since the eigenspace $\operatorname{span}\{v_0\}$ is preserved by the Laplacian, we can conclude that the control cannot change the dynamics on this space. Thus, we conclude that the rate of convergence is at most $2\nu(2n+5)^2$.

Now we set $\nu = \frac{1}{10}$, from above, we know that for $n \in \{1, 2\}$, the rate of convergence $\lambda = 20 > \frac{81}{5}$ is not achieved with the first $2n$ controls. Simulations below show that, in these examples, it is enough to add one more control to achieve the rate. In particular, we have $M = 2n+1 \le 5 < (\frac{\lambda}{\nu})^{\frac{1}{2}} = 10\sqrt{2} = M_{\mathrm{ref}} < M_{\exp} = \mathrm{e}^{10\sqrt{2}}$. In Figures 6 and 7 we see the results of the simulations for the cases $n = 1$ and $n = 2$; we can check the stabilization rate to zero of the heat system (i.e., system (3.1) with $\hat{u} = 0$).



(a) The function χ. (b) Convergence rates of v_u and v.

FIG. 6. *Case $n = 1$. The first three controls can stabilize the heat system.*



(a) The function χ. (b) Convergence rates of v_u and v.

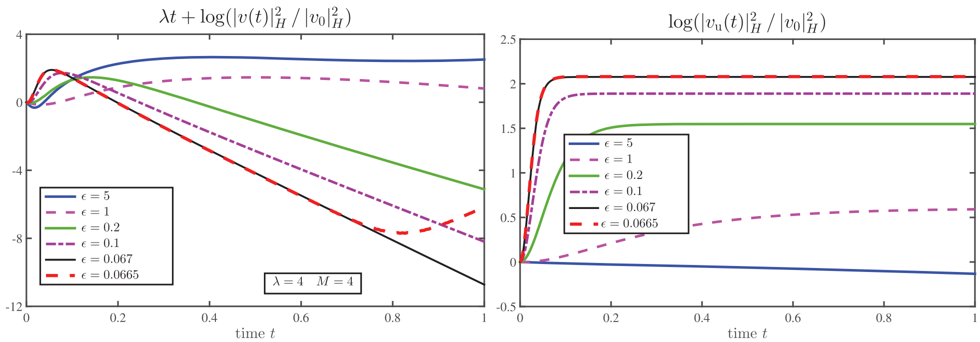
FIG. 7. *Case $n = 2$. The first five controls can stabilize the heat system.*

6.3. Instability of the system. Increasing $|\hat{u}|_{\mathcal{W}}$ and decreasing ν brings more instability to the system, which leads to the necessity to take a bigger number M of controls. To illustrate the instability of the (uncontrolled) system (3.1) and the response of the controller, we can just take a stationary reference trajectory. The main advantage is that we do not need to solve the differential Riccati equation that is the more expensive numerical step. Notice, however, that (as far as we know) an estimate depending on the norm $|\hat{u}|_{\mathcal{W}}$ is not known also in this case; for the Oseen–Stokes system estimates are known but depend on $\hat{u}$ (cf. the discussion and given references in section 1). We will set

$$\hat{u}(t, x) = \epsilon^{-1} \sin(5x) \quad \text{and} \quad v_0(x) = \sin(\pi x), \tag{6.4}$$

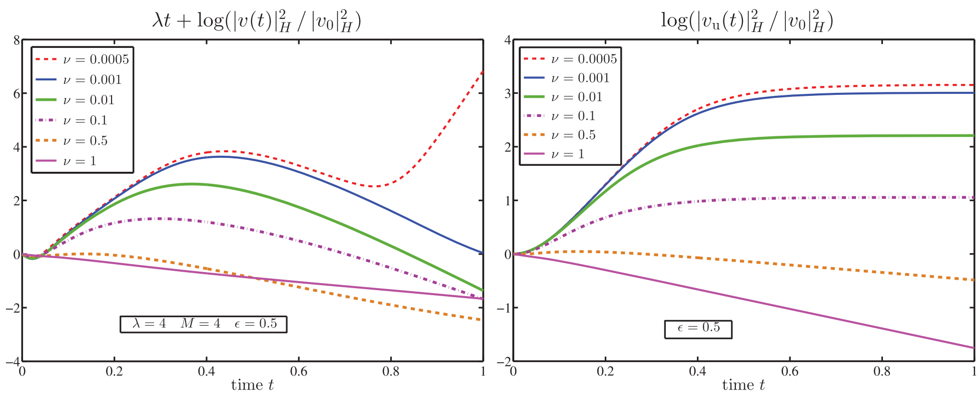
where ϵ is a constant that we will use to change the norm $|\hat{u}|_{\mathcal{W}}$ of $\hat{u}$.

6.3.1. Changing the norm of the reference trajectory. Here, we take $\lambda = 4$, $M = 4$, and $\nu = \frac{1}{10}$. In Figure 8 we see that the uncontrolled system becomes more instable as ϵ decreases, that is, as $|\hat{u}|_{\mathcal{W}}$ increases. We can also see that the four controls work up to $\epsilon = 0.067$ but not for $\epsilon = 0.0665$. This could mean that either the number of controls in not enough anymore or that our discretization is



(a) Controlled case. (b) Uncontrolled case.

FIG. 8. *Instability increases as ϵ decreases.*



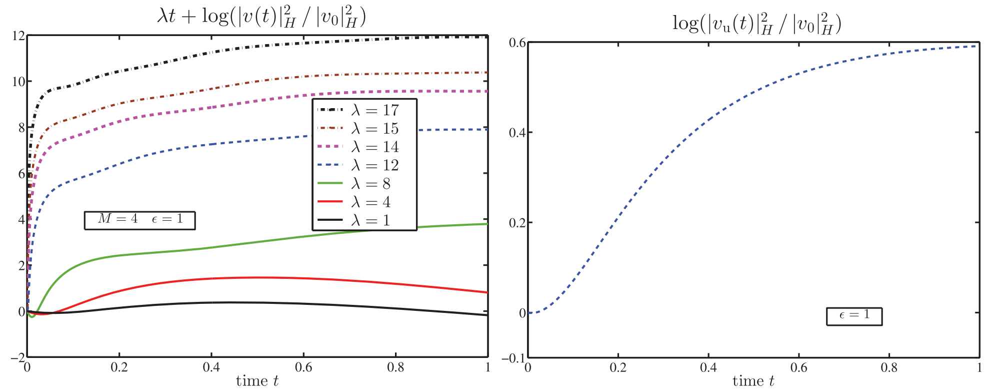
(a) Controlled case. (b) Uncontrolled case.

FIG. 9. *Instability increases as ν decreases.*

not fine enough. (Notice that for smaller ϵ, the magnitudes $|\partial_x \hat{u}(t, x)|_{\mathbb{R}}$ become bigger; we will come back to this issue hereafter in section 7.3.) Notice that in all the cases, we have $M_{\text{ref}} \geq \sqrt{44} \approx 6.63$ and $M_{\text{exp}} \geq \mathrm{e}^{\sqrt{44}} \approx 759.95$. In particular, $M_{\text{ref}}^{(\epsilon=0.067)} \approx 149.49$ is already big compared to M.

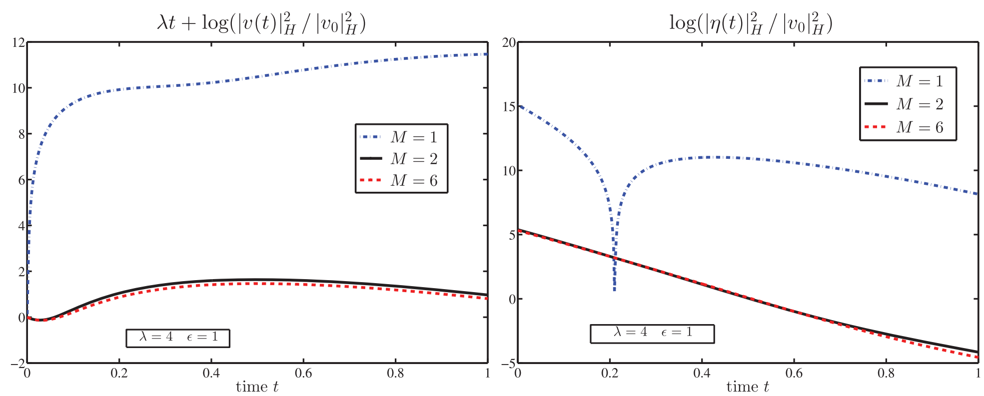
6.3.2. Changing the viscosity. Here, we take $\lambda = 4$, $M = 4$, and $\hat{u}(t, x) = 2\sin(5x)$. In Figure 9 we see that the uncontrolled system becomes more instable as ν decreases. We can also see that the four controls work up to $\nu = 0.001$ but not for $\nu = 0.0005$. Again, either the number of controls in not enough anymore or our discretization is not fine enough. (Notice that for smaller ν, the magnitudes $|\partial_x \hat{u}(t, x)|_{\mathbb{R}}$ become bigger when compared to ν.) Notice that in all the cases, we have $M_{\text{exp}} \geq \mathrm{e}^{\sqrt{8}} \approx 16.92$, $M_{\text{ref}}^{(\nu=1)} = \sqrt{8} \approx 2.83 < 4$, and $M_{\text{ref}}^{(\nu \leq 0.5)} \geq \sqrt{22} \approx 4.69 > 4$. In particular, notice that for $\nu \in \{0.5, 1\}$, we have $M \approx M_{\text{ref}}$ and the corresponding plots in Figure 9(a) remain below $0.423 \approx \log(1 + \mathrm{e}^{\frac{1}{2}})$, which suits (3.4) better than (3.16).

6.3.3. Changing the desired decreasing rate. Here, we take $M = 4$, $\hat{u} = \sin(5x)$, and $\nu = \frac{1}{10}$. In Figure 10 we see that with the four controls, we can get at least a rate of convergence $\lambda = 17$. We also see that $|v(t)|_H^2 \leq \mathrm{e}^{C_\lambda} \mathrm{e}^{-\lambda t} |v_0|_H^2$, where C_λ is the maximum of the corresponding curves. Since in all cases we have $\lambda > C_\lambda$, we



(a) Controlled case. (b) Uncontrolled case.

FIG. 10. *Behavior as the desired exponential rate changes.*



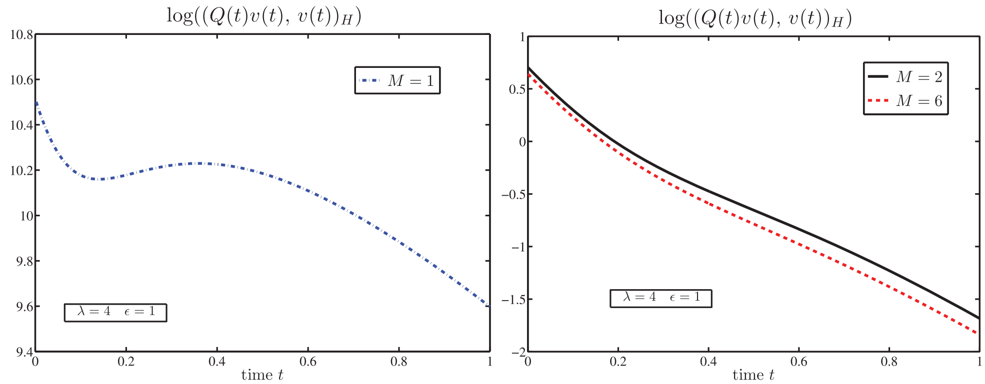
(a) The solution. (b) The control.

FIG. 11. *Behavior as M increases.*

see that the controller is effective at time $t = 1$, that is, the norm has been squeezed at time $t = 1$. On the other hand, for example, for $\lambda = 15$, up to time $t = 0.2$ we can guarantee that $|v(t)|_H^2 \le \mathrm{e}^9\mathrm{e}^{-15t}|v_0|_H^2$, in particular $|v(0.2)|_H^2 \le \mathrm{e}^6|v_0|_H^2$, that is we cannot guarantee that the norm has been squeezed, but at time $t = 0.8$ we find that $|v(0.8)|_H^2 \le \mathrm{e}^{11}\mathrm{e}^{-15\frac{8}{10}}|v_0|_H^2 = \mathrm{e}^{-1}|v_0|_H^2$, and we see that the norm is already squeezed.

6.3.4. Changing the number of controls. Here, we take $\hat{u} = \sin(5x)$, $\nu = \frac{1}{10}$, and $\lambda = 4$. In Figure 11 we see that $|v(t)|_H^2 \le \mathrm{e}^{C_M^v}\mathrm{e}^{-\lambda t}\,|v_0|_H^2$ and $|\eta(t)|_H^2 \le \mathrm{e}^{C_M^\eta}\mathrm{e}^{-\lambda t}\,|v_0|_H^2$ with C_M^v and C_M^η decreasing as M increases. Figure 12(a) could explain the cusp in the control plot in Figure 11; we guess that one control is not enough, because the cost function $t \mapsto (Q(t)v(t),\, v(t))_H$ must be a strictly decreasing function $(Q(s)v(s),\, v(s))_H = (Q(t)v(t),\, v(t))_H + \int_s^t \mathrm{e}^{\lambda\tau}(|\partial_x v(\tau)|_H^2 + |\eta(\tau)|_H^2)\,\mathrm{d}\tau$ for $s \le t$ (cf. section 3.4). In Figure 12(b) we can see that two controls can stabilize the system and that the cost decreases as M increases.

7. Numerical examples: The Burgers system. It remains to confirm that the feedback control stabilizes the system (1.1)–(1.2) to a given reference trajectory $\hat{u}$,



(a) One control is not stabilizing the system. (b) Two controls stabilize the system.

FIG. 12. *The cost decreases as the number of controls increase.*

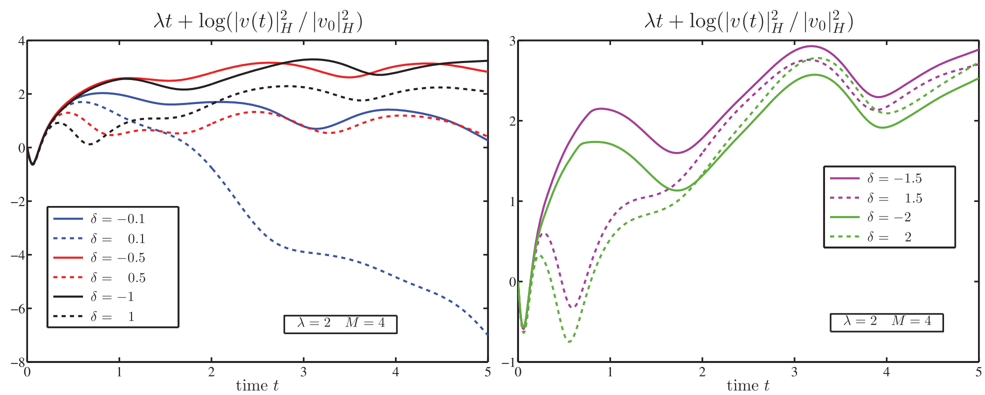


FIG. 13. *Convergence rate to $\hat{u}$ holds locally.*

provided that $|u_0 - \hat{u}(0)|_H^2$ is "small." We recall that $\hat{u}$ solves (1.1) with $\zeta = 0$ and $\hat{u}(0) = \hat{u}_0$. Below, we denote $d \coloneqq u - \hat{u}$ and $d_{\rm u} \coloneqq u_{\rm u} - \hat{u}$, where $u_{\rm u}$ solves system (1.1)–(1.2) with $\zeta = 0$, and u solves system (1.1)–(1.2) with the feedback control ζ, as in (3.22), computed to stabilize the system (3.1) to zero.

7.1. Local nature of the results and nonlinear nature of the equation. As in section 6.1, we set $\nu = \frac{1}{10}$, χ as defined in (6.1), $\lambda = 2$, and the trajectory $\hat{u} = C_{\rm nr}\,(\sin(-t)\sin(8x) - \cos(3t)\sin(8x))$ from the family (6.2). Again, we set $M = 4 < M_{\rm ref} = \sqrt{120}$. Next, we consider the family of initial conditions $u_0 = u_0^\delta \coloneqq d_0^\delta + \hat{u}_0$ with $d_0 = d_0^\delta = \delta(\sin(x) - \sin(6x))$ and $\delta \in \mathbb{R} \setminus \{0\}$.

In Figure 13 we can see that the feedback control is able to stabilize, with the desired rate, the nonlinear system (1.1)–(1.2) to the trajectory $\hat{u}$, provided that d_0 is small enough. We can see that, for $|\delta|_{\mathbb{R}} > 1$, the stabilization rate is not guaranteed; while for $|\delta| \le 1$ it holds. For example, for $|\delta|_{\mathbb{R}} \le 1$ we can see that the local maxima of the plotted curves seem either to converge to a real number or to decrease, while for $|\delta|_{\mathbb{R}} > 1$ those local maxima seem to go to infinity. Notice that the radius 1 here is suitable for this example; for other settings the stabilization may hold only for smaller $|\delta|_{\mathbb{R}}$.

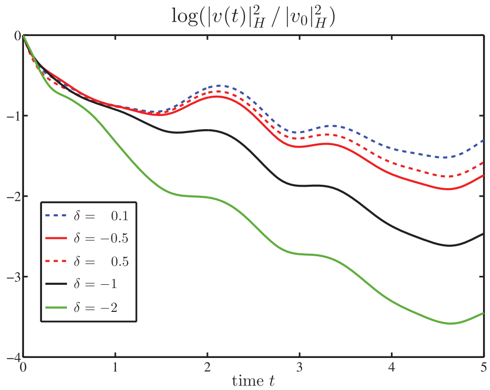


FIG. 14. *Uncontrolled case.*

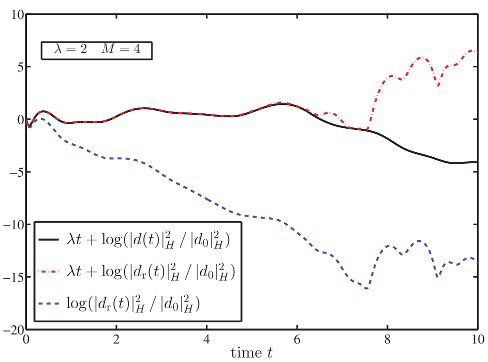


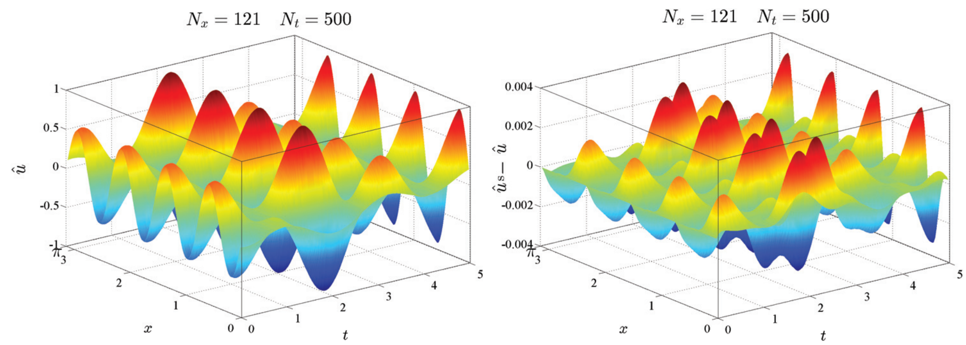
FIG. 15. *Ficticious versus real external force.*

In Figure 14 we can see that the uncontrolled systems do not go exponentially to $\hat{u}$ (at least not with the rate $\lambda = 2$); here, we have plotted the curves corresponding to some of those values of δ in Figure 13 (for the other the behavior is similar).

We can also see the nonlinear nature of the equations, because changing the sign of the initial condition leads to different curves.

Remark 7.1. The results correspond to simulations in which we have taken a fictitious external force h_{f} (i.e., an approximation of h) that makes $\hat{u}$ a solution of the discrete system (cf. section 5.5).

7.2. Real versus fictitious external force behavior. Here, we are in the same setting as in section 7.1. But now we fix $\delta = 1$ and consider $\hat{u}$ in the longer time interval $t \in [0, 10]$. We compare the numerical results in the case when we take the real external force $h = -\partial_t \hat{u} - \hat{u}\partial_x \hat{u} + \nu \partial_{xx} \hat{u}$ with those in the case when we take the fictitious external force h_{f} (cf. Remark 7.1). We denote $d = \bar{u} - \bar{\hat{u}}$ and $d_{\mathrm{r}} = \bar{u}_{\mathrm{r}} - \bar{\hat{u}}$, where $\bar{u}_{\mathrm{r}}$ solves (5.14) (that is, with the real external force $\bar{h}$) and $\bar{u}$ solves (5.14) with $\bar{h}_{\mathrm{f}}$ in the place of $\bar{h}$. In Figure 15, we confirm the rate of convergence of $\bar{u}$ to $\bar{\hat{u}}$ in the entire time interval, while for d_{r} the rate is confirmed until time $t = 6$. After time $t = 6$, we see that d_{r} remains bounded; this just means that the magnitude of $\bar{u}_{\mathrm{r}} - \bar{\hat{u}}$ has reached that of the discretization error of our solver, and consequently we cannot expect the magnitude of $\bar{u}_{\mathrm{r}} - \bar{\hat{u}}$ to decrease more.



(a) The reference trajectory $\hat{u}$. (b) The discretization error.

FIG. 16. *The difference between the discrete* $\hat{u}_\mathrm{S}$ *and exact* $\hat{u}$ *solutions.* $(h, k) = (\frac{\pi}{121}, \frac{5}{500})$.

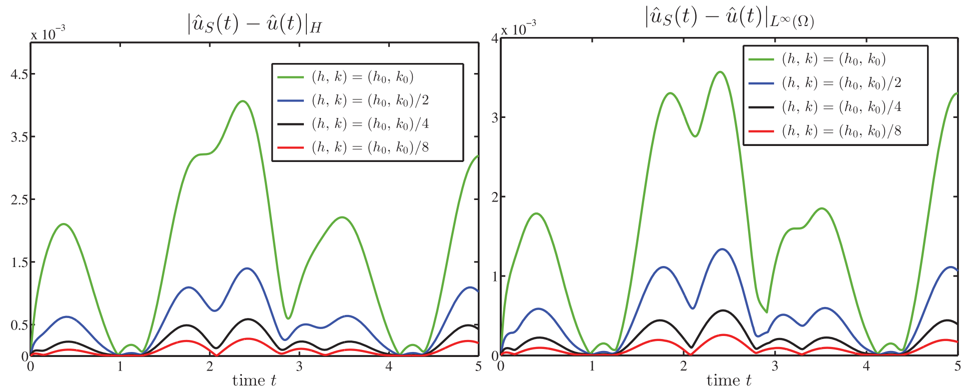


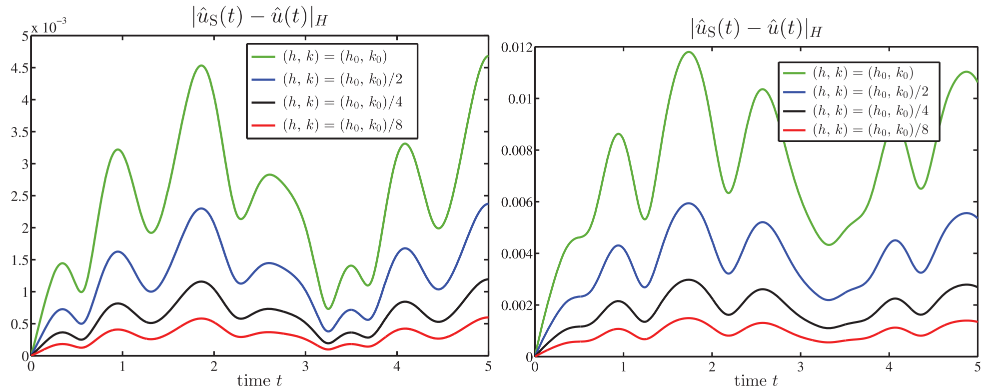
FIG. 17. *The discretization error.* $\hat{u} = \hat{u}^{(8,8)}$ *(cf.* (6.2)*)*, $(h_0, k_0) = (\frac{1}{121}, \frac{1}{100})$.

7.3. On the discretization error. Here, we are in the same setting as in section 7.1 with $\delta = 1$. We observe that, following the scheme (5.17), with the exact external force h (in the place of h_f), the discrete solution $\hat{u}_\mathrm{S}$ for (5.15) will be close to $\hat{u}$. Moreover, $\hat{u}_\mathrm{S}$ converges to $\hat{u}$ as $(k, h) \to (0, 0)$.

The main goal of the presented simulations in the previous sections is to show that an estimate like (3.3) should hold, in general, instead of (3.15), and it is not our intention to compare our algorithm/discretization to solve the Burgers and Oseen–Burgers systems with existing ones. That is why we have not performed a rigorous numerical analysis concerning the convergence of the scheme. Though, we would like to say that from numerical experiments that we have performed, we expect linear convergence. We present some results that indeed suggest that the error $\hat{u}_\mathrm{S} - \hat{u}$ is proportional to $h + k$.

Let $\hat{u} = \hat{u}^{(i,j)}$ be as in (6.2), with $(i, j) = (8, 8)$. In Figure 16 we can see the shape of the exact solution $\hat{u}$ and that of the difference $\hat{u}_\mathrm{S} - \hat{u}$ to the discrete solution $\hat{u}_\mathrm{S}$ given by the solver. In Figure 17 the error $\hat{u}_\mathrm{S} - \hat{u}$ is proportional to $h + k$; notice that as we squeeze (h, k) by the factor $\frac{1}{2}$, the plots are squeezed by a factor (not bigger than) $\frac{1}{2}$. Also, in the examples in Figure 18, corresponding to $\hat{u} = \hat{u}^{(i,j)}$ as in (6.2) with $(i, j) = (1, 4)$ and $(i, j) = (3, 2)$, the error is proportional to $h + k$.

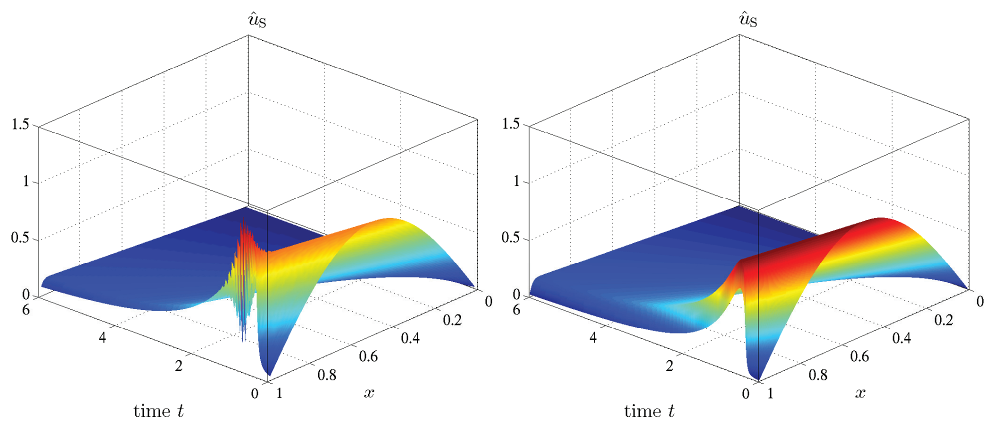
We must, however, recall that it is well known that in general, when solving the Burgers equation with small viscosity by finite elements, spurious oscillations may



(a) $\hat{u} = \hat{u}^{(1,4)}$.

(b) $\hat{u} = \hat{u}^{(3,\,2)}$.

FIG. 18. *The discretization error.* $(h_0, k_0) = (\frac{1}{121}, \frac{1}{100})$.



(a) Mesh size $(h, k) = (\frac{1}{80}, \frac{1}{100})$.

(b) Mesh size $(h, k) = (\frac{1}{320}, \frac{1}{200})$.

FIG. 19. *Spurious oscillations occur on coarser meshes.*

appear in the numerical solution if the convection part is dominant. They can be reduced by a reduction of the mesh size or by stabilization of the numerical scheme (e.g., see [11, 16, 24]). In our simulations, the reference trajectory is smooth with moderately bounded gradient. Thus, we can choose moderately small mesh parameters, such that no spurious oscillations appear. When we apply our scheme to a different example where the gradient $\partial_x u$ of the exact solution reaches big magnitudes, we will observe spurious oscillations on a coarse mesh; the oscillations vanish by decreasing the mesh parameters, see Figure 19. In Figure 19(a) we have taken the same space step and time step and viscosity as in [1, Figure 3(a)]. We see that we get the same behavior near $(t, x) = (1, 1)$, where the gradient of the exact solution has a big magnitude $|\partial_x u(1, 1)|_{\mathbb{R}}$; see also [16, section 5.6.5].

For references on numerical methods for feedback control and stabilization of the Burgers equation, we refer the reader to [3, 22, 31, 32, 33, 34, 44].

8. Final remarks. We have presented some estimates on the number of internal controls M we need to exponentially stabilize the Burgers system to a given reference trajectory $\hat{u} = \hat{u}(t, x)$. In the case that we take $\chi = 1_\Omega$, in particular, there is no constraint on the support of the control, we can derive a better estimate comparing with the general case (cf. sections 3.1 and 3.2), and we have presented the results of some numerical simulations that suggest that an estimate like that obtained in the case $\chi = 1_\Omega$ might hold also in the general case.

Throughout the paper, we consider controls given in the form $\sum_{i=1}^{M} \eta_i(t)\chi\mathbb{E}_0^{\mathcal{O}}\underline{s}_i(x)$. Usually, the controls at our disposal depend on each specific application; of course we can always consider another family of controls $\Psi = \{\psi_i \mid i \in \mathbb{N}_0\}$, perform the simulations for controls like $\sum_{i=1}^{M} \kappa_i(t)\psi_i(x)$, and perhaps also derive the corresponding estimates on M following the procedure in sections 3.1 and 3.2.

We have focused on the viscous 1D Burgers system. However, we are convinced that the challenge of finding an estimate for M, through a condition like (3.8) (preferable to a condition like (3.19)), will present analogous difficulties for the cases of $2D$ and $3D$ Burgers and Navier–Stokes systems and also for a wide class of parabolic systems.

Our results do not apply to the case of the nonviscous Burgers equation (i.e., to the case in which we take $\nu = 0$ in (1.1)); that is a completely different problem. We do not even know if a finite number M of controls is enough to stabilize the system. (In a general situation, the number M of needed controls will go to $+\infty$ as ν goes to 0.)

Showing the existence of a finite-dimensional feedback control supported on a small subset of the boundary and stabilizing the system to reference a nonstationary solution is work that is still going on. (See [40, 41] for some work in this direction.) Also in this case, it will be interesting to have an estimate on the dimension of the controller.

The value $\nu = \frac{1}{10}$ that we use in most of the simulations is (perhaps) too big for many applications. Of course we can take smaller ν, but in that case we may need to also take a finer mesh in order to guarantee that the stabilization observed for the discretized system in the numerical simulations will also hold for the continuous system. Notice that, when the numerical solution for system (1.1) goes to $\hat{u}$ as time increases, we can extrapolate that the evaluations $u(t,\, ih)$, $i \in \{1, 2, \ldots, N_x - 1\}$ of the continuous solution at the spatial mesh points will also go to $\hat{u}(t,\, ih)$ as time increases. Recall that if $|u(t) - \hat{u}(t)|_H$ goes to 0 as t increases, then $|u(t) - \hat{u}(t)|_V$ also does (provided that $\hat{u} \in \{v \in \mathcal{W} \mid \sup_{\tau \geq 0} |\partial_x \hat{u}|_{L^2((\tau,\, \tau+1),\, L^2(\Omega,\, \mathbb{R}))} < +\infty\}$, due to the smoothing property of the system (3.1); see [7, Lemma 2.1]). However, the fact that $|u(t,\, ih) - \hat{u}(t,\, ih)|_{\mathbb{R}}$ goes to 0 for all $i \in \{1, 2, \ldots, N_x - 1\}$ as time increases is in general not enough to conclude that u goes to $\hat{u}$. Indeed, from [25, Theorem 4.2] (for the case of the Navier–Stokes system in a two-dimensional Torus) we can derive that to conclude that u goes to $\hat{u}$, the space step h should be taken proportional to $\frac{\nu^2}{1-2\log(\nu)}$ (for small ν); and supposing that a similar estimate holds for the 1D Burgers system, it would follow that the number N_x of space points (determining nodes) should be proportional to $\frac{1-2\log(\nu)}{\nu^2}$. Notice that the computational effort and computational time will increase with N_x. We refer also to [26] and [19, Chapter III, section 2] and references therein concerning the estimates on the number of determining nodes.

The mathematical theory concerning stabilization to time-dependent trajectories (cf. [7]) is not as developed as for stabilization to a stationary state (cf. [2, 4, 5, 6, 8, 38, 39]). However, since these problems arise in applications, methods to solve these problems numerically have already been developed (see, e.g., [20, 28, 29] and references therein); notice that in this setting, "trajectory" will often mean a suitable evolutionary discrete process $u_0 \in Z$, $u_{i+1} = S(u_i) \in Z$, $i \in \mathbb{N}$, where Z is a Hilbert space. Other approaches can be found, for example, in [30] (in particular, see section 4 concerning trajectory tracking) and in [21] (in particular, see section 7.1 concerning linear feedback control of Navier–Stokes flows).

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