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FINITE ELEMENT APPROXIMATION OF LEVEL SET MOTION BY POWERS OF THE MEAN CURVATURE*

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Abstract. In this paper we study the level set formulations of certain geometric evolution equations from a numerical point of view. Specifically, we consider the flow by powers greater than one of the mean curvature (PMCF) and the inverse mean curvature flow (IMCF). Since the corresponding equations in level set form are quasi-linear, degenerate, and especially possibly singular a regularization method is used in the literature to approximate these equations to overcome the singularities of the equations. The regularized equations depend both on an regularization parameter and in case of the ICMF additionally on further parameters. Motivated by Feng, Neilan, and Prohl [*Numer. Math.*, 108 (2007), pp. 93–119], who study the finite element approximation of IMCF, we prove error estimates for the finite element approximation of the regularized equations for PMCF. We validate the rates with numerical examples. Additionally, the regularization error in the rotationally symmetric case for both flows is numerically analyzed in a two-dimensional setting. Therefore, in the case of IMCF we fix the additional parameters. Furthermore, having the goal to estimate the regularization error we derive barriers for the regularized level set IMCF respecting all parameters and specify them further in a rotationally symmetric simplified case. At the end of the paper we present simulations in the three-dimensional case.

Key words. mean curvature flow, level set equation, regularization, viscosity solution, finite elements

AMS subject classifications. 35J60, 35J70, 35J75, 65L60, 35D40

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1. Introduction.

The inverse mean curvature flow (cf. (5.1)) served as an important tool in Huisken and Ilmanen's proof [49] of the Riemannian Penrose inequality in general relativity. Later its level set formulation (cf. (5.3)) was extended to the flow by powers $k > 1$ of the mean curvature (cf. (2.1)) by Schulze [65], who also proved a certain inequality using the level set formulation (cf. (2.4)) of this flow. See sections 2 and 5 for a survey concerning properties of these flows. The paper [49] aroused the interest for a numerical analysis of this special level set approach to inverse mean curvature flow, which led to the paper [37] by Feng, Neilan, and Prohl, who introduced a finite element discretization for the level set formulation of inverse mean curvature flow as it appears in [49]. The starting point for their finite element method is regularized level set equations, which are defined by using a regularization parameter ε, a bounded domain Ω_L (i.e., an artificial boundary), and artificial boundary values L. These regularized equations (see section 5 for a definition) play an important role in the existence proof of [49]. Feng, Neilan, and Prohl [37] prove error estimates in the

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H^1-norm and the L^2-norm for the approximation of the regularized equations and confirm their rates by numerical examples. Furthermore, they focus on the aspect that their finite element method approximates the regularized equation (instead of the equation for level set inverse mean curvature flow) and present some numerical examples in which they study the corresponding regularization error and the overall error. For this they fix the parameters L and Ω_L and vary ε and the discretization parameter h. When computing an example where the domain is bounded away from the singularity of the equation (the singularities of the equation are points where the gradient vanishes) they obtain that a rather mild coupling of the regularization and the discretization parameter is sufficient; see [37, Tests 5 and 6].

Inspired by the paper [37] the contribution of our paper is to study a finite element approximation for the regularized level set flow by powers $k \geq 1$ of the mean curvature as considered in [65]. In the first part we prove rates for the H^1- and L^2-errors for the approximation of the regularized equations and confirm them by numerical examples. In the second part of the paper we study the regularization error in the rotationally symmetric case for the flow by powers $k \geq 1$ of the mean curvature numerically, similarly as in [37] but now the singularity of the equation is in the interior of the domain where the solution is computed. We obtain rates of the order of the corresponding theoretical estimate from [53]. Moreover, similar to this estimate we observe that this rate improves when $k \geq 1$ decreases. The third part of the paper deals with the regularization error for level set inverse mean curvature flow. First we derive barriers for the general case involving all three parameters ε, Ω_L, and L. Then we specify them in a simplified rotationally symmetric setting and confirm the obtained rate by a numerical example.

We give an overview over several publications about geometric flows. For the behavior of smooth parametric flows we refer to [3, 4, 33, 41, 48, 64, 66, 71]. For level set formulations for mean curvature flow, see, e.g., [36, 55, 60, 67, 68], and for its interpretation as the value function of a deterministic two-person game see [51]. For applications in image processing of geometric PDEs we refer to [1, 2, 19, 20]. For geometric flows describing the evolution of convex and nonconvex curves see [6, 38, 40, 45, 57, 63]. The approximation of geometric evolution equations with finite elements is considered in [8, 9, 10, 37, 59, 70, 72], by finite difference schemes in [24, 61], and by semi-Lagrangian schemes in [18].

Let us finish this overview with some specific papers about finite element approximations to mean curvature flow. A full error estimate for the finite element approximation of the parabolic level set equation for mean curvature flow can be found in [26]. An error estimate for the finite element approximation of mean curvature flow of closed surfaces by discretizing the evolution equations for the mean curvature and for the normal appeared very recently; see [52] and also [7, 27, 29, 30, 34].

Our paper is organized as follows. In section 2 we introduce the setting of the level set flow by powers of the mean curvature (level set PMCF). In section 3 we formulate the finite element approximation of regularized level set PMCF, prove error estimates, and present numerical examples. In section 4 we present numerically obtained rates for the regularization error of level set PMCF. In section 5 we introduce the regularized level set inverse mean curvature flow formulation (level set IMCF) from [49]. Then we show in section 6 theoretically and numerically obtained rates for the regularization error of regularized level set IMCF. Section 7 contains some numerical examples in which we simulate level set PMCF in the nonrotationally symmetric case and we give a short description of the implementation used for the numerical computations presented in this paper. We conclude section 7 with some remarks on an alternative level set formulation which is often used in the literature for the mean curvature flow case. Finally, in section 8 we present some simulations for two-dimensional surfaces evolving in the three-dimensional domain of definition of the level set function.

We conclude this section by introducing some notation. We denote the Euclidean norm of $\mathbb{R}^{n+1}$ by $|\cdot|$ and the inner product of $a, b \in \mathbb{R}^{n+1}$ by $a \cdot b$. For an open subset $\Omega \subset \mathbb{R}^{n+1}$ and $m \in \mathbb{N}^*$, $p \geq 1$, we denote the corresponding Sobolev spaces by $W^{m,p}(\Omega)$, $W_0^{m,p}(\Omega)$, $H^m(\Omega) = W^{m,2}(\Omega)$, and $H_0^m(\Omega) = W_0^{m,2}(\Omega)$. The dual spaces are denoted by $W^{-m,p}(\Omega) = W_0^{m,p}(\Omega)^*$ with dual pairing $\langle \cdot, \cdot \rangle$. Throughout this paper $c > 0$ denotes a generic constant.

2. Level set PMCF. In this section we focus on the level set equation of the flow by powers $k \geq 1$ of the mean curvature (PMCF). The parametric formulation of this flow in the smooth case is studied in [48, 64], where it is shown that even for all $k > 0$ convex initial hypersurfaces converge to points in finite time and that in the case $k = 1$ they converge to round points by preserving convexity. Improved convergence results (i.e., convergence to *round* points) in the case $k > 1$ can be found in [66] under certain pinching assumptions (i.e., the ratio of the largest and the smallest principal curvature is bounded from above by a constant larger than 1 and the distance of this constant from 1 measures how strong the pinching is) for the initial hypersurface. Such a pinching is necessary, as can be seen from [5], where it is shown that without such a pinching even convexity might get lost in the case $k > 1$. The parametric definition of this flow for a power $k > 0$ of the mean curvature is as follows. Let $M \subset \mathbb{R}^{n+1}$ be a closed, embedded hypersurface with positive mean curvature. A classical solution of the PMCF is a smooth family $x : M \times [0, T] \to \mathbb{R}^{n+1}$ of hypersurfaces $M_t := x(M, t)$ satisfying the parabolic evolution equation

$$\frac{\partial x}{\partial t} = -H^k \nu, \quad x \in M_t, \quad 0 \leq t \leq T, \tag{2.1}$$

where H is the mean curvature of M_t at the point x and ν the outward unit normal of M_t in x. When the initial hypersurface has positive mean curvature then, in the case $k > 1$, the flow might develop singularities before the enclosed volume of the moving flow hypersurface goes to zero but, for the case $k = 1$, there is a well-developed theory of parametric smooth mean-convex mean curvature flow; see [46]. For general $k \geq 1$ there is a useful notion of solutions for initial hypersurfaces with positive mean curvature which can handle singularities; see the level set equation (2.4) below.

For completeness and without further relevance for this paper we mention that it is also possible to derive for certain values of k short time extistence for solutions of the parametric formulation of the variant of PMCF (2.1)

$$\frac{\partial x}{\partial t} = -|H|^{k-1} H \nu \tag{2.2}$$

(cf. [57]), which allows also flow hypersurfaces which do not necessarily have positive mean curvature.

Our motivation is to provide numerical analysis for the level set flow corresponding to (2.1) for cases where theoretical results concerning the behavior of solutions exist and have turned out to be useful. Namely, the starting point will be the level set ansatz in [65], where certain inequalities are derived by using level set PMCF in case $k \geq 1$. We remark that there exists a certain level set theory for the cases $0 < k \leq 1$ which models the level sets as zero level sets of certain time dependent functions; see [58] and the references therein. In our paper we use stationary level set equations.

Naturally, it might happen thereby that the level sets of the level set function develop an interior so that it turned out to be useful to consider "level sets" of the form (2.3) where the inequality is chosen depending on the flow direction. While PMCF is a contracting flow we correspondingly later define the level sets for IMCF which is an expanding flow with opposite inequalities; cf. (5.2). Additionally to the analysis of the flow itself, [65] shows that a certain isoperimetrical difference is monotone under the level set flow and proves by using this monotonicity an isoperimetrical inequality.

Our numerical analysis concerns the level set formulation of PMCF and its regularization which we introduce in the following. Let $\Omega \subset \mathbb{R}^{n+1}$ be open, connected, and bounded having smooth boundary $\partial\Omega$ with positive mean curvature which we consider as initial hypersurface. We call the level sets

$$\Gamma_t := \partial\{x \in \Omega : u(x) > t\}, \tag{2.3}$$

$t \geq 0$, of the continuous function $0 \leq u \in C^0(\bar{\Omega})$ a level set PMCF, if u is a viscosity solution of

$$\begin{cases} \operatorname{div}\left(\dfrac{Du}{|Du|}\right) = -\dfrac{1}{|Du|^{\frac{1}{k}}} & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

This equation can be found in [65, section 4], where the notion of weak solutions, as formulated in (2.5)–(2.7), is introduced and existence and uniqueness are proven for $k \geq 1$ and $n \leq 6$. The weak solution is also a viscosity solution, which can be seen by a direct consideration. For a definition of viscosity solutions we refer to the classical literature; see, e.g., [22, 23]. For the specialization of the definition to this case see [53, section 2]. The relation between (2.4) and (2.2) can be seen as follows. If u is smooth in a neighborhood of $x \in \Omega$ with nonvanishing gradient and satisfies in this neighborhood (2.4), then the level set $\{u = u(x)\}$, $x \in \Omega$, is locally at x a smooth hypersurface and moves at x in the direction of its outer normal with speed H^k where H is its mean curvature in x. The notion of a weak solution of (2.4) is as follows (cf. [65]): Using elliptic regularization of level set PMCF we obtain the equation

$$\begin{cases} \operatorname{div}\left(\dfrac{Du^\varepsilon}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}}\right) = -(\varepsilon^2 + |Du^\varepsilon|^2)^{-\frac{1}{2k}} & \text{in } \Omega, \\ \qquad\qquad u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

which has a unique smooth solution u^ε for sufficiently small $\varepsilon > 0$ (cf. [65, section 4]); moreover, there is $c_0 > 0$ such that

$$\|u^\varepsilon\|_{C^1(\bar{\Omega})} \leq c_0 \tag{2.6}$$

uniformly in ε and (for a subsequence)

$$u^\varepsilon \to u \tag{2.7}$$

uniformly in $\bar{\Omega}$ where $u \in C^{0,1}(\bar{\Omega})$ is a suitable limit function and is called a weak solution of (2.4).

2.1. The linearized operator. We define for $\varepsilon > 0$ and $z \in \mathbb{R}^{n+1}$

$$|z|_\varepsilon := f_\varepsilon(z) := \sqrt{|z|^2 + \varepsilon^2}. \tag{2.8}$$

Then we have for the first and second derivatives of f_ε with respect to the variable $z = (z_i)$ that

$$(\text{grad}_{f_\varepsilon}(z))_i = D_i f_\varepsilon(z) = \frac{z_i}{|z|_\varepsilon}, \quad (\text{hess}_{f_\varepsilon}(z))_{ij} = D_i D_j f_\varepsilon(z) = \frac{\delta_{ij}}{|z|_\varepsilon} - \frac{z_i z_j}{|z|_\varepsilon^3}. \tag{2.9}$$

Moreover, for $p > 1$ and $\frac{1}{p} + \frac{1}{p^*} = 1$ we introduce the operator Φ_ε by

$$\Phi_\varepsilon : W_0^{1,p}(\Omega) \to W^{-1,p^*}(\Omega), \quad \Phi_\varepsilon(v) := -D_i\left(\frac{D_i v}{|Dv|_\varepsilon}\right) - \frac{1}{|Dv|_\varepsilon^{\frac{1}{k}}}, \tag{2.10}$$

where we use the convention to sum over repeated indices, so that (2.5) can be written as

$$\Phi_\varepsilon(u^\varepsilon) = 0. \tag{2.11}$$

Denoting the derivatives of Φ_ε in u^ε by

$$L_\varepsilon := D\Phi_\varepsilon(u^\varepsilon) \tag{2.12}$$

we have for all $\varphi \in W_0^{1,p}(\Omega)$ that

$$\begin{aligned} L_\varepsilon \varphi &= -D_i\left((\text{hess}_{f_\varepsilon}(Du^\varepsilon))_{ij} D_j \varphi\right) + \frac{1}{k} f_\varepsilon(Du^\varepsilon)^{-1-\frac{1}{k}} (\text{grad}_{f_\varepsilon}(Du^\varepsilon))_j D_j \varphi \\ &=: -D_i(a^{ij} D_j \varphi) + b^i D_i \varphi. \end{aligned} \tag{2.13}$$

The coefficients a^{ij} and b^i are in $C^\infty(\bar{\Omega})$. Note that the estimate (2.6) is not available for higher order derivatives of u^ε uniformly in ε but since ε is fixed in the present section, this does not have an effect on the following considerations. The linear operator

$$L_\varepsilon : W_0^{1,p}(\Omega) \to W^{-1,p^*}(\Omega) \tag{2.14}$$

and its adjoint operator L_ε^* are topological isomorphisms; cf. Corollary A.2 in the appendix.

3. Discretization and error estimate for regularized level set PMCF.

In this section we present a finite element discretization of the regularized equation (2.5) and prove error estimates. We consider the case that the space dimension $n+1$ is 2 or 3 and that Ω is convex. The latter is only a restriction if $n+1 = 3$ since $\partial\Omega$ has positive mean curvature by assumption (cf. section 1), which agrees with convexity in the case that $\partial\Omega$ is a curve.

3.1. Discretization.

Let (T_h, Ω_h) be a quasi-uniform triangulation of Ω with mesh size $0 < h < h_0$, h_0 sufficiently small and $V_h \subset H^1(\Omega_h)$ the finite element space given by

$$V_h := \left\{ v \in C^0(\bar{\Omega}_h) : v_{|\partial\Omega_h} = 0,\ v_{|T} \text{ linear for all } T \in T_h \right\}. \tag{3.1}$$

In view of the convexity of Ω there holds $\Omega_h \subset \Omega$. A function $u_h \in V_h$ will also be considered as a function on Ω by extending it by zero in $\Omega \setminus \Omega_h$. Clearly, the extended u_h then lies in $H^1(\Omega)$. The variational formulation of (2.5) is given by

$$\int_{\Omega_h} \frac{Du_h^\varepsilon \cdot Dv_h}{\sqrt{\varepsilon^2 + |Du_h^\varepsilon|^2}} \mathrm{d}x = \int_{\Omega_h} (\varepsilon^2 + |Du_h^\varepsilon|^2)^{-\frac{1}{2k}} v_h \mathrm{d}x \quad \text{for all } \ v_h \in V_h, \tag{3.2}$$

where we fix $\varepsilon > 0$ from now on and denote the finite element solution by $u_h^\varepsilon \in V_h$. For formal reasons we will consider boundary tetrahedra (boundary triangles in the case $n = 1$) to be extended to boundary tetrahedra with one "curved face." Therefore we will replace a boundary element $T \in T_h$ (i.e., $n + 1$ vertexes of T lie on $\partial\Omega$) by $\tilde{T} = T \cup B$ with

$$B{:=}\{tq + (1-t)Pq \mid 0 \le t \le 1, q \in bf\}, \tag{3.3}$$

where bf is the boundary face of T, i.e., $n+1$ vertexes of bf lie on $\partial\Omega$ and Pq is the unique minimizer of $\operatorname{dist}(q,\cdot)_{|\partial\Omega}$. We denote the resulting triangulation by $\tilde{T}_h$. This leaves the space of finite element functions we use (namely, V_h) unchanged. Note that the boundary strip $\Omega \setminus \Omega_h$ has measure $O(h^2)$.

From [17, Theorem 8.5.3] we deduce for $L = L_\varepsilon$ or $L = L_\varepsilon^*$ and $F \in W^{-1,p^*}(\Omega)$ that there is a unique solution $u_h \in V_h$ of

$$\langle Lu_h, \varphi_h\rangle = F\varphi_h \quad \text{for all } \varphi_h \in V_h, \tag{3.4}$$

where $u \in H^1(\Omega)$ is the unique solution of $Lu = F$ and we have the estimate

$$\|u_h\|_{W^{1,p}(\Omega)} + \|u - u_h\|_{W^{1,p}(\Omega)} \le c\|u\|_{W^{1,p}(\Omega)}. \tag{3.5}$$

Furthermore, if $F \in L^p(\Omega)$ we have

$$\|u - u_h\|_{W^{1,p}(\Omega)} + h\|u - u_h\|_{L^p(\Omega)} \le ch^2\|F\|_{L^p(\Omega)}. \tag{3.6}$$

Remark 3.1. Note that we used the assertion of [17, Theorem 8.5.3] under slightly different assumptions:

(i) We assume a right-hand-side $F \in W^{-1,p^*}(\Omega)$ (instead $F \in L^p(\Omega)$).
(ii) We consider the equation on Ω (instead of a polygonal domain) and use as discretization the triple $(\tilde{T}_h, \Omega, V_h)$.

3.2. Error estimate. The following theoretical error analysis for the finite element approximation in the L^p- and $W^{1,p}$-norms of the regularized equation is along the lines of the corresponding error analysis in [37] with some minor adaptions to include the general exponent k, i.e., setting $k = -1$ reproduces the corresponding error estimate in [37] for the finite element approximation of the regularized level set inverse mean curvature flow. Having in mind that these estimates are the key theoretical observation in [37] and since our experimental error analysis also studies these estimates for general exponents we repeat the argument here in a condensed fashion for the convenience of the reader.

We have the following error estimate in the $W^{1,p}$-norm.

THEOREM 3.2. *For every $p > n + 1$ and sufficiently small $h > 0$ there exists a positive constant $c = c(\|u^\varepsilon\|_{W^{2,p}(\Omega)})$ such that (3.2) has a solution $u_h^\varepsilon \in V_h$ satisfying*

$$\|u^\varepsilon - u_h^\varepsilon\|_{W^{1,p}(\Omega)} \le ch. \tag{3.7}$$

This solution is unique in a sufficiently small $W^{1,p}$-neighborhood of u^ε in V_h.

Proof. Let $I_h u^\varepsilon$ be the interpolation of u^ε, i.e., the continuous piecewise linear function on Ω_h which is equal to u^ε at all nodes of Ω_h. We extend $I_h u^\varepsilon$ by zero to a function on Ω. Then, we have

$$\|I_h u^\varepsilon - u^\varepsilon\|_{W^{1,p}(\Omega)} \le ch \tag{3.8}$$

with $c = c(\|u^\varepsilon\|_{W^{2,p}(\Omega)})$; cf. [35, Corollary 1.109]. Setting

$$\bar{B}_p^h := \{v_h \in V_h : \|u^\varepsilon - v_h\|_{W^{1,p}(\Omega)} \le ch\} \tag{3.9}$$

with positive constant c (which results from the following considerations), we obtain u_h as the unique fixed point in $\bar{B}_p^h$ of the operator $T : V_h \to V_h$ with

$$L_\varepsilon(w_h - Tw_h) = \Phi_\varepsilon(w_h), \quad w_h \in V_h. \tag{3.10}$$

For the proof we proceed in three steps. We show (i) $\bar{B}_p^h \neq \emptyset$, (ii) T is a contraction, and (iii) $T(\bar{B}_p^h) \subset \bar{B}_p^h$.

(i) For sufficiently small h we have $I_h u^\varepsilon \in \bar{B}_p^h$.

(ii) Let $v_h, w_h \in \bar{B}_p^h$ and $\xi_h = v_h - w_h$; then using (3.10) we conclude

$$\begin{aligned} L_\varepsilon(Tv_h - Tw_h) &= L_\varepsilon \xi_h + \Phi_\varepsilon(w_h) - \Phi_\varepsilon(v_h) \\ &= (L_\varepsilon - D\Phi_\varepsilon(v_h + \Theta\xi_h))\xi_h =: F. \end{aligned} \tag{3.11}$$

In order to estimate $\|F\|_{W^{-1,p^*}(\Omega)}$ which leads to an estimate of $\|Tv_h - Tw_h\|_{W^{1,p}(\Omega)}$ in view of Corollary A.2 we choose $\psi \in W_0^{1,p^*}(\Omega)$ with $\|\psi\|_{W^{1,p^*}(\Omega)} \le 1$ and estimate $\langle F, \psi \rangle$. To do so we use a mean value theorem for which we need the following auxiliary estimate:

$$\begin{aligned} \|Du^\varepsilon - (Dv_h + \Theta D\xi_h)\|_{L^\infty(\Omega)} &\le \|Du^\varepsilon - DI_h u^\varepsilon\|_{L^\infty(\Omega)} + \|DI_h u^\varepsilon - D\tilde{v}_h\|_{L^\infty(\Omega)} \\ &\le ch + ch^{1-\frac{n+1}{p}}, \end{aligned} \tag{3.12}$$

where $\tilde{v}_h = v_h + \Theta\xi_h \in \bar{B}_p^h$ and where we used an inverse estimate. The resulting estimate implies

$$\|Tv_h - Tw_h\|_{W^{1,p}(\Omega)} \le c(h + h^{1-\frac{n+1}{p}})\|\xi_h\|_{W^{1,p}(\Omega)} \le \frac{1}{4}\|\xi_h\|_{W^{1,p}(\Omega)} \tag{3.13}$$

for sufficiently small h.

(iii) Let $w_h \in \bar{B}_p^h$. We have

$$\begin{aligned} \|Tw_h - u^\varepsilon\|_{W^{1,p}(\Omega)} &\le \|Tw_h - TI_h u^\varepsilon\|_{W^{1,p}(\Omega)} + \|TI_h u^\varepsilon - I_h u^\varepsilon\|_{W^{1,p}(\Omega)} \\ &\quad + \|I_h u^\varepsilon - u^\varepsilon\|_{W^{1,p}(\Omega)} \\ &\le \frac{h}{2} + \|TI_h u^\varepsilon - I_h u^\varepsilon\|_{W^{1,p}(\Omega)} + ch. \end{aligned} \tag{3.14}$$

It remains to estimate the norm on the right-hand side. We have

$$\begin{aligned} \|TI_h u^\varepsilon - I_h u^\varepsilon\|_{W^{1,p}(\Omega)} &\le c\|\Phi_\varepsilon(I_h u^\varepsilon)\|_{W^{-1,p^*}(\Omega)} \\ &= c\|\Phi_\varepsilon(I_h u^\varepsilon) - \Phi_\varepsilon(u^\varepsilon)\|_{W^{-1,p^*}(\Omega)} \le ch \end{aligned} \tag{3.15}$$

again by a mean value theorem estimate and we obtain

$$T(\bar{B}_p^h) \subset \bar{B}_p^h. \tag{3.16}$$

□

Employing a duality argument as in [37] we obtain an L^p-error estimate in the following theorem.

THEOREM 3.3. *For $p > n+1$ we have*

$$\|u^\varepsilon - u_h^\varepsilon\|_{L^p(\Omega)} \leq ch^2 \tag{3.17}$$

with $c = c(\|u^\varepsilon\|_{W^{2,2p}(\Omega)}) > 0$.

Proof. From the definitions of u^ε and u_h^ε we get

$$\int_\Omega \left(\frac{Du^\varepsilon}{|Du^\varepsilon|_\varepsilon} - \frac{Du_h^\varepsilon}{|Du_h^\varepsilon|_\varepsilon} \right) \cdot D\varphi_h \mathrm{d}x + \int_\Omega \left(|Du^\varepsilon|_\varepsilon^{-\frac{1}{k}} - |Du_h^\varepsilon|_\varepsilon^{-\frac{1}{k}} \right) \varphi_h \mathrm{d}x = 0 \quad \text{for all } \varphi_h \in V_h. \tag{3.18}$$

This equation can be written equivalently as

$$\int_\Omega (A_h^\varepsilon De_h^\varepsilon) \cdot D\varphi_h \mathrm{d}x + \int_\Omega (a_h^\varepsilon \cdot De_h^\varepsilon)\, \varphi_h \mathrm{d}x = 0 \quad \text{for all } \varphi_h \in V_h \tag{3.19}$$

with

$$\begin{aligned} A_h^\varepsilon &:= \int_0^1 D^2 f_\varepsilon(Du^\varepsilon + tD(u_h^\varepsilon - u^\varepsilon))dt, \\ a_h^\varepsilon &:= -\frac{1}{k} \int_0^1 f_\varepsilon(Du^\varepsilon + tD(u_h^\varepsilon - u^\varepsilon))^{-\frac{1}{k}-1} Df_\varepsilon(Du^\varepsilon + tD(u_h^\varepsilon - u^\varepsilon))dt, \\ e_h^\varepsilon &:= u_h^\varepsilon - u^\varepsilon. \end{aligned} \tag{3.20}$$

For later purposes we set

$$\begin{aligned} \bar{A}_h^\varepsilon &:= D^2 f_\varepsilon(Du^\varepsilon), \\ \bar{a}_h^\varepsilon &:= \frac{1}{k} f_\varepsilon(Du^\varepsilon)^{\frac{1}{k}-1} Df_\varepsilon(Du^\varepsilon). \end{aligned} \tag{3.21}$$

Let $\varphi \in W_0^{1,p^*}(\Omega)$ be given by

$$L_\varepsilon^* \varphi := |e_h^\varepsilon|^{p-1} \operatorname{sgn}(e_h^\varepsilon) \tag{3.22}$$

with sign-function sgn. Furthermore, let $\varphi_h \in V_h$ be the corresponding finite element solution of this equation. We test (3.22) with e_h^ε and get by symmetry of $\bar{A}_h^\varepsilon$ that

$$\int_\Omega |e_h^\varepsilon|^p \mathrm{d}x = \int_\Omega (\bar{A}_h^\varepsilon De_h^\varepsilon) \cdot D\varphi_h \mathrm{d}x + \int_\Omega (\bar{a}_h^\varepsilon \cdot De_h^\varepsilon) \varphi_h \mathrm{d}x. \tag{3.23}$$

By (3.19) we have further

$$\begin{aligned} \int_\Omega |e_h^\varepsilon|^p \mathrm{d}x &= \int_\Omega \left((\bar{A}_h^\varepsilon - A_h^\varepsilon) De_h^\varepsilon \right) \cdot D\varphi_h \mathrm{d}x + \int_\Omega \left((\bar{a}_h^\varepsilon - a_h^\varepsilon) \cdot De_h^\varepsilon \right) \varphi_h \mathrm{d}x \\ &\leq c \int_\Omega |De_h^\varepsilon|^2 |D\varphi_h| \mathrm{d}x + c \int_\Omega |De_h^\varepsilon|^2 \varphi_h \mathrm{d}x \\ &\leq c \|\varphi_h\|_{W^{1,p^*}(\Omega)} \|e_h^\varepsilon\|_{W^{1,2p}(\Omega)}^2. \end{aligned} \tag{3.24}$$

In view of Corollary A.2 and (3.22) we get

$$\|\varphi_h\|_{W^{1,p^*}(\Omega)} \leq c \left(\int_\Omega |e_h^\varepsilon|^{(p-1)p^*} \mathrm{d}x \right)^{\frac{1}{p^*}} = c \|e_h^\varepsilon\|_{L^p(\Omega)}^{\frac{p}{p^*}}. \tag{3.25}$$

Using (3.23) and Theorem 3.2 to estimate $\|e_h^\varepsilon\|_{W^{1,2p}(\Omega)}$ we complete the proof. ∎

3.3. Implementation. The discretization errors in the H^1- and L^2-norms have been computed numerically by solving (2.4) for the case of a unit circle as initial curve with $\varepsilon = 0.1$ and $k = 1$ on meshes with $h_i = 0.4 \cdot 0.5^i$, $i = 0, \ldots, 6$. The solutions have been computed iteratively by linearizing (3.2) in the following way:

$$\int_{\Omega_h} V_j (Du^\varepsilon_{h,j} \cdot Dv_h) \mathrm{d}x = \int_{\Omega_h} f_j v_h \mathrm{d}x \quad \text{for all} \quad v_h \in V_h, \tag{3.26}$$

where $j \in \mathbb{N}$ is the iteration index and V_j and f_j have been updated in each iteration,

$$\begin{aligned} V_j &:= \begin{cases} 1, & j = 1, \\ \gamma V_{j-1} + (1-\gamma)\frac{1}{\sqrt{\varepsilon^2 + |Du^\varepsilon_{h,j}|^2}}, & j > 1, \end{cases} \\ f_j &:= \begin{cases} 1, & j = 1, \\ \gamma f_{j-1} + (1-\gamma)(\varepsilon^2 + |Du^\varepsilon_{h,j}|^2)^{-\frac{1}{2k}}, & j > 1. \end{cases} \end{aligned} \tag{3.27}$$

The value of γ affects the convergence of the iterations and was set to 0.1. In each iteration the linearized equation has been solved using FreeFem [47]. These solutions were compared to a solution obtained by solving (2.4) in radial coordinates. Note that due to radial symmetry (2.4) simplifies to an ODE which was solved on a one-dimensional grid of $h = 0.001$. Figure 1 shows the numerical discretization errors as well as the rates proven in Theorems 3.2 and 3.3. Figure 2 shows the discretization errors $u^\varepsilon_h - u^\varepsilon_{0.001}$ and the corresponding meshes.

4. Numerical study of the regularization error for PMCF. This section shows some numerically obtained rates for the regularization error of regularized level set PMCF. Since small values of the regularization parameter ε are difficult to handle we restrict the setting to the rotationally symmetric case in order to have very high accuracy.

We state the following theoretical regularization error estimate.

THEOREM 4.1. *Let u be solution of* (2.4) *and u^ε of* (2.5). *Then for $0 < \lambda < \frac{1}{2k}$ the regularization error of the regularized level set PMCF satisfies*

$$|u - u^\varepsilon|_{C^0(\Omega)} = O(\varepsilon^\lambda). \tag{4.1}$$

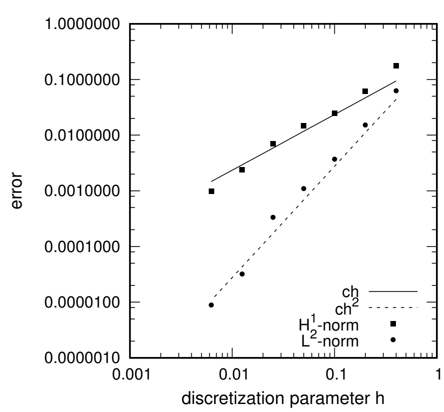


FIG. 1. *Discretization error for the unit circle as initial curve with $k = 1$, $\varepsilon = 0.1$.*

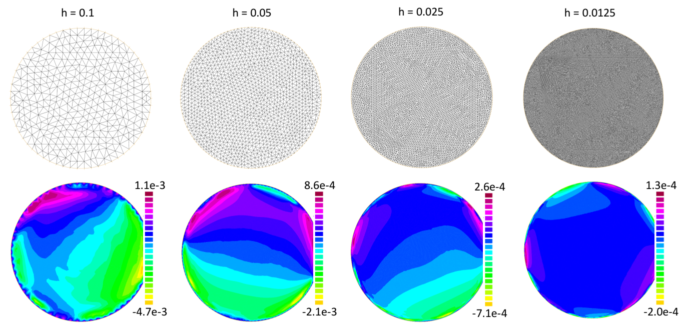


FIG. 2. *Top: mesh with decreasing mesh size h. Bottom: the corresponding error $u_h^\varepsilon - u_{0.001}^\varepsilon$.*

Proof. See [53] for the proof. ∎

For completeness we mention that we actually have an estimate of the Hölder norm which results from a well-known interpolation inequality and the uniform boundedness of the first derivatives of u^ε.

COROLLARY 4.2. *Assume the situation of Theorem* 4.1 *and fix* $0 < \gamma < 1$ *and* $0 < \lambda < \frac{1}{2k}$. *Then there holds*

$$\|u - u^\varepsilon\|_{C^{0,\gamma}(\Omega)} \leq c(\lambda, \gamma)\varepsilon^{\lambda(1-\gamma)}. \tag{4.2}$$

Proof. In the following we state a well-known interpolation inequality. For $0 < \beta < \alpha \leq 1$ and a function $v \in C^0(\Omega)$ we have

$$[v]_\beta \leq 2^{1-\frac{\beta}{\alpha}} [v]_\alpha^{\frac{\beta}{\alpha}} (\|v\|_{C^0(\Omega)})^{1-\frac{\beta}{\alpha}}, \tag{4.3}$$

where these expressions might become infinity and

$$[v]_\alpha := \sup_{x \neq y} \frac{|v(x) - v(y)|}{|x-y|^\alpha}. \tag{4.4}$$

Since u^ε is uniformly bounded in the C^1-norm (cf. (2.6)) we can use inequality (4.3) to conclude the claim of the corollary. ∎

Our rotationally symmetric example for which we study the regularization error is a shrinking circle. Let the circle $\partial B_{r_0}(0) \subset \mathbb{R}^2$ with radius $r_0 > 0$ serve as our initial curve so that the solution u of (2.4) is given as

$$u(r) = \frac{r_0^{k+1} - r^{k+1}}{k+1}, \tag{4.5}$$

where $r \geq 0$ denotes the radius variable in polar coordinates in $\mathbb{R}^2$ with center in the origin. The regularized equation reduces to the following one-dimensional equation when formulated in radial coordinates, i.e., $u^\varepsilon = u^\varepsilon(r)$:

$$\begin{cases} \dfrac{1}{r}\dfrac{\mathrm{d}}{\mathrm{d}r}\left(\dfrac{r\frac{\mathrm{d}}{\mathrm{d}r}u^\varepsilon}{\sqrt{\varepsilon^2 + \left|\frac{\mathrm{d}}{\mathrm{d}r}u^\varepsilon\right|^2}}\right) = -\left(\varepsilon^2 + \left|\dfrac{\mathrm{d}}{\mathrm{d}r}u^\varepsilon\right|^2\right)^{-\frac{1}{2k}} \quad \text{in } [0, r_0], \\ \dfrac{\mathrm{d}}{\mathrm{d}r}u^\varepsilon(0) = 0 \quad \text{and} \quad u^\varepsilon(r_0) = 0. \end{cases} \tag{4.6}$$

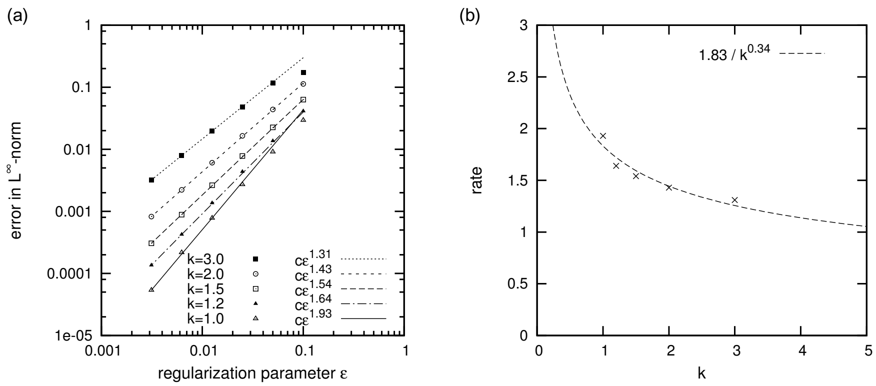


FIG. 3. (a) *Regularization error in the case of a circle as initial curve for* $k = 1, 1.2, 1.5, 2, 3$. (b) *Rate of regularization error as function of* k.

We solve this one-dimensional boundary value problem by a Newton algorithm combined with the Thomas algorithm as a direct solver. In Figure 3(a) the error $\|u^\varepsilon - u\|_{L^\infty(\Omega_h)}$ is plotted for $k \in \{1.0, 1.2, 1.5, 2.0, 3\}$. By fitting the function $c\varepsilon^{r_k}$ to the computed L^∞-errors we obtain the rate of the regularization error r_k (Figure 3(b)) as a function of k, namely, $r_k \approx 1.83/k^{0.34}$, which means indeed that the rate depends on the power k—the larger the k, the smaller the rate. However, this is a little bit better than $1/(2k)$.

Figure 4 shows the corresponding solutions of the regularized equations.

An important property of our test is that the singularity of the (not regularized) equation lies in our numerical example in the domain of the computation. The regularization error has also been studied numerically in the case of regularized level set IMCF in [37, Tests 5, 6]. Note that in these tests the singularity is not in the domain of the computation; see the end of subsection 6.2, where we describe these tests in more detail.

5. Level set IMCF. We recall some background information about the inverse mean curvature flow; details and precise statements can be found in [49]. Geroch [42] introduced the inverse mean curvature flow and observed that the so-called Hawking quasi-local mass of a 2-surface in an asymptotically flat 3-manifold is monotone nondecreasing (Geroch monotonicity). Jang and Wald [50] discovered that in the presence of classical solutions to inverse mean curvature flow starting from the inner boundary of an asymptotically flat 3-manifold and converging to large coordinate spheres the Geroch monotonicity implies the—at the time of the paper [50] conjectured—Penrose inequality in general relativity. In general such classical solutions do not exist. Huisken and Ilmanen [49] developed a theory of weak solutions which allows that the flow hypersurfaces jump (over a positive 3-volume, so-called fattening occurs but their enclosed volume behaves continuously) instead to propagate continuously; furthermore, they could show that the Geroch monotonicity carries over to the weak flow. This enabled them to prove the Penrose inequality.

The convergence of smooth classical solutions of IMCF in Euclidean space starting from closed, star-shaped initial hypersurfaces with positive mean curvature to expanding round spheres was shown in [41, 71].

Let $M \subset \mathbb{R}^{n+1}$ be a closed, embedded hypersurface. A classical solution of the inverse mean curvature flow is a smooth family $x : M \times [0, T] \to \mathbb{R}^{n+1}$ of hypersurfaces

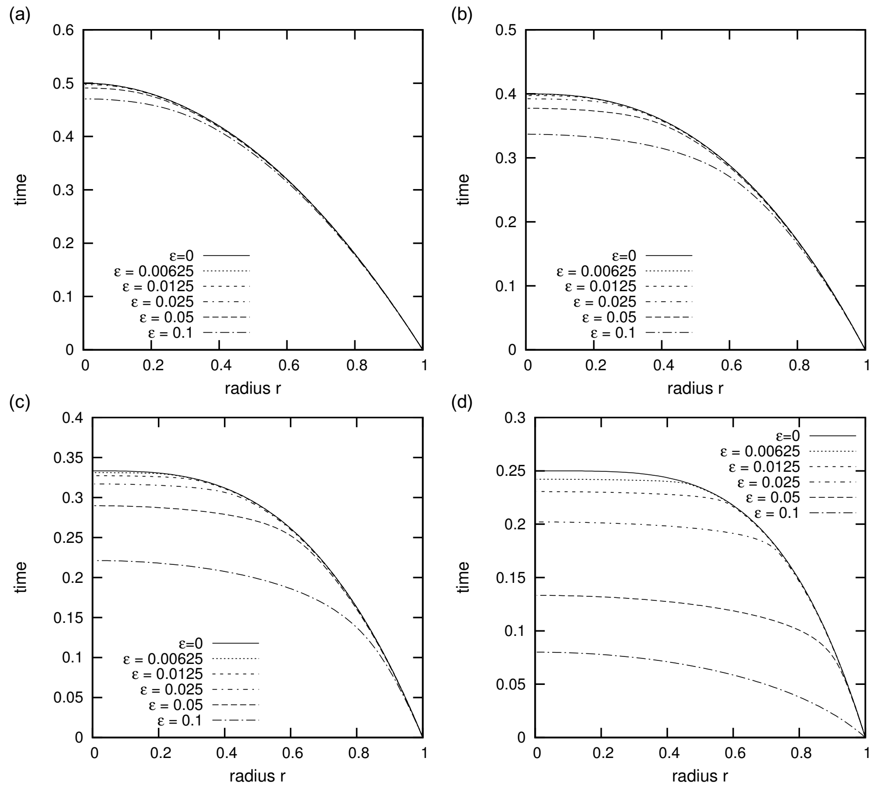


FIG. 4. *Radial solution for a circle as initial curve for* (a) $k = 1$, (b) $k = 1.5$, (c) $k = 2$, *and* (d) $k = 3$.

$M_t := x(M, t)$ satisfying the parabolic evolution equation

$$\frac{\partial x}{\partial t} = \frac{\nu}{H}, \quad x \in M_t, \quad 0 \le t \le T, \tag{5.1}$$

where H, assumed to be positive, is the mean curvature of M_t at the point x and ν is the outward unit normal. If the flow is given by the level sets of a function $u : \mathbb{R}^{n+1} \to \mathbb{R}$ via

$$E_t := \{x \in \mathbb{R}^{n+1} : u(x) < t\}, \quad M_t := \partial E_t, \tag{5.2}$$

then wherever u is smooth with $Du \neq 0$, (5.1) is equivalent to

$$\operatorname{div}\left(\frac{Du}{|Du|}\right) = |Du| \tag{5.3}$$

and the left-hand side of (5.3) is the mean curvature of the level set $\{u = t\}$ and the right-hand side is the inverse normal speed.

Following [49] we set $v(x) := n \log |x|$ and define the domains

$$F_L := \{v < L\}, \quad \Omega_L := F_L \setminus \bar{E}_0, \tag{5.4}$$

where $E_0 \subset \mathbb{R}^{n+1}$ is an open set with $\partial E_0 \in C^1$, $E_0 \subset\subset F_0$, and $L > 0$. For $n = 1$ we have $F_L = \{x \in \mathbb{R}^2 \ : \ |x| \le e^L\}$ and for $n = 2$ we have $F_L = \{x \in \mathbb{R}^3 \ : \ |x| \le e^{\frac{L}{2}}\}$. For L and ε positive we consider the following regularized level set equation given by

$$
\begin{cases} E^\varepsilon u^\varepsilon := \operatorname{div}\left(\dfrac{Du^\varepsilon}{\sqrt{|Du^\varepsilon|^2+\varepsilon^2}}\right) - \sqrt{|Du^\varepsilon|^2+\varepsilon^2} = 0 & \text{in } \Omega_L, \\ u^\varepsilon = 0 & \text{on } \partial E_0, \\ u^\varepsilon = L-2 & \text{on } \partial F_L. \end{cases} \tag{5.5}
$$

In the case that there exists a (hence unique) solution u^ε we will denote it by $u^{\varepsilon,L}$. From [49, Lemma 3.4] we know the following existence result.

LEMMA 5.1. *For every $L > 0$ there is $\varepsilon(L) > 0$ such that for $0 < \varepsilon \le \varepsilon(L)$ a smooth solution u^ε of* (5.5) *exists.*

Furthermore, [49, Example 2.3] shows that one can expect that L is at most $\frac{c}{\varepsilon(L)}$, or equivalently,

$$
\varepsilon(L) \le \frac{c}{L}. \tag{5.6}
$$

From [49, p. 365] we recall the definition of a weak solution of (5.3).

DEFINITION 5.2. (i) *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set; then $u \in C^{0,1}(\Omega)$ is a weak solution of* (5.3) *on Ω if*

$$
J_u^K(u) \le J_u^K(v), \tag{5.7}
$$

where

$$
J_u^K(v) := \int_K |Dv| + v|Du| \mathrm{d}x \tag{5.8}
$$

for all $v \in C^{0,1}(\Omega)$ with $\{v \neq u\} \subset\subset \Omega$ and compact $\{v \neq u\} \subset K \subset \mathbb{R}^{n+1}$.

(ii) *u is a weak solution of* (5.3) *with initial condition E_0 if*

$$
u \text{ satisfies (i) with } \Omega = \mathbb{R}^{n+1} \setminus \bar{E}_0 \text{ and } E_0 = \{u < 0\}. \tag{5.9}
$$

We state the following existence theorem which holds due to [49, Theorem 3.1] and remark that the latter contains further information like a gradient estimate.

THEOREM 5.3. *For every open and bounded set $\Omega \subset \mathbb{R}^{n+1}$ there is a weak solution of* (5.3) *with initial condition E_0 which is unique on $\mathbb{R}^{n+1} \setminus E_0$.*

Furthermore, from the proof of [49, Theorem 3.1] we conclude that there exist $\mathbb{R} \ni L_i \to \infty$, $0 < \varepsilon_i \to 0$, solutions $u_i = u^{\varepsilon_i, L_i}$ of (5.5) (i.e., with $\varepsilon = \varepsilon_i$ and $L = L_i$) and $u \in C^{0,1}(\mathbb{R}^{n+1} \setminus E_0)$ so that

$$
u_i \to u \tag{5.10}
$$

locally uniformly on $\mathbb{R}^{n+1} \setminus E_0$ and u is a weak solution of (5.3). Here, we may assume that ε_i is small compared to $\frac{1}{L_i}$; compare the proof of [49, Theorem 3.1].

6. General barriers and regularization error for rotationally symmetric level set IMCF. Having the estimate of Theorem 4.1 for level set PMCF in mind we would like to prove something similar for IMCF. But for IMCF the situation is different since the regularized equation (5.5) depends on the triple $(\varepsilon, \Omega_L, L)$, where Ω_L and L are coupled explicitly and ε is chosen according to Lemma 5.1, or more explicitly, according to (5.6). We derive in a first step upper and lower bounds for the solution of (5.5) depending on the data. Then we simplify the setting by assuming that E_0 is a ball and that the boundary values on $\partial\Omega_L$ coincide with (exact) IMCF starting from ∂E_0. Our general barriers imply in this special case that the regularization error is of order ε^2, which we confirm with a numerical example. We remark that in general the "regularization error" is mainly dominated by the artificial boundary values. We think that it is interesting to study the regularization error for more general E_0.

6.1. The general barriers. Let positive $\bar{L}$ and $\bar{\varepsilon}$ be given so that problem (5.5) has a solution $\bar{u} = u^{\bar{\varepsilon},\bar{L}}$. As stated above we want to estimate the regularization error, which means here to estimate

$$|u - \bar{u}|_{C^0(\bar{\Omega}_l)} \tag{6.1}$$

for some fixed $0 < l < \bar{L}$, where u is a weak solution of (5.3). Since the boundary values on ∂F_L in (5.5) are rather artificial, (6.1) can only be expected to be small for $0 < l \ll \bar{L}$. The idea to derive an estimate for (6.1) is as follows. Let $\tilde{\delta} > 0$ be arbitrary; then there exists $i = i(\tilde{\delta}) \in \mathbb{N}$ so that

$$|u - u^{\varepsilon_i,L_i}|_{C^0(\Omega_l)} \leq \tilde{\delta} \quad \text{and} \quad \varepsilon_i \ll \bar{\varepsilon} \quad \text{and} \quad L_i \gg \bar{L}. \tag{6.2}$$

The strategy is to construct an upper barrier $b_1 = b_1(\varepsilon, L)$ and a lower barrier $b_2 = b_2(\varepsilon, L)$ for (5.5) for general $\varepsilon > 0$ and $L > 0$ for which (5.5) has a solution, i.e., the barriers satisfy per definition

$$b_2(\varepsilon, L) \leq u^{\varepsilon} \leq b_1(\varepsilon, L) \tag{6.3}$$

in Ω_L. Using inequality (6.3) with the pairs $(\varepsilon, L) = (\bar{\varepsilon}, \bar{L})$ and $(\varepsilon, L) = (\varepsilon_i, L_i)$ we deduce in the cases $(\varepsilon, L) = (\bar{\varepsilon}, \bar{L})$ and $(\varepsilon, L) = (\varepsilon_i, L_i)$ bounds for $\bar{u}$ and u^{ε_i,L_i}, respectively, and combined we conclude that

$$b_2(\bar{\varepsilon}, \bar{L}) - b_1(\varepsilon_i, L_i) \leq \bar{u} - u^{\varepsilon_i,L_i} \leq b_1(\bar{\varepsilon}, \bar{L}) - b_2(\varepsilon_i, L_i) \tag{6.4}$$

in $\Omega_{\bar{L}}$.

Now we construct the barriers for general $\varepsilon > 0$ and $L > 0$ so that (5.5) has a solution. We use the ansatz $\varphi(v)$, where $\varphi \in C^\infty(\mathbb{R})$ will be chosen appropriately and v is chosen according to (5.4), i.e.,

$$v(x) = n \log |x|. \tag{6.5}$$

We have

$$D_i v(x) = n \frac{x_i}{|x|^2}, \quad D_i D_j v(x) = n \left(\frac{\delta_{ij}}{|x|^2} - 2\frac{x_i x_j}{|x|^4} \right) \tag{6.6}$$

and noting that $x \in \mathbb{R}^{n+1}$

$$\Delta v(x) = \frac{n(n-1)}{|x|^2}. \tag{6.7}$$

Setting $r = |x|$ and $w = (|\varphi'|^2 |Dv|^2 + \varepsilon^2)^{\frac{1}{2}}$ we have

(6.8)
$$\begin{aligned} E^{\varepsilon}\varphi(v) &= \operatorname{div}\left(\frac{\varphi' Dv}{(|\varphi'|^2|Dv|^2 + \varepsilon^2)^{\frac{1}{2}}} \right) - \left(|\varphi'|^2 |Dv|^2 + \varepsilon^2 \right)^{\frac{1}{2}} \\ &= w^{-3}(\varphi''|Dv|^2 w^2 + \varphi' \Delta v w^2 - |\varphi'|^2 \varphi'' |Dv|^4 - (\varphi')^3 D_i v D_j v D_i D_j v - w^4). \end{aligned}$$

Using $w = (\frac{n^2}{r^2}|\varphi'|^2 + \varepsilon^2)^{\frac{1}{2}}$ we obtain further

(6.9)
$$\begin{aligned} E^\varepsilon\varphi(v) &= w^{-3}\left(\varphi''\frac{n^2}{r^2}w^2 + \varphi'\frac{n(n-1)}{r^2}w^2 - |\varphi'|^2\varphi''\frac{n^4}{r^4} + (\varphi')^3\frac{n^3}{r^4} - w^4\right) \\ &= w^{-3}\left(\frac{(\varphi')^3}{r^4}n^4 + \frac{\varepsilon^2}{r^2}(\varphi''n^2 + \varphi'n(n-1)) - \frac{n^4}{r^4}|\varphi'|^4 - 2\varepsilon^2\frac{n^2}{r^2}|\varphi'|^2 - \varepsilon^4\right). \end{aligned}$$

In order to give the right-hand side of (6.9) a sign the coupling of ε and L stated in (5.6) has to be made more explicit.

ASSUMPTION 6.1. *For the following we assume that*

$$\varepsilon = \alpha L^{-1} \tag{6.10}$$

with $0 < \alpha < \frac{1}{2}$.

Our aim is that the leading term of the right-hand side of (6.9) is

$$\frac{n^4}{r^4}(\varphi')^3(1-\varphi') \tag{6.11}$$

and this is enforced by exploiting the fact that all remaining terms

$$\frac{\varepsilon^2}{r^2}(\varphi''n^2 + \varphi'n(n-1) - 2n^2|\varphi'|^2 - \varepsilon^2 r^2) \tag{6.12}$$

have the factor ε^2. We will choose φ with $|\varphi''| < 1$ and $0 < |1-\varphi'| = \delta < 1$ so that the term (6.11) becomes leading. This can be achieved by using on the one hand that

$$|(\varphi')^3(1-\varphi')| \geq (1-\delta)^3\delta \tag{6.13}$$

and on the other hand that

$$\frac{r^2}{n^4}\varepsilon^2(\varphi''n^2 + \varphi'n(n-1) - 2n^2|\varphi'|^2 - \varepsilon^2 r^2) \leq 7\frac{\alpha^2}{n^2}. \tag{6.14}$$

Namely, if we couple now α and δ so that

$$(1-\delta)^3\delta \geq 7\frac{\alpha^2}{n^2}, \tag{6.15}$$

then we can put this together to an inequality between the left-hand sides of (6.13) and (6.14) and the leading term of (6.9) is then (6.11). In order to specify concrete barriers we need the following assumption.

ASSUMPTION 6.2. *We assume that*

$$\partial E_0 \subset B_{r_2}(0) \setminus B_{r_1}(0) \tag{6.16}$$

with appropriate $0 < r_1 < r_2$.

Our ansatz for the upper barrier is $\varphi_1(v)$ and for the lower barrier $\varphi_2(v)$ with φ_i, $i = 1, 2$, a linear function with slope $s_i = 1 + (-1)^{i+1}\delta$ which lies above (case $i = 1$), respectively, below (case $i = 2$), and touches the line segment which connects the points $(n\log r_i, 0)$ and $(L, L-2)$ and δ satisfies (6.13). From a comparison principle we know that these barriers are bounds for u^ε from above and below in Ω_L; furthermore, these bounds are obtained explicitly for given data L, α as follows. For the upper bound we obtain

$$\varphi_1(x) = \begin{cases} s_1(x - n\log r_1) & \text{if} \quad s_1 > \frac{L-2}{L-n\log r_1}, \\ s_1(x-L) + L - 2 & \text{else,} \end{cases} \tag{6.17}$$

where $x \in \mathbb{R}$. Analogously, we have for the lower bound

$$\varphi_2(x) = \begin{cases} s_2(x - n\log r_2) & \text{if} \quad s_2 < \frac{L-2}{L-n\log r_2}, \\ s_2(x-L) + L - 2 & \text{else.} \end{cases} \tag{6.18}$$

These barriers provide good estimates in Ω_l, $0 < l \ll L$ if L is large, α sufficiently small, and $r_i < 1$, $i = 1, 2$, are both close to 1. Then especially the initial hypersurface has "small oscillation." Note that since the boundary values $L - 2$ of u^ε on ∂F_L are rather artificial we also expect good estimates only in Ω_l (and not in Ω_L). Summarizing these considerations leads to the following theorem.

THEOREM 6.3. *Let Assumption* 6.2 *be valid and let triple* (ε, δ, L) *satisfy Assumption* 6.1 *together with inequality* (6.15). *Then defining*

$$b_i := \varphi_i \circ v, \quad i = 1, 2, \tag{6.19}$$

with φ_i *as in* (6.17), (6.18) *yields an upper barrier* b_1 *and a lower barrier* b_2 *for* (5.5).

This theorem, (6.2), and (6.4) can be combined to an estimate of the regularization error.

COROLLARY 6.4. *Let* $\tilde{\delta} > 0$ *be sufficiently small and the pair* (ε_i, L_i) *so that* (6.2) *holds. If there exist positive* $\bar{\delta}$, δ_i *so that both triples* $(\varepsilon, \delta, L) = (\bar{\varepsilon}, \bar{\delta}, \bar{L})$ *and* $(\varepsilon, \delta, L) = (\varepsilon_i, \delta_i, L_i)$ *satisfy the assumptions of Theorem* 6.3 *yielding pairs of barriers* $b_r(\bar{\varepsilon}, \bar{L})$, $b_r(\varepsilon_i, L_i)$, $r = 1, 2$, *then we have*

$$b_2(\bar{\varepsilon}, \bar{L}) - b_1(\varepsilon_i, L_i) - \tilde{\delta} \le \bar{u} - u \le b_1(\bar{\varepsilon}, \bar{L}) - b_2(\varepsilon_i, L_i) + \tilde{\delta}. \tag{6.20}$$

In the notation of the corollary we dropped the implicit dependence of the barriers on δ_i, $\bar{\delta}$. In the next section we derive concrete barriers in a special case which illustrates the approximation error due to the regularization parameter ε.

6.2. Discussion of the barriers for a special case. We apply the barriers from the previous section to the special situation when E_0 is a ball with center in the origin, radius r, and boundary values on $\partial\Omega_L$ given by the exact level set IMCF. We choose φ_i, $i = 1, 2$, as a linear function with slope $1 + (-1)^{i+1}\delta$ which lies above (case $i = 1$), respectively, below (case $i = 2$), and touches the line segment which connects the points $((n-1)\log r, 0)$ and $(L, \tilde{L})$, where $\tilde{L} \in \mathbb{R}$ is suitable (i.e., equal to the "arrival time" of IMCF at the boundary of $B_L(0)$) and δ satisfies (6.15). It is clear that this yields a regularization error which is "purely due to ε" and which is given by ε^2. To see this note that δ involved in the definition of the slopes of the barriers is of size $\delta \approx \alpha^2 \approx \varepsilon^2$.

In the implementation we solved the following equivalent one-dimensional problem on the interval $[0, \log L]$ given by

$$\frac{(\varphi')^3}{e^{4x}} + \frac{\varepsilon^2}{e^{2x}}\varphi'' - \frac{1}{e^{4x}}|\varphi'|^4 - 2\varepsilon^2\frac{1}{e^{2x}}|\varphi'|^2 - \varepsilon^4 = 0 \tag{6.21}$$

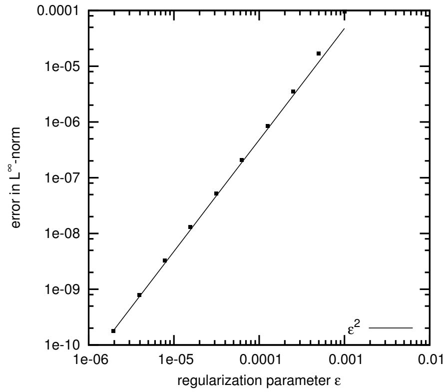


FIG. 5. *Regularization error in case of a circle as initial curve.*

with boundary values $\varphi(0) = 0$, $\varphi(\log L) = \log L$. For $\varepsilon = 0$ the function $\varphi(x) = x$ is the solution of this equation. This is illustrated in Figure 5. Note that in our numerical example we study the regularization error in the rotationally symmetric setting which models the evolution of expanding circles. Let us compare this with the numerical examples in [37, Tests 5 and 6]. Here the authors take solely the regularized operator for IMCF, generate a right-hand side with the prescribed exact solution x^2+y^2 in the domain $(1,2)^2$, and derive an experimentally obtained rate of order $O(\varepsilon)$. As in our example for level set IMCF the singularity (i.e., the stationary point of the solution) is not in the domain $(1,2)^2$. Note also that the level sets of x^2+y^2 in $(1,2)^2$ are pieces of but not closed curves. The experimentally obtained rate is then $O(\varepsilon)$; see also our remarks at the end of section 4. That means that our example and [37, Tests 5 and 6] illustrate different (not opposed) features of the operator for regularized level set IMCF.

7. Simulations and further remarks. In this section we present some simulations in the nonrotationally symmetric case for PMCF. A short description of the implementation used for the numerical examples is presented, and finally for reasons of completeness we make a short comment on an alternative level set formulation where the level set function is time dependent and which is often used in the literature.

7.1. Simulations in a nonrotationally symmetric case for PMCF. The phenomenon of becoming round can be measured by the isoperimetrical deficit

$$l(t)^2 - 4\pi a(t), \tag{7.1}$$

where $l(t)$ denotes the length of the curve and $a(t)$ the enclosed area at time t. According to theoretical results in [65] we confirm the monotonicity of this deficit during the evolution in the special case of the ellipse with half axes equal to 1 and 2 as the initial curve; see Figure 6. Furthermore, for $\varepsilon = 0.05$, we see that with increasing k the curves transform faster into a circle (Figure 9). When comparing the exact solutions for the circle for different values of k (Figure 4 and Figure 7–8) and the approximate solutions for the ellipse with $\varepsilon = 0.15$ for different values of k, respectively, we see that the flow reaches the singularity earlier for larger k. In Figure 4 we plot a section (along the long and short half axes of the initial curve) of the solution u^ε in the case of the circle and in Figures 7–8 in the case of an ellipse as initial curve for different values of ε. Figure 9 shows level sets of $u^{0.1}$ for the case of

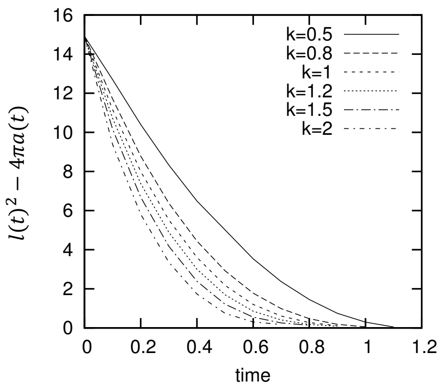
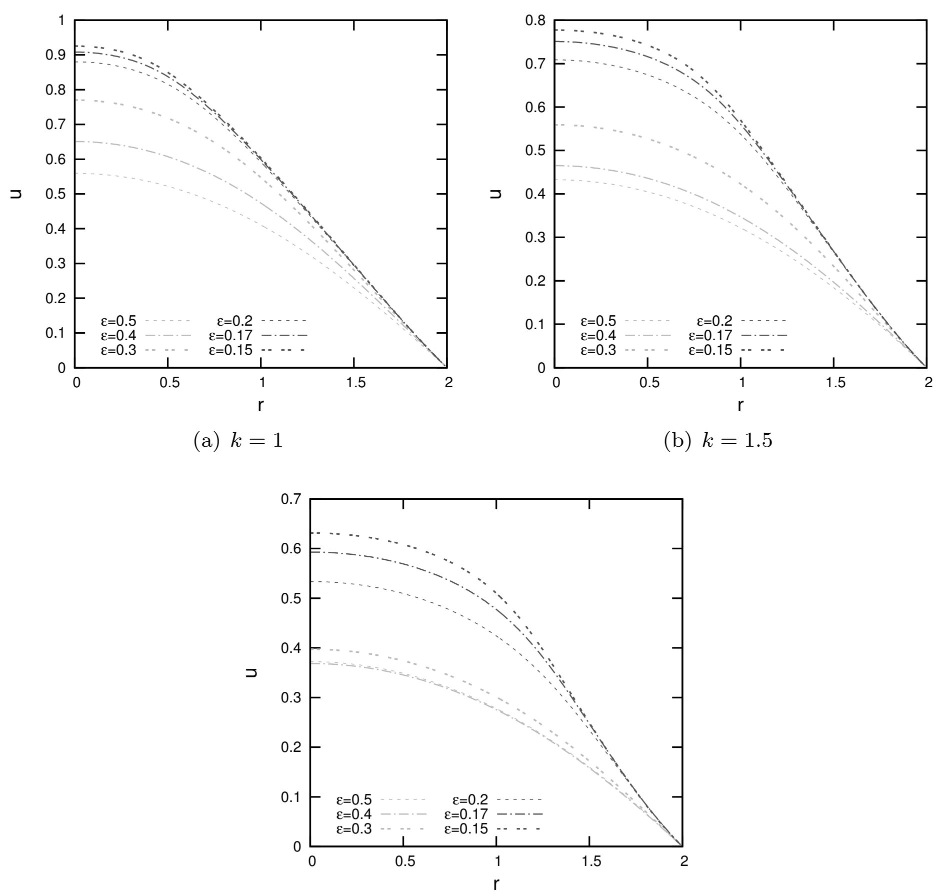


FIG. 6. *Isoperimetrical deficit.*

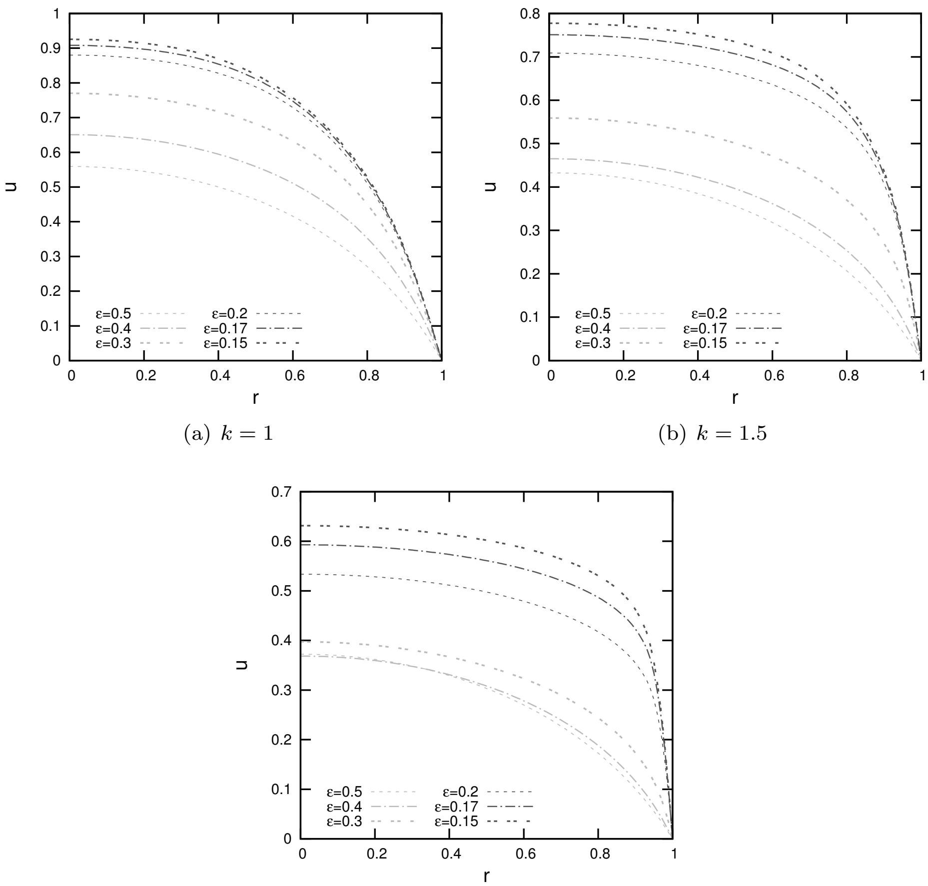


(a) $k = 1$

(b) $k = 1.5$

(c) $k = 2$

FIG. 7. *Solution for the ellipse as initial curve. Section in direction of the long half axis of the initial curve.*



(a) $k = 1$

(b) $k = 1.5$

(c) $k = 2.$

FIG. 8. *Solution for the ellipse as the initial curve. Section in direction of the short half axis of the initial curve.*

the ellipse as the initial curve and different values of k. We remark that our theory covers only the case $k \geq 1$ but there is a well-defined and well-known behavior for the flow of convex curves with speeds given by general positive powers of the curvature; see [4]. Our observations are as follows. For $k = 0.5$ we see for $\varepsilon = 0.1$ a quite good approximation of the phenomenon of shrinking to a "round point" and further lessening of ε does not show significant improvements. For all k the inner level line for $\varepsilon = 0.1$ seems to be already "round," while for $k = 2$ this seems to be far from a "point."

7.2. On the implementation. To compute the finite element approximation u_h^ε of u^ε we used a discretization with unstructured grids; see Figure 10. These were generated by the mesh generator Gmsh; see [43]. We solved the nonlinear equation (3.2) with a Newton method which uses a biconjugate gradient stabilized solver (BiCGSTAB) and SSOR preconditioning. For the implementation we used PDELab, a discretization module for solving PDEs which depends on the Distributed and Unified Numerics Environment (DUNE). As further references concerning PDELab we

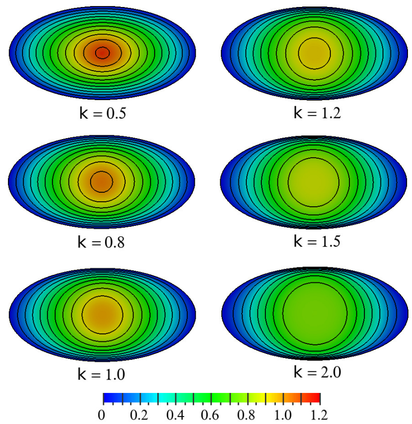


FIG. 9. *Solution for the ellipse for* $\varepsilon = 0.1$.

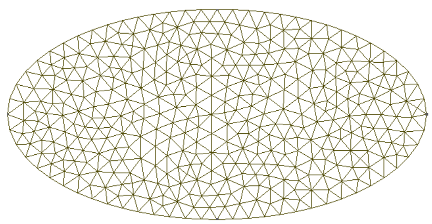


FIG. 10. *Mesh for the discretization with size* $h = 0.15$ *for the ellipse with half axes* 1 *and* 2.

refer to [62, 13]; information about DUNE can be found in [15, 11, 12, 32]. In order to get solutions for small ε we used a warm-start, i.e., we decreased ε stepwise to the desired small value and performed on each stage a computation with the solution for the previous ε as initial value.

7.3. Alternative level set formulation. For completeness we mention an alternative formulation of the motion by powers of the mean curvature which uses a level set formulation with a level set function which depends on the time. Let $M_0 \subset \mathbb{R}^{n+1}$ be a given initial hypersurface and $u_0 : \mathbb{R}^{n+1} \to \mathbb{R}$ a continuous function such that

$$M_0 = \{x \in \mathbb{R}^{n+1} : u_0(x) = 0\}. \tag{7.2}$$

Let $u : [0, \infty) \times \mathbb{R}^{n+1} \to \mathbb{R}$ be the unique viscosity solution of

$$\frac{\partial u}{\partial t} = |Du| \operatorname{div} \left(\frac{Du}{|Du|} \right)^k \tag{7.3}$$

in $\mathbb{R}^{n+1} \times (0, \infty)$ with $u(0, \cdot) = u_0$ in $\mathbb{R}^{n+1}$. Here, we only assume that $k > 0$ and we specify k in what follows where necessary. We call the family of the

$$M(t) := \{x \in \mathbb{R}^{n+1} : u(t, x) = 0\}, \quad t > 0, \tag{7.4}$$

a (time dependent) level set PMCF. Equation (7.3) is a fully nonlinear, degenerate, and possibly singular (if $Du = 0$) parabolic equation. In the case $k > 1$ the elliptic main part of (7.3) is not in divergence form and fully nonlinear in the second spatial derivatives which is of disadvantage having our finite element approach in mind. For the existence of viscosity solutions in this case see [56]. Furthermore, (7.3) is higher dimensional than the equation we used. Nevertheless, this formulation is quite common in the literature in the cases $0 < k \leq 1$ and in general also available when the speed is not necessarily positive. We give a short overview. Existence and uniqueness of a solution for (7.3) is proved in [21, 22, 36] in the case $k = 1$. In [58], (7.3) in case $0 < k \leq 1$ is approximated by a family of regularized equations and rates of convergence of the corresponding solutions are obtained. Concerning the regularization of a stationary geometric partial differential equation we also refer to [14]. The time dependent formulation (7.3) in the case $k = \frac{1}{3}$, i.e., the affine curvature equation, is used for image processing; cf. [1, 39]. In the case $k = 1$, i.e., mean curvature flow, (7.3) has been studied intensively analytically and numerically; cf., e.g., [18, 24, 26, 28, 51].

For the rest of this section we focus on some selected papers and fix therefore $k = 1$. We want to point out the paper [25] by Deckelnick, where the solution u^ε of a regularized version of (7.3) is approximated by a finite difference scheme which was originally proposed by Crandall and Lions [24]. In Deckelnick's paper, rates for the convergence of the discrete solution to the solution u of the (not regularized) level set equation are proved. The total error consists of a regularization error of the form

$$\|u - u^\varepsilon\|_{L^\infty(\Omega)} \leq c_\alpha \varepsilon^\alpha \tag{7.5}$$

with $\alpha \in (0, \frac{1}{2})$ arbitrary and c_α a positive constant (see [25, Theorem 1.2] for details) and a discretization error which is a *polynomial* expression in the discretization parameter and the reciprocal regularization parameter. Furthermore, the value for the order of convergence of the discretization error (and hence for the total approximation error) is very low. The main point here is that this is an *overall* error estimate and that the obtained rate is of polynomial order in powers of the discretization parameter and the inverse regularization parameter.

The paper [54] studies the dependence of the constants $c = c(\varepsilon)$ from ε in the error estimates for the finite element approximation of the regularized equations for level set PMCF (cf. (2.5)) and level set IMCF (cf. (5.5)). The obtained order of this dependence is an exponential expression in inverse powers of ε when a suitable coupling of the order of the finite elements and the space dimension is assumed. The method of the proof is completely different from [25] since in our case where the equation is elliptic and not parabolic this dependence is rather *implicit*. Such an exponential dependence is not unusual, as can be seen in the paper [31]. There the viscosity solution u of (7.3) is approximated by a solution u_ε of the regularized equation and then the regularized equation is approximated by a solution $u_{\varepsilon,h}$ of a semidiscrete problem. The regularization error is again of the form (7.5) but the error $u_\varepsilon - u_{\varepsilon,h}$ measured in a certain energy norm (cf. [28, Theorem 6.4]) is only of order $c_\varepsilon h$, where, and this is the important point, the constant c_ε depends *exponentially* on $\frac{1}{\varepsilon}$. Numerical tests from that reference, however, suggest that the resulting bound overestimates the error. In the special case of two dimensions, i.e., the moving hypersurfaces are curves, Deckelnick and Dziuk [31] prove L^∞-convergence (without rates) of the discrete solution provided $h = h(\varepsilon)$ is sufficiently small, where "sufficiently small" is not quantified by an explicit or even polynomial expression.

8. Three-dimensional simulations of evolving surfaces. While previous sections deal with the numerical and theoretical analysis of convergence rates and approximation rates, in this section we show a few results of evolving surfaces simulated

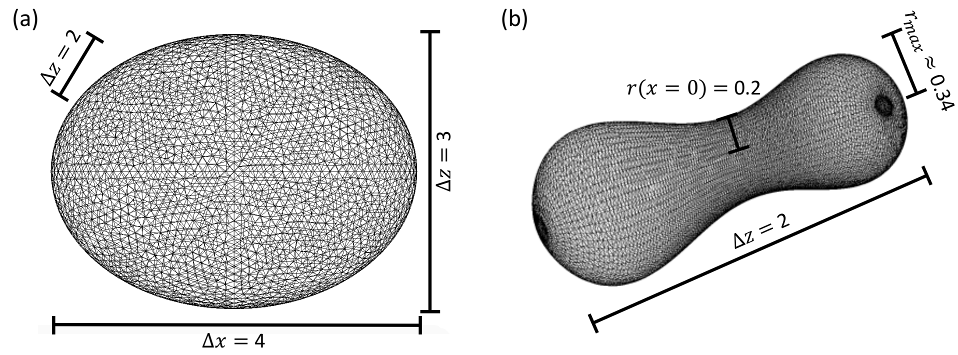


FIG. 11. *Mesh showing the discretization* (a) *of an ellipsoid (with half axes:* 2, 1.5, *and* 1*) and* (b) *of a rotationally symmetric object.*

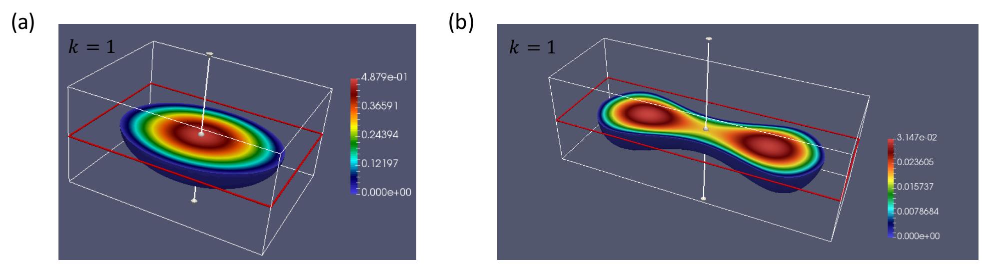


FIG. 12. *Level sets of the arrival time in the three-dimensional domain for* $k = 1$*.*

in a three-dimensional domain. Evolving surfaces are computed by solving (2.4) for the case of an ellipsoid and a rotationally symmetric surface (Figure 11) as initial condition, with the parameters set to $\varepsilon = 0.05$ and $k = 1$, $k = 1.5$, and $k = 2$. The solutions are computed iteratively by linearizing (3.2) as described in (3.26) and (3.27) and solving the resulting PDE using FreeFem [47].

The arrival time plotted along a cross section is provided in Figure 12. Figure 13 shows the arrival time along the axes of symmetry.

Appendix A. Some auxiliary observations. Since $L_\varepsilon : H_0^1(\Omega) \to H^{-1}(\Omega)$ is a topological isomorphism by the classical L^2-theory this also holds for $L_\varepsilon^* : H_0^1(\Omega) \to H^{-1}(\Omega)$. We define the uniformly elliptic, regular Dirichlet form of order 1, which is associated to L_ε, by

$$B : W_0^{1,p}(\Omega) \times W_0^{1,p^*}(\Omega) \to \mathbb{R}, \quad B[u,v] = \int_\Omega a^{ij} D_i u D_j v + b^i D_i u v \,\mathrm{d}x \tag{A.1}$$

and set

$$\begin{aligned} N_{p^*} &:= \{v \in W_0^{1,p^*}(\Omega) : B[\psi, v] = 0 \text{ for every } \psi \in C_0^\infty(\Omega)\}, \\ N_p &:= \{v \in W_0^{1,p}(\Omega) : B[v, \phi] = 0 \text{ for every } \phi \in C_0^\infty(\Omega)\}. \end{aligned} \tag{A.2}$$

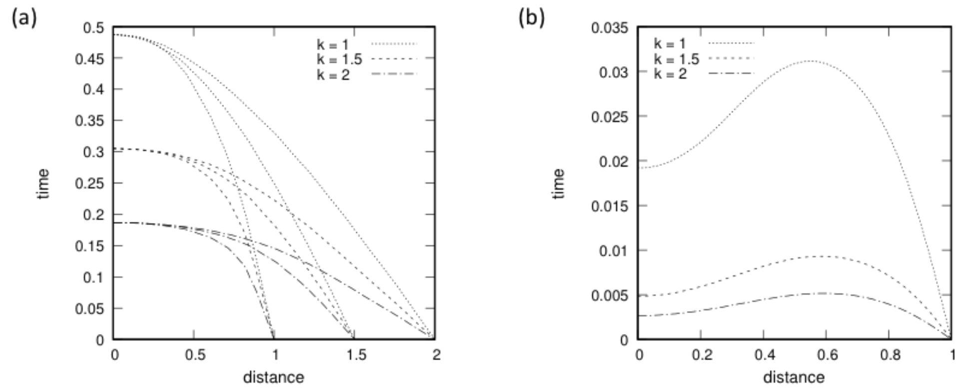


FIG. 13. *Arrival time for* $k = 1; 1.5; 2$ (a) *in the direction of the three half axes of the ellipse;* (b) *in the direction of the symmetry axis of the rotationally symmetric object.*

From Fredholm's alternative (cf. [69, Theorem 10.7]) we deduce that for every $F \in W^{-1,p^*}(\Omega)$ the equation

$$B[u,\varphi] = F\varphi \quad \text{for all } \varphi \in W_0^{1,p^*}(\Omega) \tag{A.3}$$

has a solution $u \in W_0^{1,p}(\Omega)$ if and only if

$$v \in N_{p^*} \quad \text{implies} \quad Fv = 0. \tag{A.4}$$

If $\dim N_{p^*} = \dim N_p = 0$, then for every $F \in W^{-1,p^*}(\Omega)$, (A.3) has a unique solution.

LEMMA A.1. $\dim N_{p^*} = \dim N_p = 0$.

Proof. Let $v \in N_{p^*}$. From [69, Theorem 7.6] we get $v \in W_0^{1,p'}(\Omega)$ for all $1 < p' < \infty$, especially for $p' = 2$. Since we know from L^2-theory that (A.3) has a unique solution $u \in W_0^{1,2}(\Omega)$ if $p = 2$ and $F = 0$ we deduce that $v = 0$. Analogously we obtain the remaining claim. ☐

By the bounded inverse theorem we conclude the following result.

COROLLARY A.2. $L_\varepsilon, L_\varepsilon^*$ *are topological isomorphisms.*

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