Localized modes in the nonlinear Schrödinger equation with periodic nonlinearity and periodic potential

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- **Yu. V. Bludov, VVK, Phys. Rev. A 74, 043616 (2006)**
- **Yu. V. Bludov, V.A. Brazhnyi, VVK, Phys Rev A 76 (2007) (in press; cond-mat:0706.0079)**
- **F. Kh. Abdullaev, Yu. V. Bludov, S. V. Dmitriev, P. G. Kevrekidis, VVK, (submitted; cond-mat: 0707.2512)**

Outline

- Physical applications
- Modulational instability and localized modes
- Delocalizing transition
- Lattices: "Tight-binding" approximation

EM waves in stratified media

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial z^2} + \frac{\omega^2}{c^2} n^2 E = 0,$$

where $n = n_0 + n_1(x,z) + n_2(x,z)|E|^2 + n_4(x,z)|E|^4$.

In the parabolic approximation $E(x,z) = e^{ikz} A(x,z)$, $k = \frac{\omega}{c} n_0$, $A_z \ll kA$

$$2ik\frac{\partial A}{\partial z} + \frac{\partial^2 A}{\partial x^2} + k^2 \left(\frac{2n_1}{n_0} + \frac{n_1^2}{n_0^2} + \frac{2n_2}{n_0}|A|^2 + \left(\frac{n_2^2}{n_0^2} + \frac{2n_4}{n_0} \right) |A|^4 \right) A = 0$$

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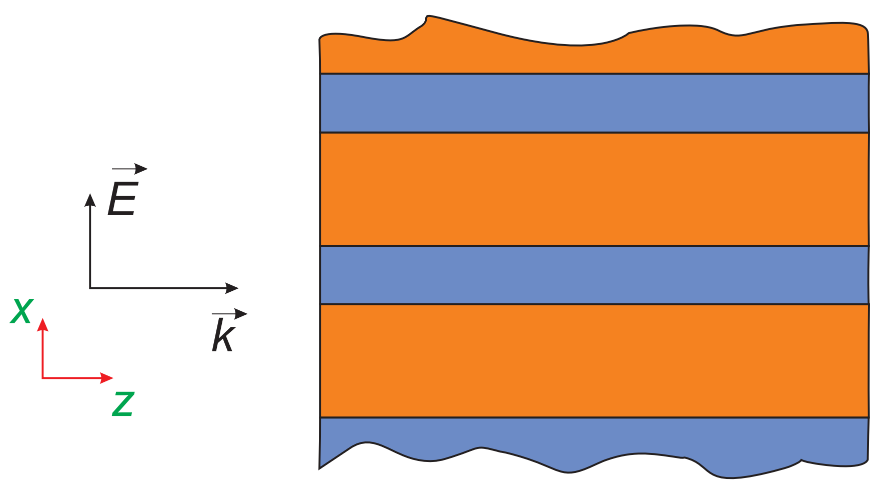

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Let $n_1(x,z) \equiv n_1(x)$, $n_2(x,z) \equiv n_2(x)$, and $n_4 \equiv 0$, then

$$2ik\frac{\partial A}{\partial z} + \frac{\partial^2 E}{\partial x^2} + 2k^2 \left(\frac{n_1(x)}{n_0} + \frac{n_2(x)}{n_0}|A|^2 \right) A = 0$$

or after renormalization $i\psi_t = -\psi_{xx} + V(x)\psi + G(x)|\psi|^2\psi$

BEC in an optical lattice

Heisenberg equation

$$i\hbar\boldsymbol{\Psi}_t = -\frac{\hbar^2}{2m}\Delta\boldsymbol{\Psi} + V_{ext}(\mathbf{r})\boldsymbol{\Psi} + g(\mathbf{r})\boldsymbol{\Psi}^\dagger\boldsymbol{\Psi}\boldsymbol{\Psi}$$

Assumption: $\boldsymbol{\Psi} = \Psi + \hat{\psi}$ with $\Psi \approx \langle N|\boldsymbol{\Psi}|N+1\rangle$

Mean-field approximation (Gross-Pitaevskii equation)

$$i\hbar\Psi_t = -\frac{\hbar^2}{2m}\Delta\Psi + V_{ext}(\mathbf{r})\Psi + g(\mathbf{r})|\Psi|^2\Psi$$

One-dimensional limit

$$i\psi_t = -\psi_{xx} + \mathcal{U}(x)\psi + \mathcal{G}(x)|\psi|^2\psi$$

[Fedichev, Kagan, Shlyapnikov, PRL, **77**, 2913 (1996); Abdullaev and Garnier, PRA **72**, 061605 (2005)]

Boson-fermion mixture in OL

Heisenberg equations

$$i\hbar\hat{\Psi}_t = -\frac{\hbar^2}{2m_b}\Delta\hat{\Psi} + V_b\hat{\Psi} + g_1\hat{\Psi}^\dagger\hat{\Psi}\hat{\Psi} + g_2\hat{\Phi}^\dagger\hat{\Phi}\hat{\Psi}\,,$$

$$i\hbar\hat{\Phi}_t = -\frac{\hbar^2}{2m_f}\Delta\hat{\Phi} + V_f\hat{\Phi} + g_2\hat{\Psi}^\dagger\hat{\Psi}\hat{\Phi}$$

Mean-field approximation: $\Psi = \langle\hat{\Psi}\rangle$, $\langle\hat{\Phi}^\dagger\hat{\Phi}\rangle = \rho_0(\mathbf{r}) + \rho_1(\mathbf{r},t)$

[Tsurumi, Wadati, J. Phys. Soc. Jap. **69** 97 (2000); Bludov, Konotop, PRA **74**, 043616 (2006)]

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m_b}\Delta\Psi + V_b(\mathbf{r})\Psi + g_{bb}|\Psi|^2\Psi + g_{bf}\rho\Psi$$

$$\frac{\partial^2\rho_1}{\partial t^2} = \nabla\left[\rho_0(\mathbf{r})\nabla\left(\frac{(6\pi^2)^{2/3}\hbar^2}{3m_f^2\rho_0^{1/3}(\mathbf{r})}\rho_1 + \frac{g_{bf}}{m_f}|\Psi|^2\right)\right].$$

$\rho_0(\mathbf{r})$ is an unperturbed (Thomas-Fermi) distribution of fermions

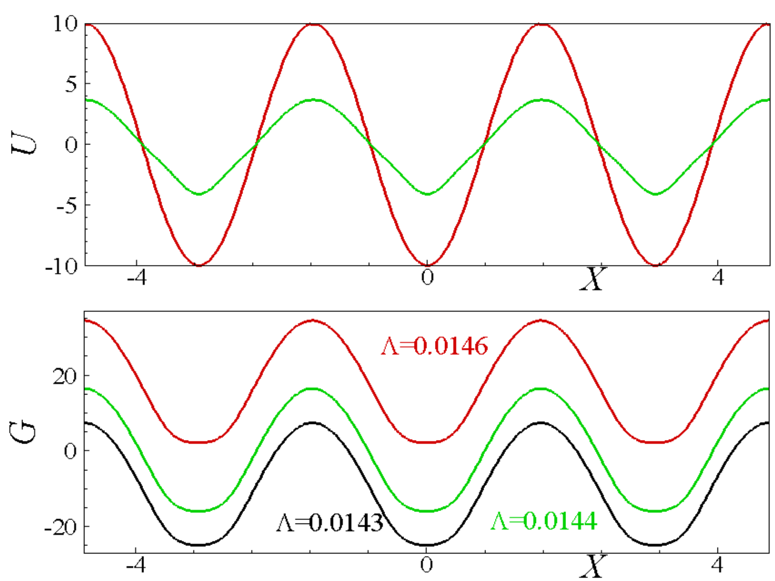
BF mixture in OL

1D limit [Bludov, Konotop, PRA **74**, 043616 (2006)]

$$i\psi_t = -\psi_{xx} + \mathcal{U}(x)\psi + \mathcal{G}(x)|\psi|^2\psi$$

$$\mathcal{U}(x) \equiv \mathcal{U}_0(x) + \mathcal{U}_1\varrho(x), \quad \mathcal{G}(x) = \mathcal{G}_0 - \mathcal{G}_1\varrho^{1/3}(x),$$

Here $\mathcal{U}_0(x) = V_b(x)/(\hbar^2\kappa^2/2m_b)$, $\mathcal{U}_1 = 4\pi\kappa a_{bf}m_b/m$,
$\mathcal{G}_0 = 4a_{bb}N_b/\kappa a^2$, and $\mathcal{G}_1 = 2\,(6/\pi)^{1/3}\,(a_{bf}/a)^2(m_f m_b/m^2)N_b$



$$\varrho(x) = \rho_0(x)/\kappa$$

$$\Lambda = \kappa a_{bb}$$

NLS equation with periodic nonlinearity

$$i\psi_t = -\psi_{xx} + \mathcal{U}(x)\psi + \mathcal{G}(x)|\psi|^2\psi,$$

$$\mathcal{U}(x) = \mathcal{U}(x+\pi), \quad \mathcal{G}(x) = \mathcal{G}(x+\pi)$$

The linear eigenvalue problem

$$-\frac{d^2\varphi_n^{(\sigma)}}{dx^2} + \mathcal{U}(x)\varphi_n^{(\sigma)} = \mathcal{E}_n^{(\sigma)}\varphi_n^{(\sigma)}, \qquad \left(\mathcal{E}_\alpha^{(-)}, \mathcal{E}_\alpha^{(+)}\right)$$ is α's gap

Multiple scale expansion: $\psi \approx \epsilon A(\tau,\xi)\varphi_n^{(\sigma)}(x)$ where $\xi = \epsilon x$ and $\tau = \epsilon^2 t$ are the slow variables, $\epsilon \sim |\mu - \mathcal{E}_n^{(\sigma)}| \ll 1$

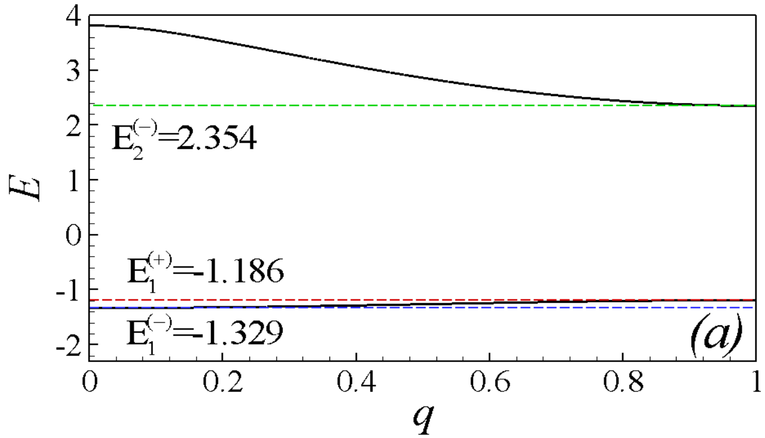
NLS equation: $iA_\tau = -(2M_n^{(\sigma)})^{-1}A_{\xi\xi} + \chi_n^{(\sigma)}|A|^2A$

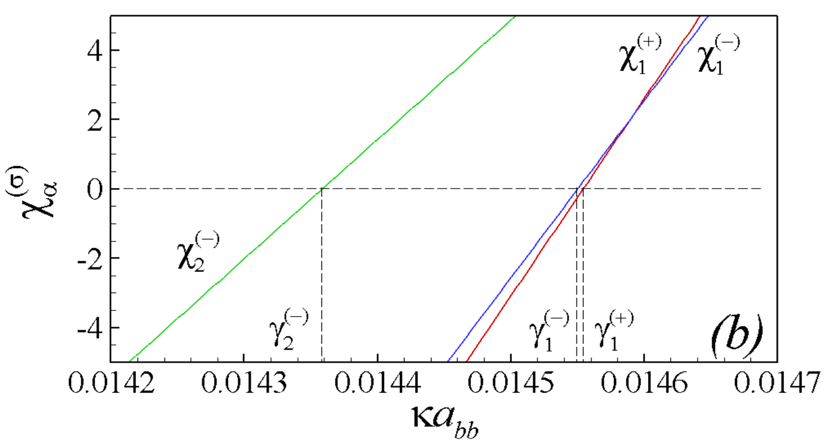
$M_n^{(\sigma)} = [d^2\mathcal{E}_n^{(\sigma)}/dk^2]^{-1}$ is the effective mass

$\chi_n^{(\sigma)} = \int_0^\pi \mathcal{G}(x)|\varphi_n^{(\sigma)}(x)|^4dx$ is the effective nonlinearity

For the modulational instability: $M_\alpha^{(\sigma)}\chi_\alpha^{(\sigma)} < 0$

NLS equation with periodic nonlinearity





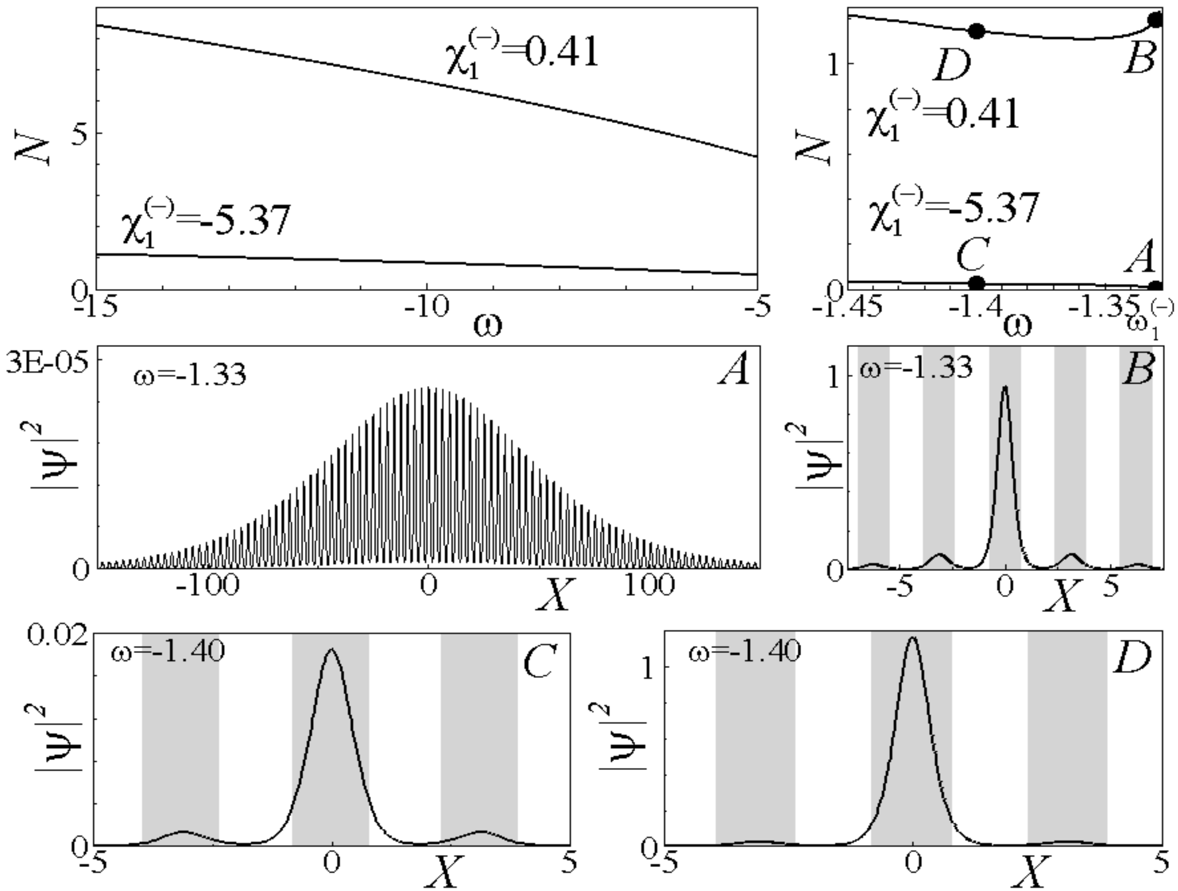
- If $\gamma_2^{(-)} < \Lambda < \gamma_1^{(+)}$: gap solitons do not exist
- Since $\mathcal{G}_m = \min \mathcal{G}(X) < 0$, in the limit $\mathcal{E} \to -\infty$, there exists a soliton:

$$\phi_S(X) \approx e^{-i\mathcal{E}T} \frac{\sqrt{2|\mathcal{E}|/|\mathcal{G}_m|}}{\cosh(X\sqrt{|\mathcal{E}|})}$$

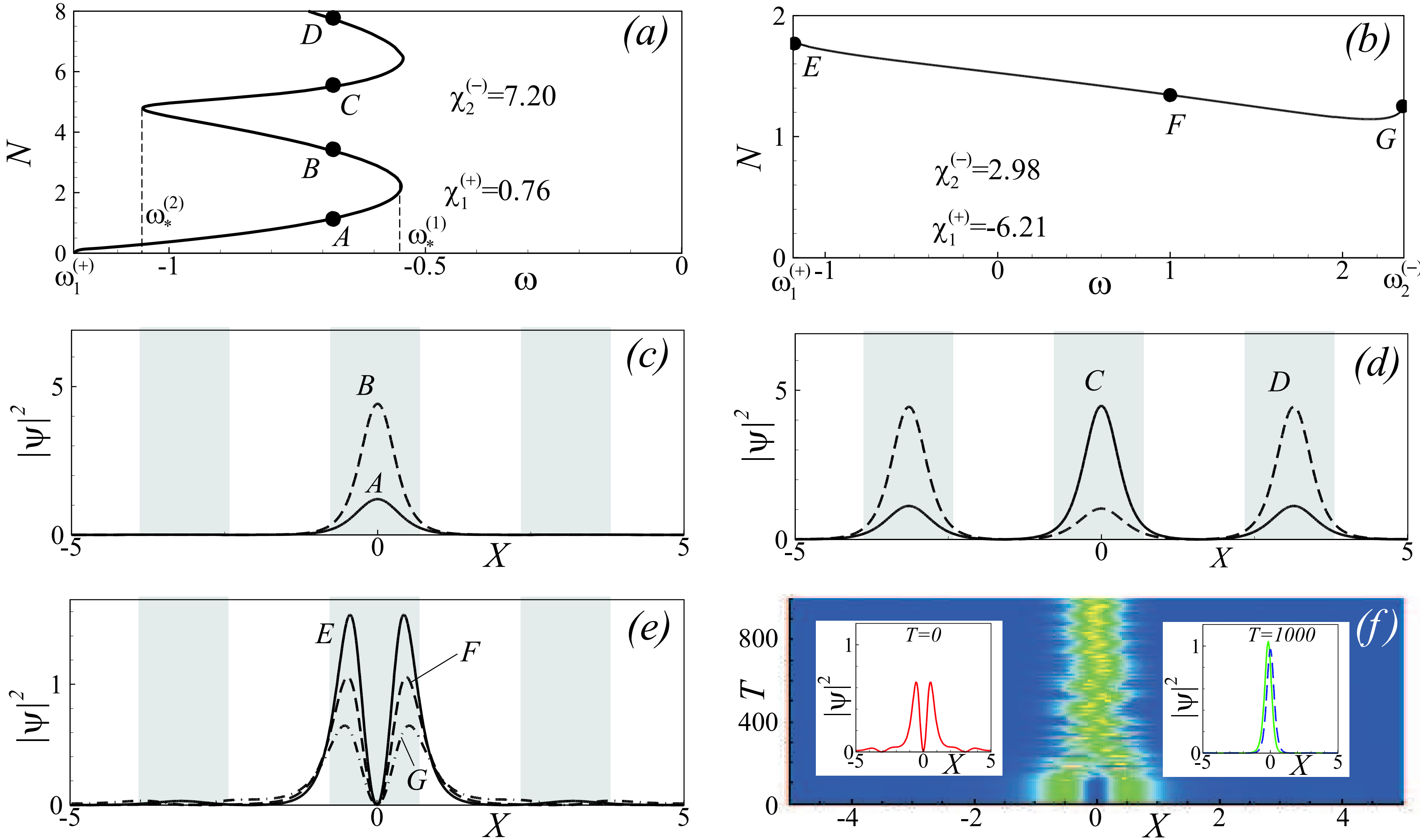
- If $\chi_1^{(-)} > 0$ in the semi-infinite band there exists a minimal number of bosons necessary for creation of a localized mode [Sakaguchi, Malomed, Phys. Rev. A 72, 046610 (2005)]

NLS equation with periodic nonlinearity

All bright localized modes are real [Alfimov, VVK, Salerno, Europhys. Lett. **58**, 7 (2002), review Brazhnyi, VVK, Mod. Phys. Lett. B **14** 627 (2004)]



NLS equation with periodic nonlinearity



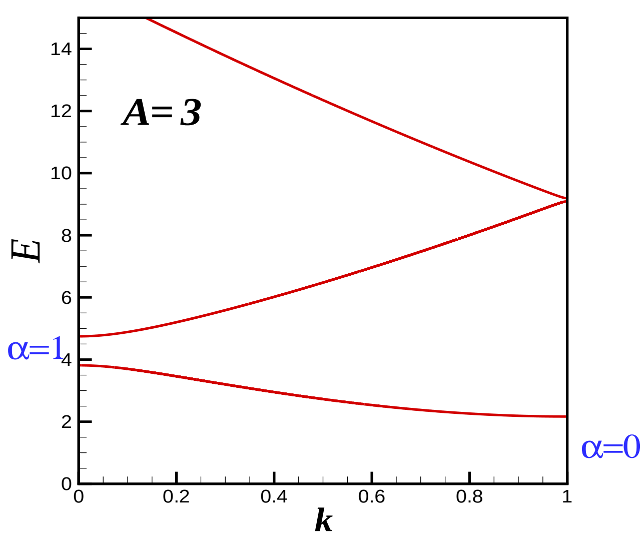
The first lowest gap

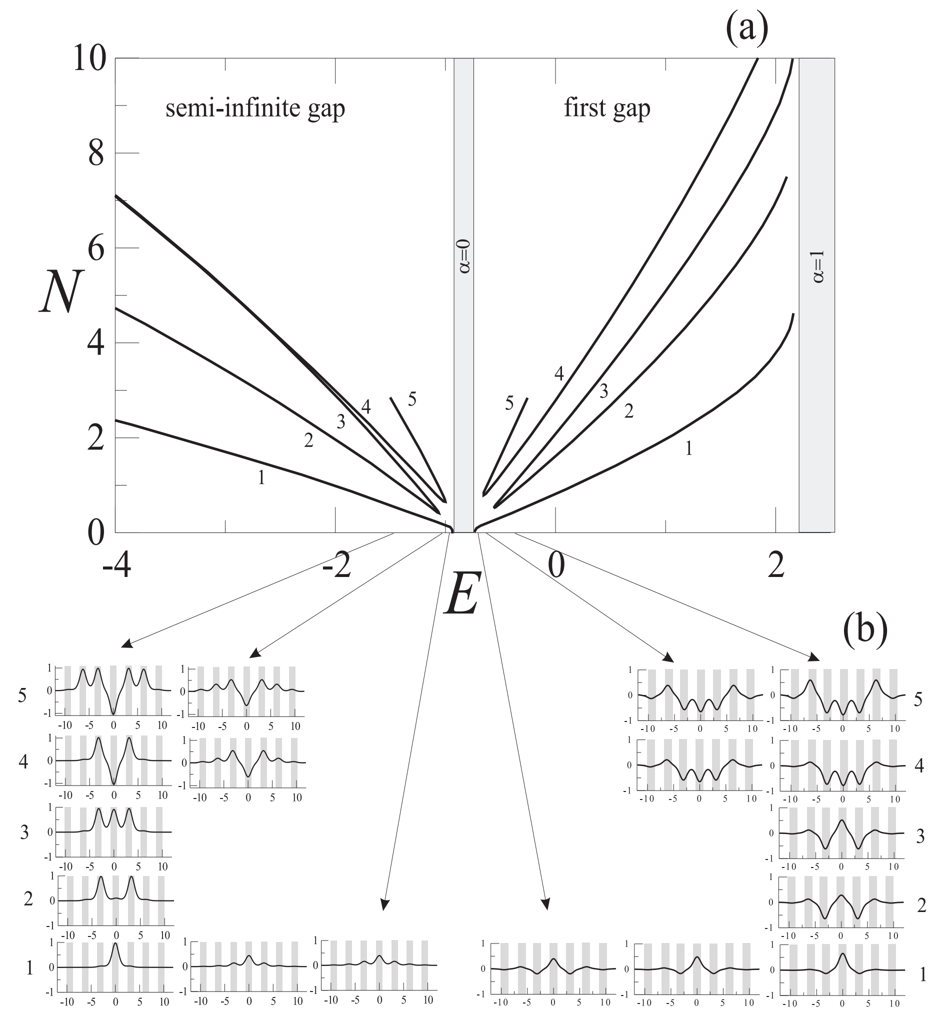
Delocalizing transition

- Decrease of the linear lattice potential in the NLS equation results in delocalizing transition in 2D and 3D, but no transition occurs in 1D. [Kalosakas, Rasmussen, Bishop, PRL **89**, 030402 (2002); Baizakov, Salerno, PRA **69**, 013602 (2004).]

$$i\psi_t = -\psi_{xx} + \mathcal{U}(x)\psi + |\psi|^2\psi$$

$$V(x) = -A\cos(2x)$$





Delocalizing transition

Consider $i\psi_t = -\psi_{xx} + \mathcal{U}(x)\psi + \mathcal{G}(x)|\psi|^2\psi$

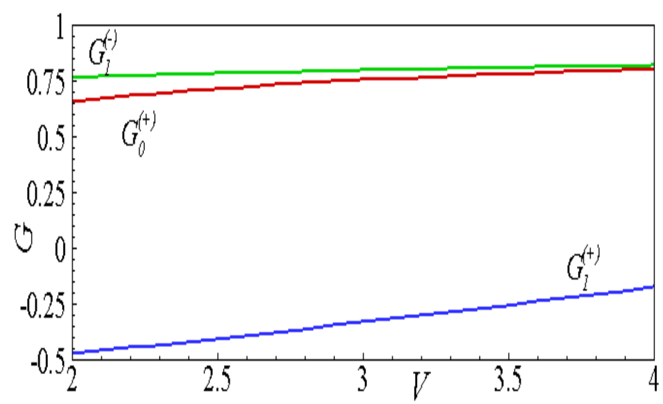
Recall $M_n^{(\sigma)} = [d^2\mathcal{E}_n^{(\sigma)}/dk^2]^{-1}$, and $\chi_n^{(\sigma)} = \int_0^\pi \mathcal{G}(x)|\varphi_n^{(\sigma)}(x)|^4 dx$
If $M_n^{(+)}\chi_n^{(+)} > 0$ and $M_n^{(-)}\chi_n^{(-)} > 0$, small amplitude solitons cannot exist at the both gap edges.

Let

$$\mathcal{U}(x) = -V\cos(2x)$$
$$\mathcal{G}(x) = G - \cos(2x)$$

A stationary solution:

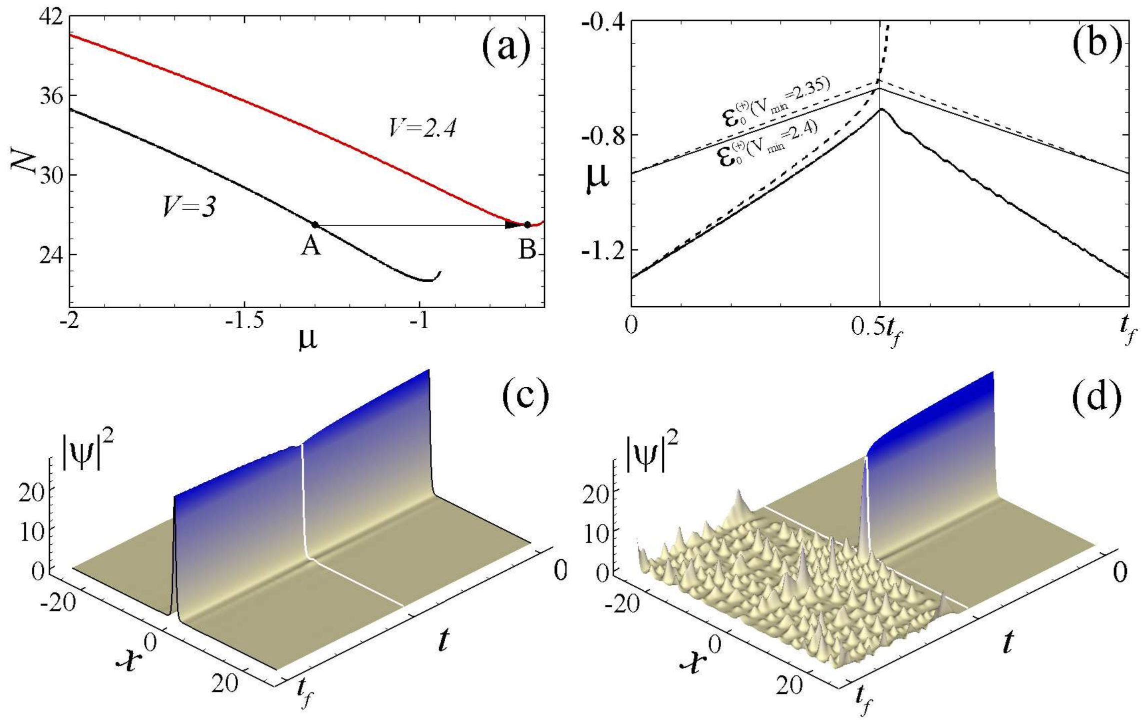
$$\psi(x,t) \to \psi(x)e^{-i\mu t}$$



$G_n^{(\sigma)} = G_n^{(\sigma)}(V)$ is the value of G at which $\chi_n^{(\sigma)}$ becomes zero.

Delocalizing transition

Let V is changing and G is fixed



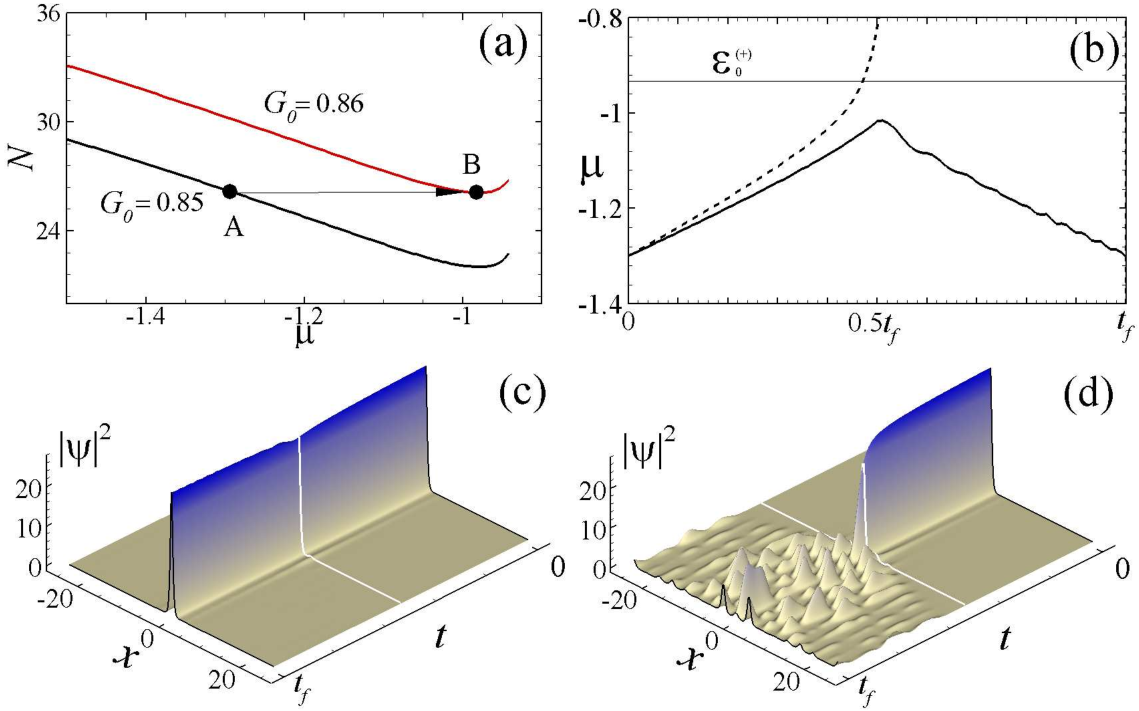
The chemical potential is $\mu = (E + E_{nl})/N$ where

$$E = \int \left(|\psi_x|^2 + \mathcal{U}(x)|\psi|^2 + \frac{1}{2}\int \mathcal{G}(x)|\psi|^4 \right) dx$$

$$E_{nl} = \frac{1}{2}\int \mathcal{G}(x)|\psi|^4 dx$$

Delocalizing transition

Let G is changing and V is fixed



Tight-binding approximation

Wannier functions

$$w_{n\alpha}(x) = \frac{1}{\sqrt{2}} \int_{-1}^{1} \varphi_{\alpha q}(x) e^{-i\pi n q}\, dq$$

The expansion: $\psi(x,t) = \sum_{n,\alpha} c_{n\alpha}(t) w_{n\alpha}(x)$ leads to

$$i\dot{c}_{n\alpha} - c_{n\alpha}\omega_{0\alpha} - \left(c_{n-1,\alpha} + c_{n+1,\alpha}\right)\omega_{1\alpha} - \sum_{n_1,n_2,n_3} c_{n_1\alpha}\bar{c}_{n_2\alpha}c_{n_3\alpha}W^{nn_1n_2n_3}_{\alpha\alpha\alpha\alpha} = 0$$

where $\mathcal{E}_{\alpha q} = \sum_n \omega_{n\alpha} e^{i\pi n q}$, $\omega_{n\alpha} = \frac{1}{2}\int_{-1}^{1} \mathcal{E}_{\alpha q} e^{-i\pi n q} dq$

$$W^{nn_1n_2n_3}_{\alpha\alpha_1\alpha_2\alpha_3} = \int_{-\infty}^{\infty} \mathcal{G}(x) w_{n\alpha}(x) w_{n_1\alpha_1}(x) w_{n_2\alpha_2}(x) w_{n_3\alpha_3}(x) dx$$

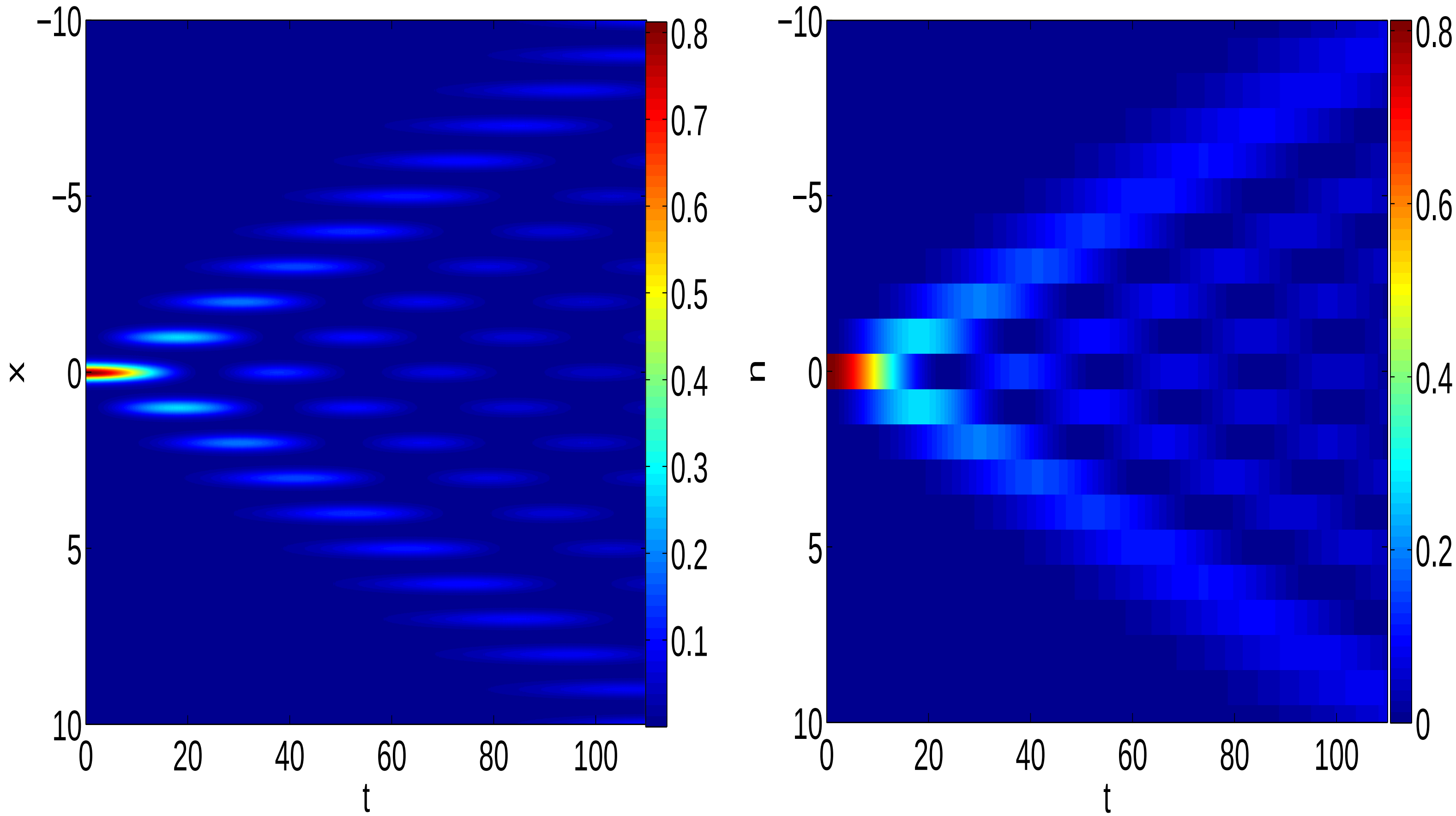
Tight-binding approximation

$$
\begin{aligned}
i\dot{c}_n &= \omega_0 c_n + \omega_1 (c_{n-1} + c_{n+1}) + W_0 |c_n|^2 c_n \\
&+ W_1 \left(|c_{n-1}|^2 c_{n-1} + \sigma \bar{c}_{n-1} c_n^2 + 2\sigma |c_n|^2 c_{n-1} \right. \\
&\left. + 2|c_n|^2 c_{n+1} + \bar{c}_{n+1} c_n^2 + \sigma |c_{n+1}|^2 c_{n+1} \right) \\
&+ W_2 \left(2|c_{n-1}|^2 c_n + \bar{c}_n c_{n-1}^2 + \bar{c}_n c_{n+1}^2 + 2|c_{n+1}|^2 c_n \right),
\end{aligned}
$$

with

$$
W_j = \int_{-\infty}^{\infty} \mathcal{G}(x) w_{1\alpha}^{j}(x) w_{0\alpha}^{4-j}(x) dx \quad j = 1, 2
$$

Tight-binding approximation



$$c_n(t) = A\,(-i)^n \exp(-i\omega_0 t)\, J_n(2\omega_1 t)$$