Multi-dimensional Compactons in Nonlinea Wave Equations

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1D Dispersive Solitary Waves

$$u_t + [u^2 + u_{xx}]_x = 0$$

The balance between the dispersion and nonlineari stabilizes localized solitons in the KdV Equation.

Dispersive Solitary Waves
Zakharov-Kuznetsov Equation

$$u_t + [u^2 + \Delta u]_x = 0$$

In 2D and 3D the dispersion is stronger than in 1I

The 2 or 3D cylindrically or spherically symmetric solitary waves are only weakly stable.

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The 2 or 3D cylindrically or spherically symmetric solitary waves are only weakly stable.

Possible solutions:

1. Weaken the dispersive forces
2. Strengthen the nonlinear forces

1D Dispersive Solitary Waves

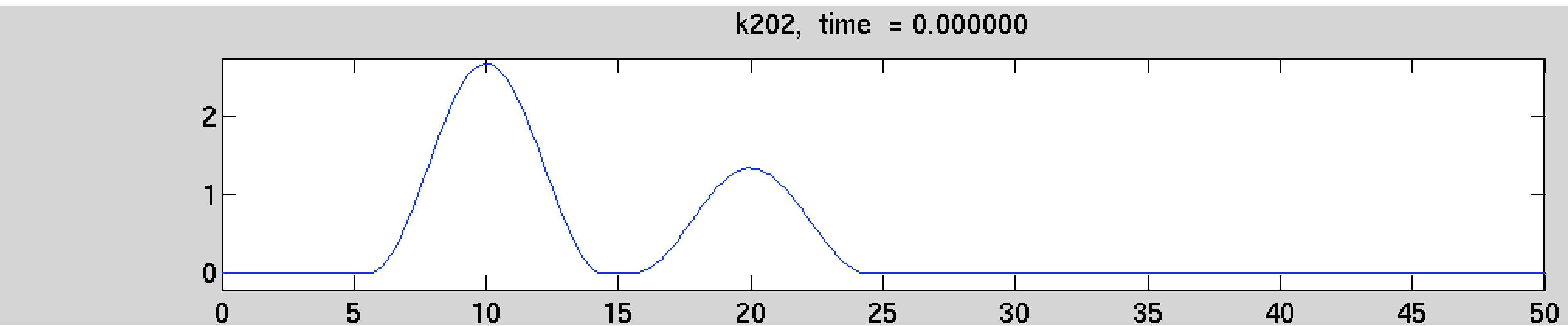
$$u_t + [u^2 + (u^2)_{xx}]_x = 0$$

The balance between the dispersion and nonlineari stabilizes localized compact solitons in the KdV compacton equation.

$$\frac{4}{3}\lambda cos^2(\frac{x-\lambda t}{4})$$

K22 Compacton Collision

$$u_t + [u^2 + (u^2)_{xx}]_x = 0$$



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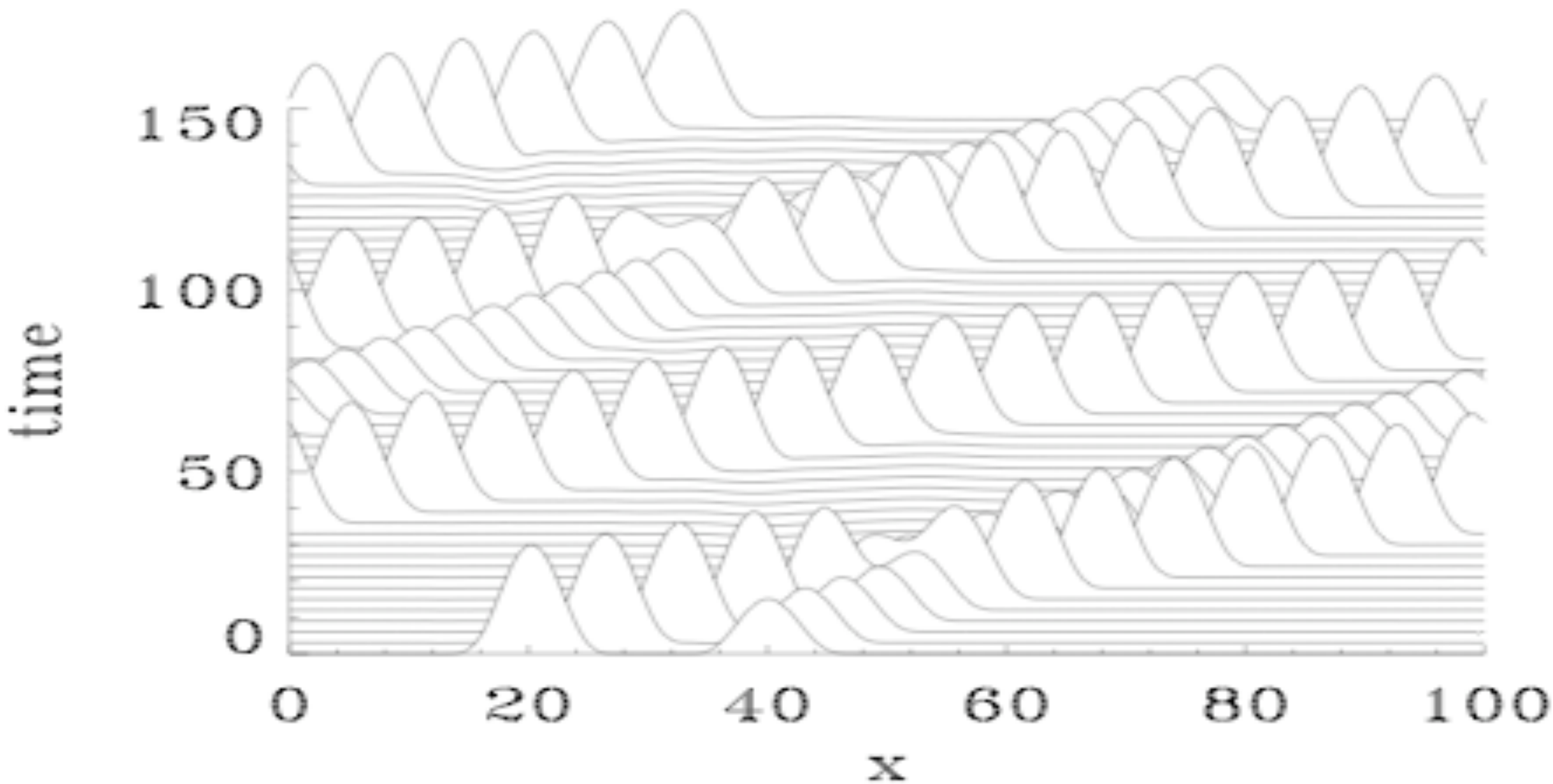
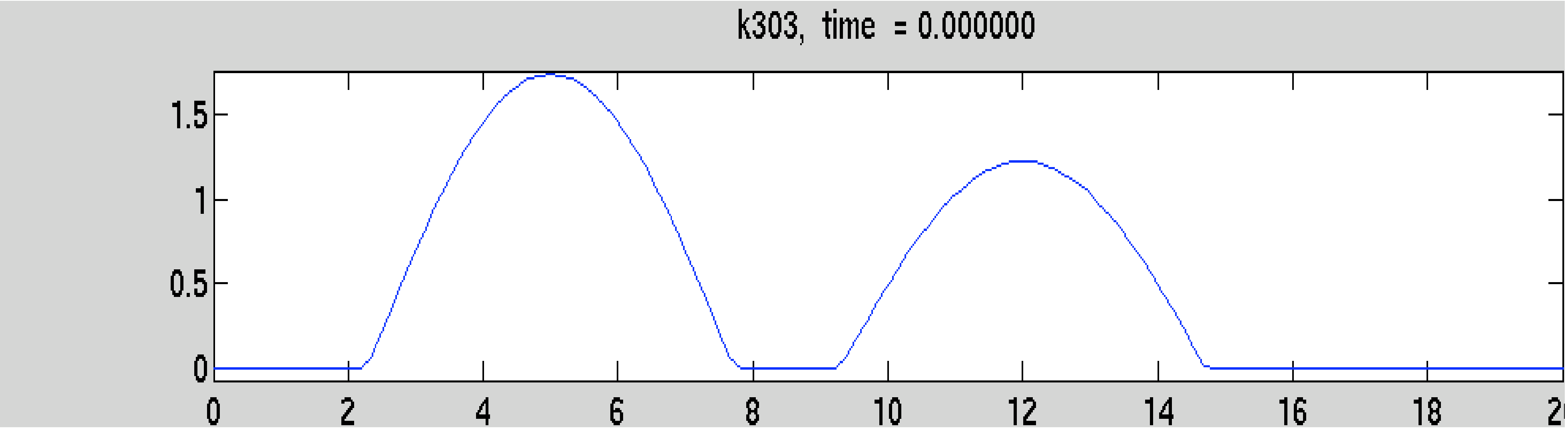


K33 Compacton Collision

$$u_t + [u^3 + (u^3)_{xx}]_x = 0$$



$$\sqrt{\frac{3\lambda}{2}} cos(\frac{x-\lambda t}{3})$$

Dispersive Solitary Waves

$$C_N(m, a+b)\textbf{:}$$

$$u_t + (u^m)_x + \frac{1}{b}[u^a(\nabla^2 u^b)]_x = 0$$

For example, the 2D and 3D symmetric solitary waves for the $C_2(2, 0+2)$ equation

$$u_t + [u^2 + \frac{1}{2}\Delta u^2]_x = 0$$

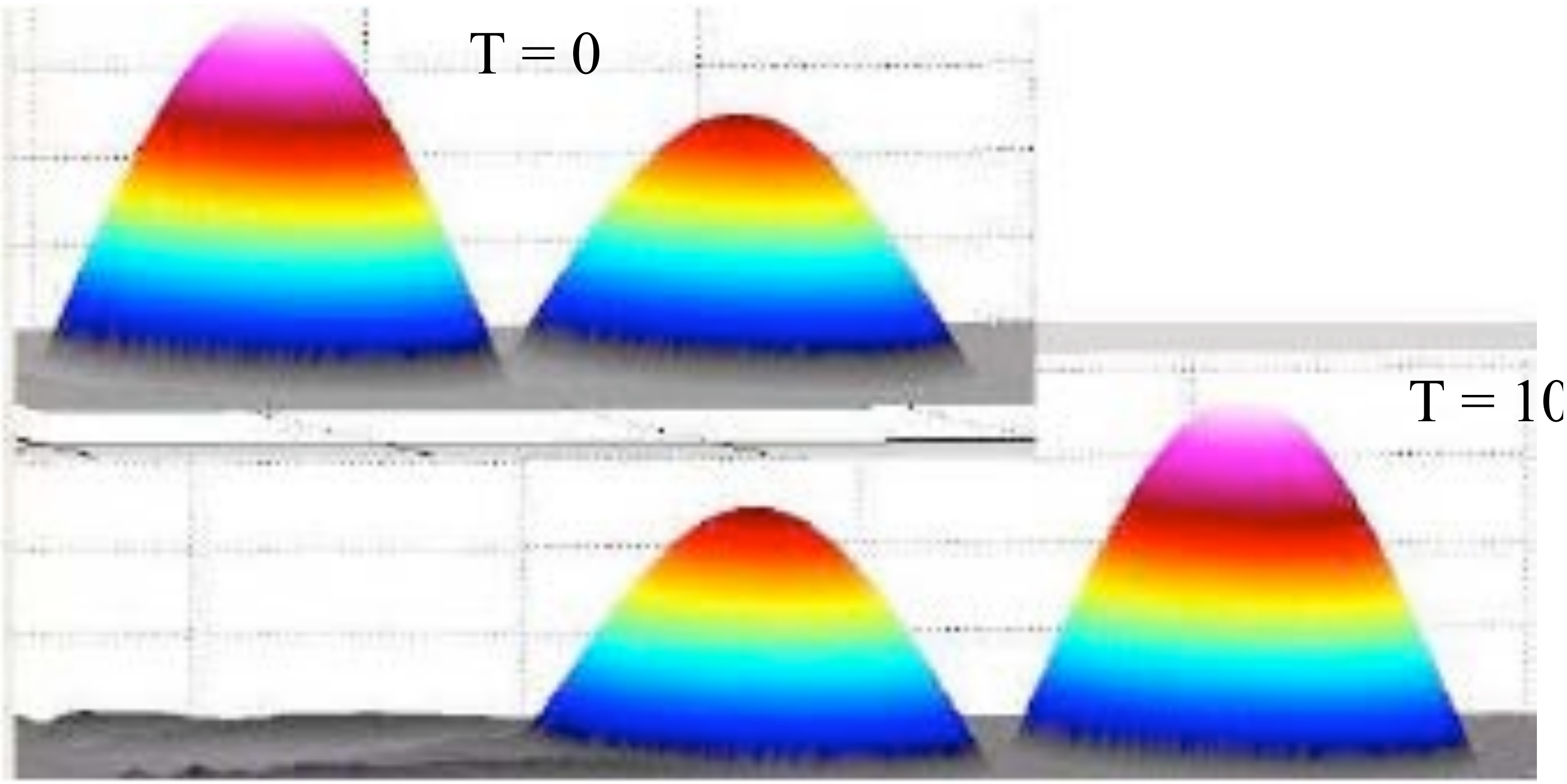
are

$$\textbf{2D:}\ u = \sqrt{\lambda[1 - cJ_0(\sqrt{2}R)]}$$

$$\textbf{3D:}\ u = \sqrt{\lambda[1 - c\sin(\sqrt{4}R)/R]}$$

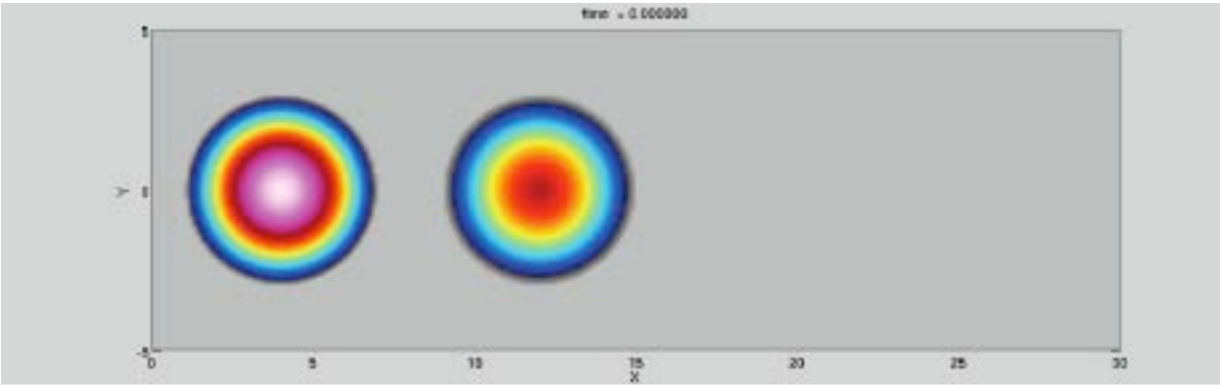
$C_2(3, 0+3)$ Solitary Waves

$$u_t + [u^3 + \frac{1}{3}\Delta u^3]_x = 0$$

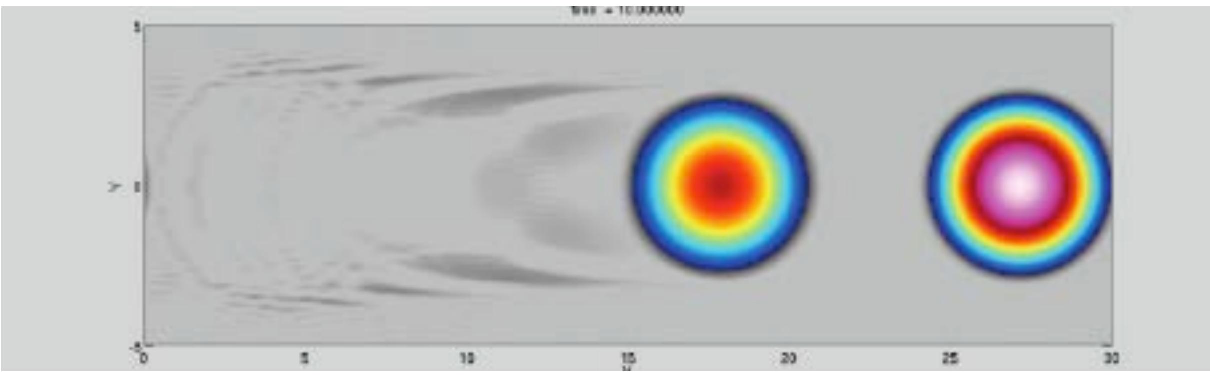


$C_2(3, 0+3)$ Hard Collision

$$u_t + [u^3 + \frac{1}{3}\Delta u^3]_x = 0$$



T = (



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Spatial Discretiztion

$$u_t + [u^2 + \frac{1}{2}\Delta u^2]_x = 0$$

M_2 and M_4 = Number of points per wavelength need for a fixed accuracy for 2-nd or 4-th order finite difference methods for linear dispersive PDEs, then

$$M_2 = 0.4 M_4^2$$

If M_4 = 50 mesh points (4th order) in each direction, then you would need M_2 = 1000 mesh points (2nd order) to achieve the same accuracy.

We use 4-th order finite difference methods

Spatial Discretiztion: Aliasing Erro

$$u_t + [u^2 + \frac{1}{2}\Delta u^2]_x = 0$$

If the solution has N nonzero Fourier modes, then th quadratic nonlinear terms create 2N nonzero Fourie modes for U_t, and hence the energy in U quickly cascades to higher and higher modes.

The numerical simulation only calculates a finite number of modes (half the number of mesh points) a the balance between dispersion and the nonlinearity lost in the highest modes. This energy in the higher modes is aliased and reappears in the lower modes.

Spatial Discretiztion: Aliasing Erro

$$u_t + [u^2 + \frac{1}{2}\Delta u^2]_x = F_{hpf}(\delta h \Delta u^2$$

δ = artificial dissipation parameter, typically order 1
h = mesh spacing
F_{hpf} = is a high pass filter in Fourier Space

$$F_{hpf} = 0 \;\; for \;\; k < \phi K_{max}$$ **and**
$$F_{hpf} = 1 \;\; for \;\; k \geq \phi K_{max}$$

The artificial dissipation on the right hand side of this equation dissipates the solution in the high modes where they are not bei calculated accurately and eliminates the dissipation in the lower

Numerical Solution: Time Integratio

$$u_t + [u^2 + \frac{1}{2}\Delta u^2]_x = 0$$

The Time Step is restricted by

Accuracy Constraints: The balance between the nonlinear inertial effects the nonlinear dispersive effects must be maintained. This balance is both critical at large values of U and as U approaches zero and restricts the time steps, even for implicit time integration methods.

Numerical Stability Constraints: The third derivative restricts explicit methods so that

$$\frac{\Delta T}{\Delta X^3} = O(1)$$

Doubling the number of spatial points in each direction requires 8 times r work in space and 8 times more integration steps = 64 times more CPU ti

Numerical Solution: Time Integratio

$$u_t + F_{lpf}\ [u^2 + \tfrac{1}{2}\Delta u^2]_x = 0$$

F_{lpf} = <u>L</u>ow <u>P</u>ass <u>F</u>ilter where the higher frequencies are reduce and the lower fraction, ϕ, of the frequencies unchanged. E.g.:

$$F_{lpf} = 1 \;\; for \;\; k < \phi K_{max} \;\; and$$
$$F_{lpf} = (1 - \alpha^2\Delta)^{-1} \ \textbf{otherwise}$$

This slows down higher wave speeds and allows a larger time ste The new time step restriction is:

$$\phi^3 \frac{\Delta T}{\Delta X^3} = O(1)$$

If the upper 1/2 of the frequencies of the time derivative are redu (ϕ=0.5) then the explicit time step can be 8 times larger

Numerical Solution Procedure

$$u_t + F_{lpf}\ [u^2 + \tfrac{1}{2}\Delta u^2]_x = F_{hpf}(\delta h \Delta u$$

High Order Finite Difference Methods in Space

High Order Spectral Dealiasing in Space

Aritificial Dissipation (HPF)

High Order Spectral Wave Regularization in Time (LPF)

Explicit High Order Time Integration Method

4th Order Adams-Bashforth-Moulton PECE meth

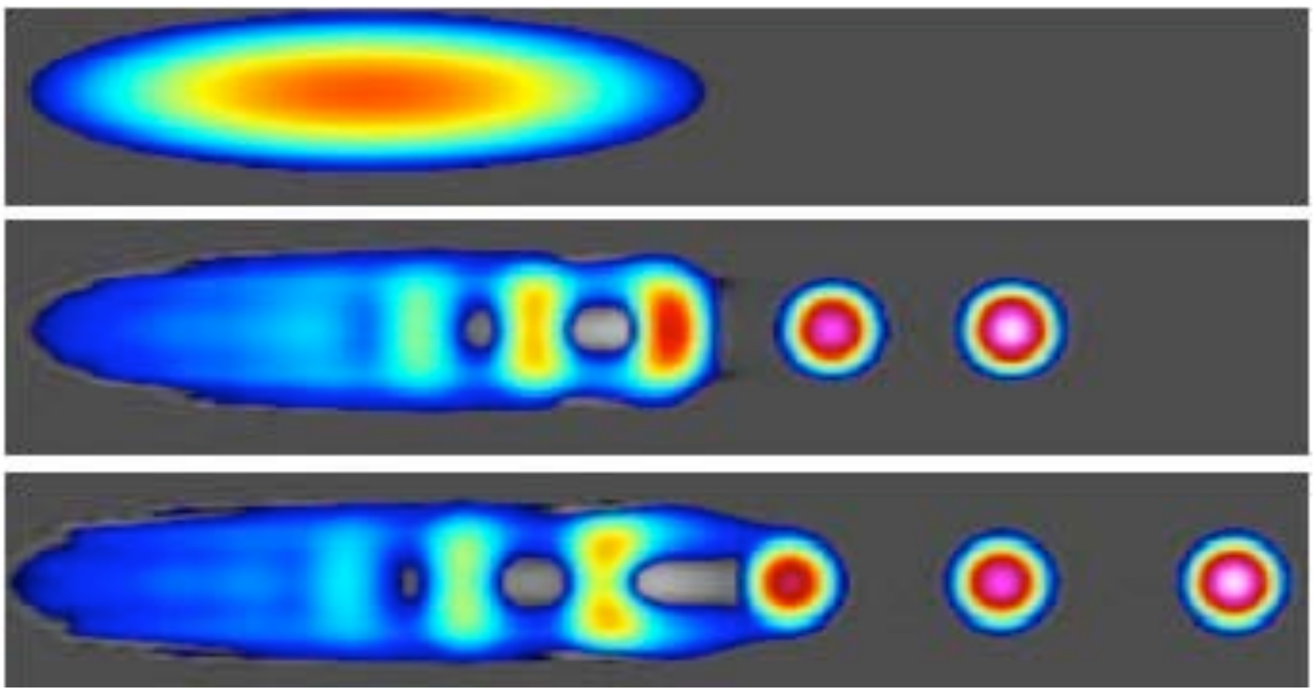
Validation and Verification:

Monitor Conservation Laws

Verify Traveling Wave Solutions

C_2(3, 1+2) Solitary Waves

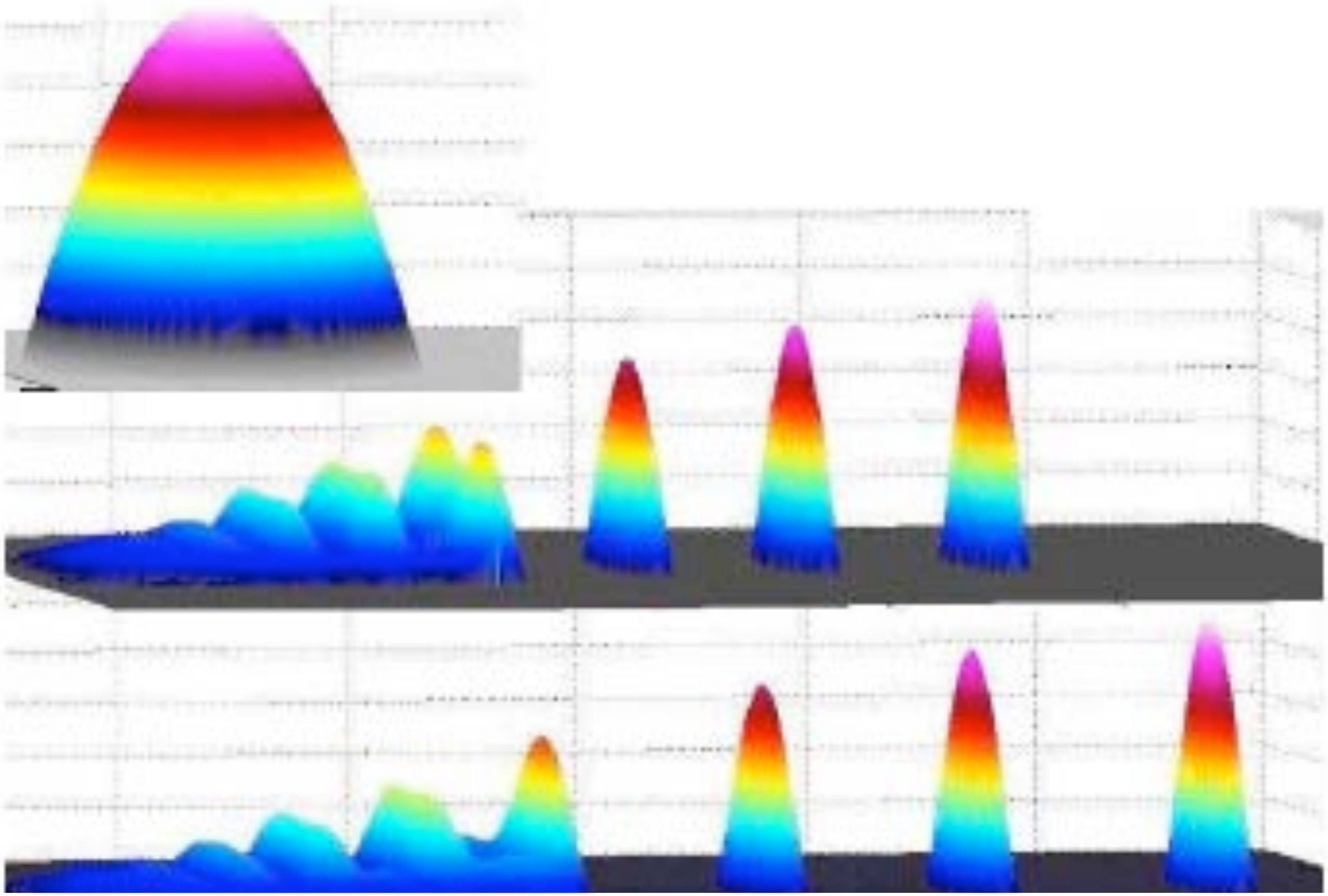
$$u_t + [u^3 + \frac{1}{2} u \Delta u^2]_x = 0$$



A series of compactons emerge from a long initial conditio

$C_2(3, 1+2)$ Solitary Waves

$$u_t + [u^3 + \frac{1}{2} u \Delta u^2]_x = 0$$



N-Dimensional Traveling Wave Compactons

$$C_N(m, a+b):$$
$$u_t + (u^m)_x + \frac{1}{b}[u^a(\nabla^2 u^b)]_x = 0$$

Spherically symmetric compactons traveling in the x direction satisfy

$$u^a[-\lambda u^{1-a} + u^{m-a} + \frac{1}{bR^{N-1}}\frac{d}{dR}R^{N-1}\frac{d}{dR}u^b] = 0$$

Where $s = x - \lambda t$ and $R = \sqrt{s^2 + y^2 + z^2}$

λ = speed

Numerical Solution: Traveling Waves

$$u^a[-\lambda u^{1-a} + u^{m-a} + \frac{1}{bR^{N-1}}\frac{d}{dR}R^{N-1}\frac{d}{dR}u^b] =$$

Solve the Boundary Value Problem for U_0 and R_{max} for $\lambda = 1$:

$U(0) = U_0$

$U'(0) = 0$

$U(R_{max}) = 0$

Regularize the equation near R=0 and U = 0

Solve for U_0 and R_{max} by shooting for Traveling Wave

Once you have solved for U_0 and R_{max} , then:

Rescale to U_0 and R_{max} for other velocities

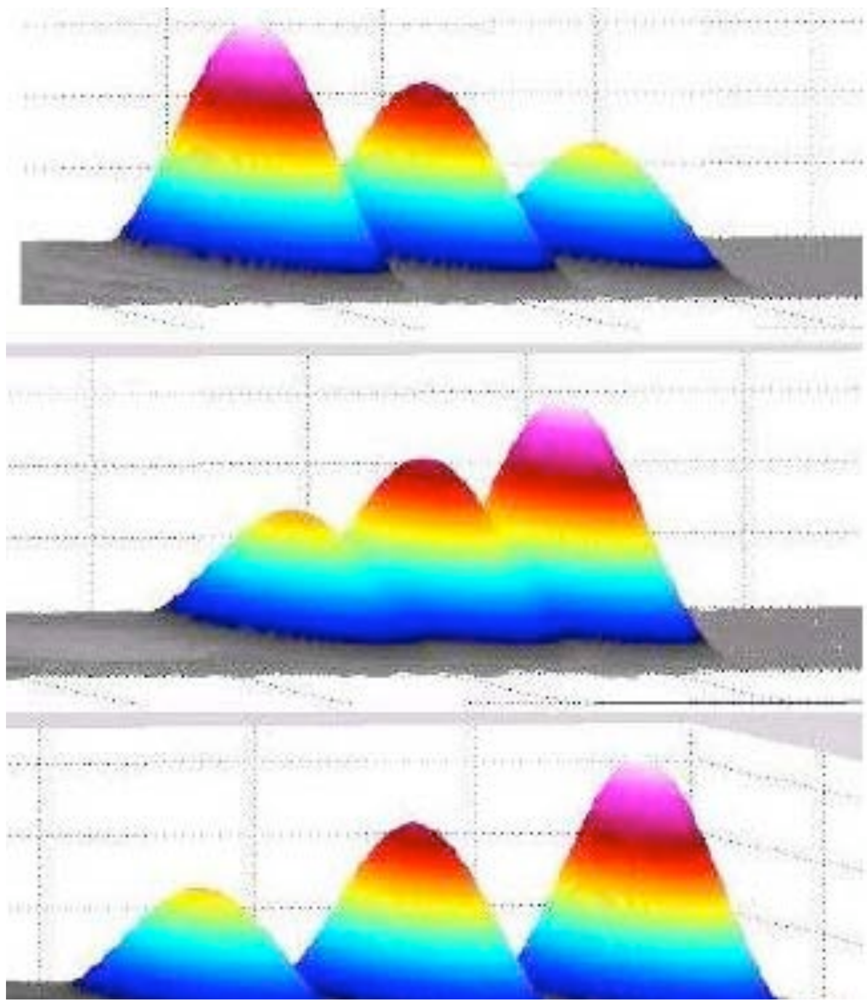
Solve and spin the solution in 2 and 3D for cylindrically and spherically symmetric compactons

$C_2(2, 0+2)$ Bessel Fcn Compactons

$$u_t + [u^2 + \frac{1}{2}\Delta u^2]_x = 0$$

$$u = \sqrt{\lambda[1 - cJ_0(\sqrt{2}R)]}$$

Traveling Wav



Initial conditi

mid collisior

post collisi

N-Dimensional Traveling Wave Compactons

$C_N(m, a+b)$:

$$u_t + (u^m)_x + \frac{1}{b}[u^a(\nabla^2 u^b)]_x = 0$$

Explicit solution for m = 1+b, a =1

$C_N(m = 1+b, 1+b)$ **solution:**

$$u = \lambda^{\frac{1}{b}}[1 - \frac{F(R)}{F(R_*)}]^{\frac{1}{b}}, \quad 0 < R \leq R_*$$

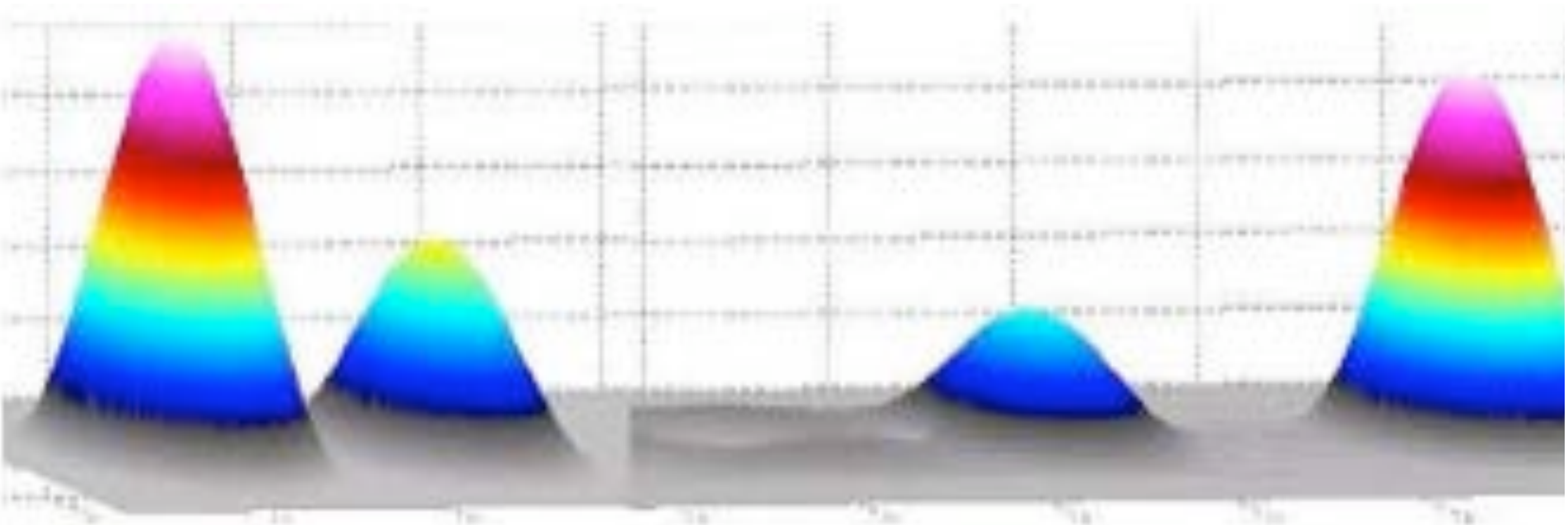
2D: $F(R) = J_0(\sqrt{b}R)$ $\qquad R = \sqrt{s^2 + y^2 + z^2}$

3D: $F(R) = \sin(\sqrt{b}R)/\sqrt{b}R$

$u = 0$ **beyond** R_***, the first trough of** $F(R)$

C_2(2, 1+1) collisions are less elastic

$$u_t + [u^2 + u\Delta u]_x = 0$$



Initial conditions

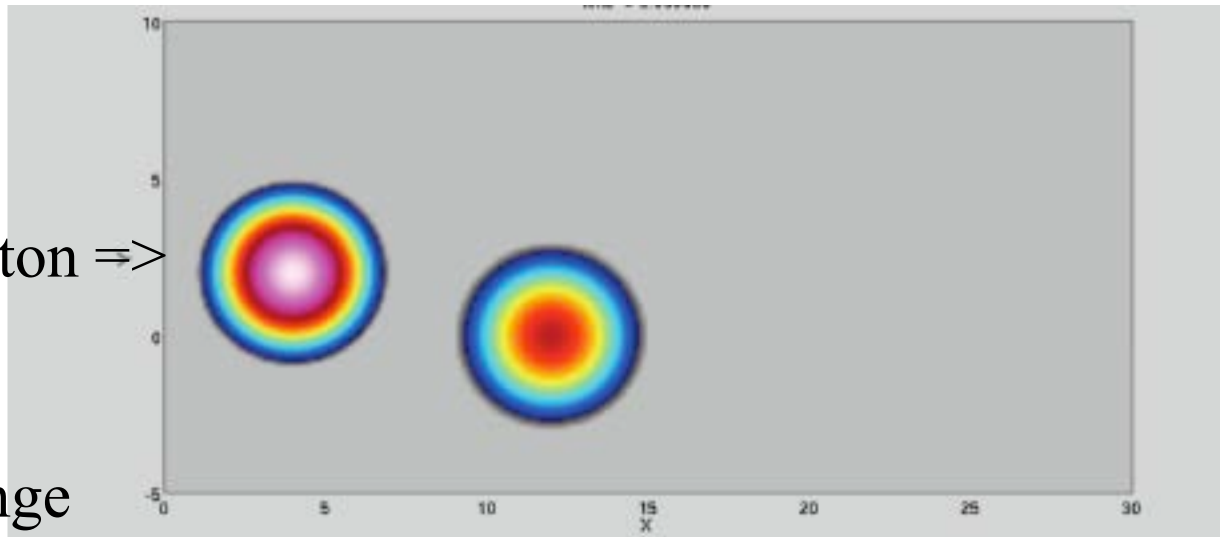
Post collision.
Note the reduced height of the slow comp.

$C_2(3, 0+3)$ Soft Collision

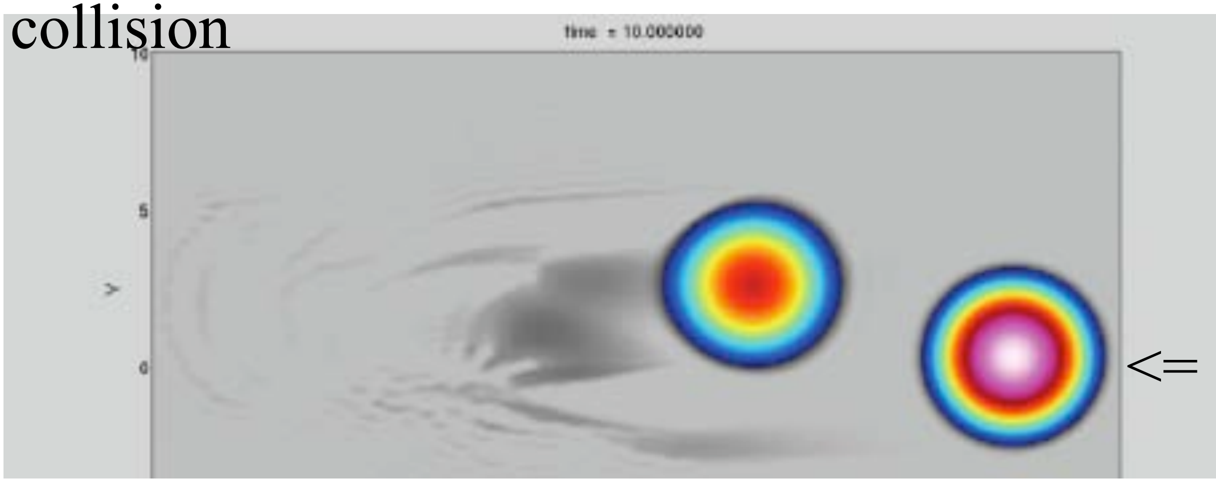
$$u_t + [u^3 + \frac{1}{3}\Delta u^3]_x = 0$$

Fast compacton =>

T = 0



Note the exchange
of energy in the collision



T = 10

<= fast compact

Traveling Wave Compactons

$C_N(m, a+b)$:

$$u_t + (u^m)_x + \frac{1}{b}[u^a(\nabla^2 u^b)]_x = 0$$

Explicit solution for m = 2, a + b = 3

$C_N(m=2, a+b=3)$ **parabolic solution:**

$$u = \kappa_N[\lambda A_N - R^2], \quad 0 < R \leq R_* = \sqrt{\lambda A_N}$$

$$R = \sqrt{s^2 + y^2 + z^2}$$

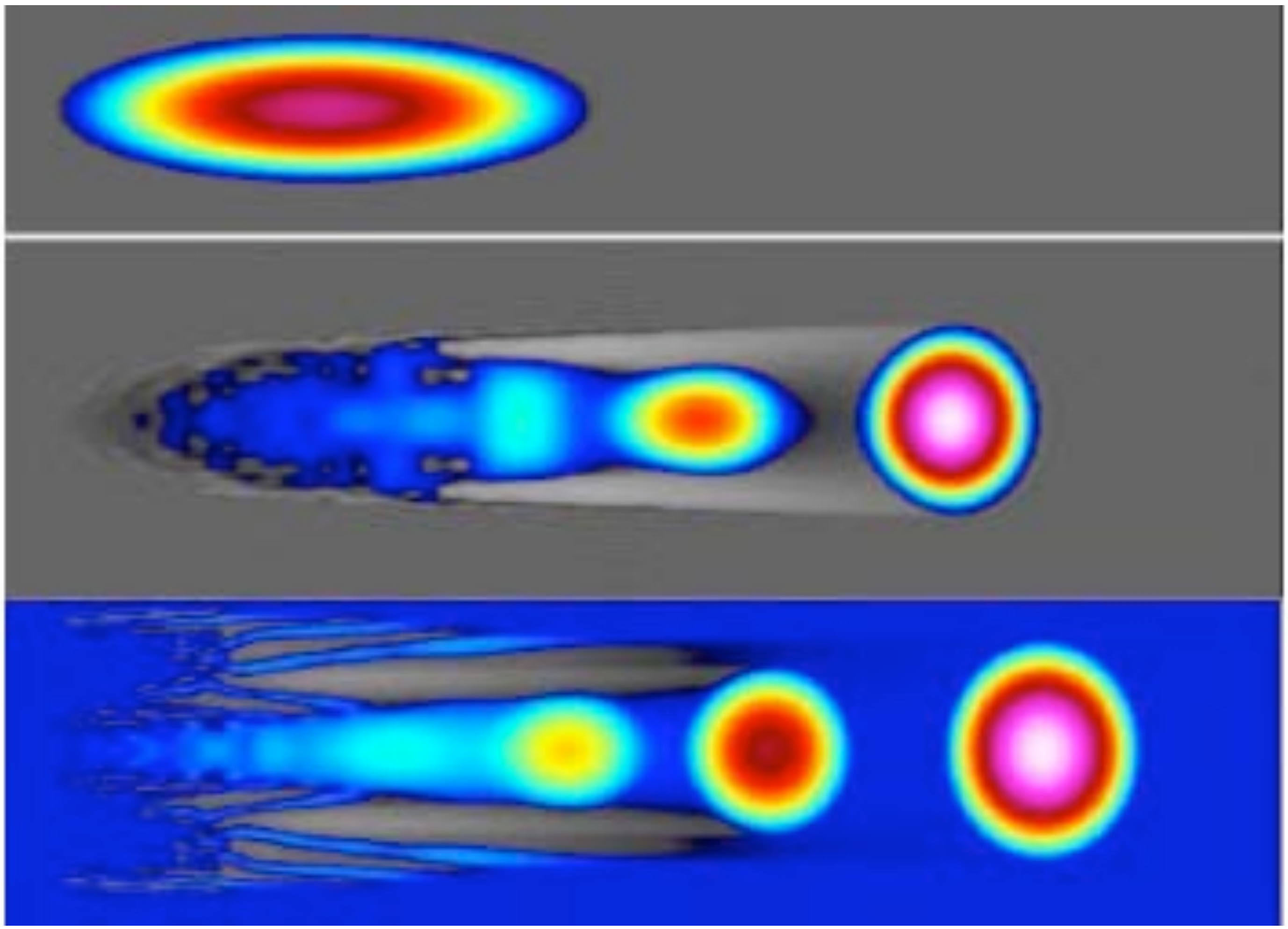
For example in 2D: $C_N(2, 0+3)$:

$A_N = \frac{3}{2}(4+N)^2$

$\kappa_N = [6(4+N)]^{-1}$

$C_2(2, 0+3)$ Parabolic Compactons

$$u_t + [u^2 + \frac{1}{3}\Delta u^3]_x = 0$$



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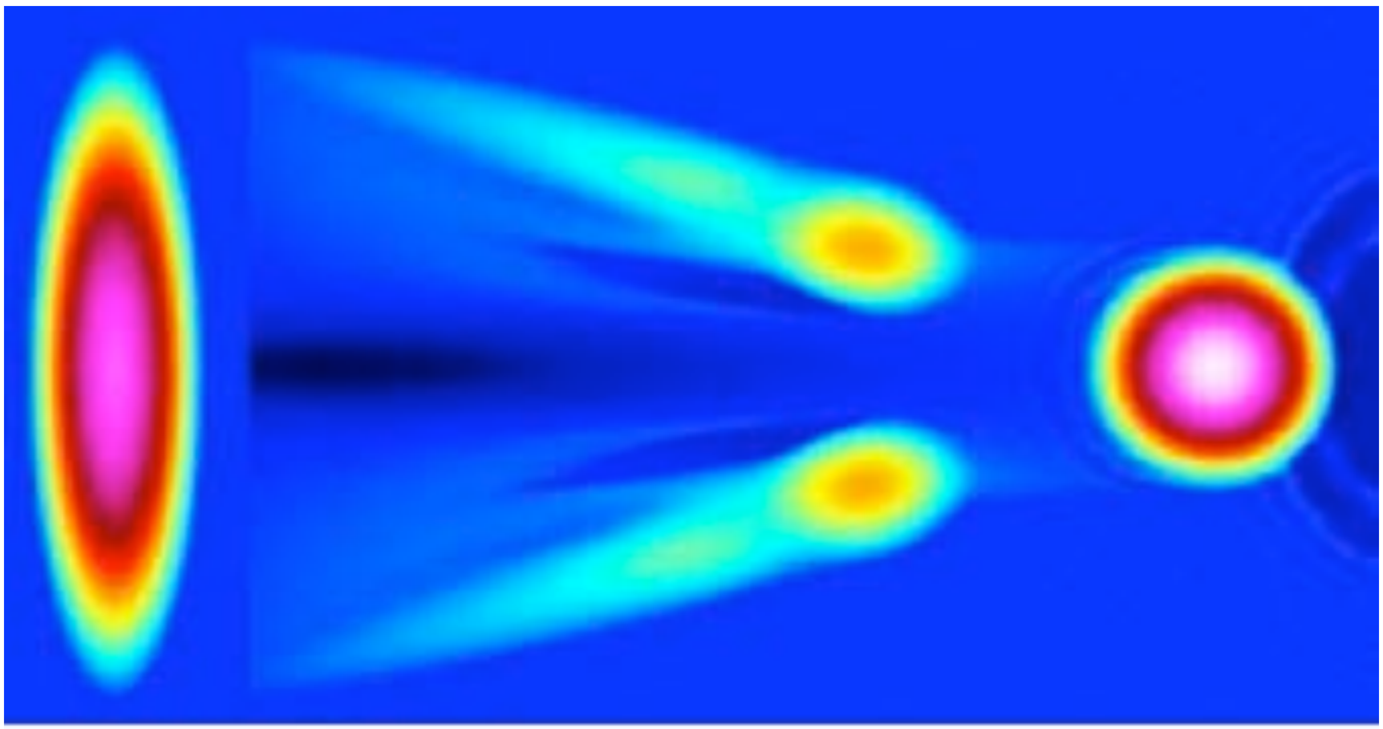
$C_2(2, 0+3)$ Parabolic Compactons

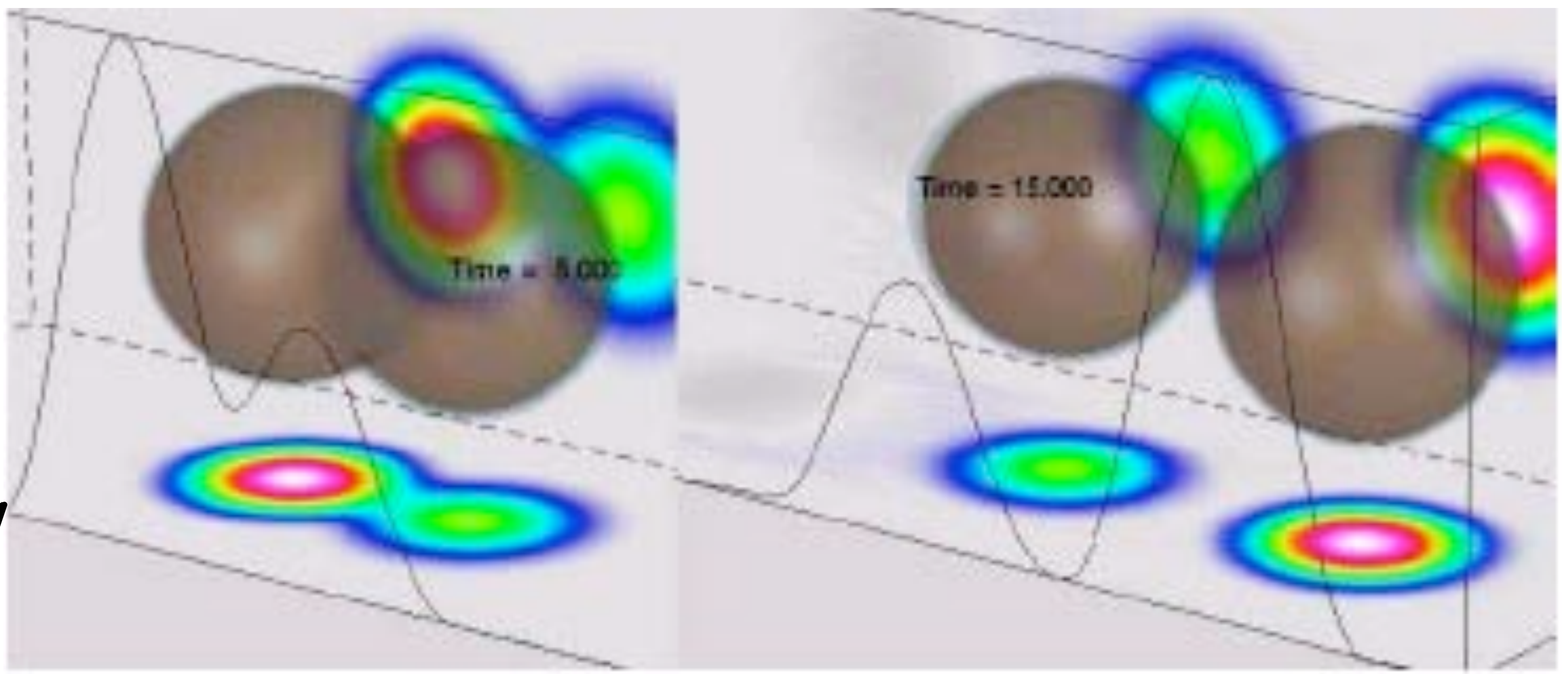
$$u_t + [u^2 + \frac{1}{3}\Delta u^3]_x = 0$$



T=0

$C_3(2, 1+2)$ Parabolic Compactons

$$u_t + [u^2 + \frac{1}{2} u \Delta u^2]_x = 0$$



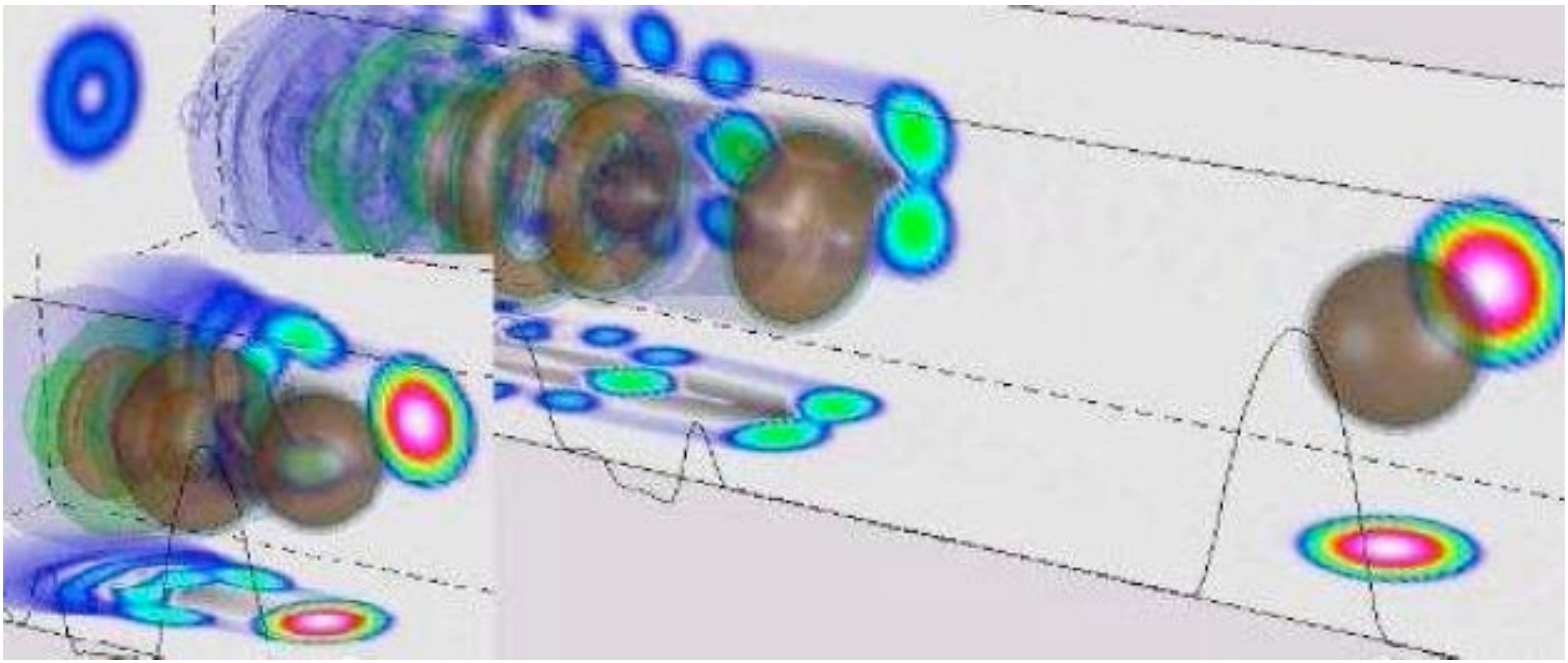
Plot down the center line

Time = 5

Time = 15

$C_3(2, 0+3)$ Parabolic Compactons

$$u_t + [u^2 + \frac{1}{3}\Delta u^3]_x = 0$$



The emergence of 3-dimensional compactons out of an initial 3-D ball wh
breaks into a sequence of rings, each which later collapses into a compact

Additional Movies available at:

http://math.lanl.gov/~mac/compacton

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