Towards the Z_3^*-Theorem

Imke Toborg, Rebecca Waldecker

Abstract.
In this article we prove a version of Glauberman's Z_3^*-Theorem for the prime 3 for finite groups G where certain local subgroups are soluble.

1. Introduction

The motivation for our work stems from Glauberman's Z^*-Theorem ([12]), published in 1966. It is a result about elements of order 2 in finite groups, and its impact on the structure theory in general and on the Classification of Finite Simple Groups (CFSG) in particular was so significant that it is a natural question to ask whether similar results hold for elements of odd prime order.
Before we state the theorem that we are ultimately interested in, we recall that some element x in a finite group G is **isolated in** G if and only if $x^G \cap C_G(x) = \{x\}$.
Then the so-called Z_p^*-Theorem is the following:

The Z_p^*-Theorem.
Suppose that G is a finite group and that p is a prime. Suppose further that $x \in G$ has order p and that x is isolated in G. Then $x \in Z_p^*(G)$.

Here the statement $x \in Z_p^*(G)$ means that x is central in G modulo $O_{p'}(G)$, where $O_{p'}(G)$ denotes the largest normal subgroup of G of order prime to p.
While the only known proof for the Z_p^*-Theorem in full generality relies on the CFSG (see for example Remark 7.8.3 on page 402 in [15]), there have been proofs for special cases that require less machinery. Work in this area has been done, for example, by Broué ([8]), using methods from representation theory. For the prime 3 there are special results by Rowley [20] and Toborg [26] relying on arguments from local group theory.
In this paper we prove a version of the Z_p^*-Theorem for the prime $p = 3$ under some solubility assumptions for local subgroups, and we only state the full result for motivation and because we want to give meaning to the expression that "the Z_p^*-Theorem holds" in some group.
The strategy with which we ultimately plan to prove the general Z_p^*-Theorem will probably invoke some so-called $\mathcal{K}$-group-hypothesis. This means that we might have to suppose that some simple sections involved in a minimal counterexample are groups from the lists of the Classification Theorem.
Our current approach does not need any such hypothesis, but instead works with the hypothesis that certain subgroups are soluble.

Theorem A.
Let G be a finite group and let $x \in G$ be an isolated element of order 3 such that $C_G(x)$ is soluble. Suppose further that $r_3(G) \geq 3$ and that the centraliser of every involution in every section of G is soluble. If the Z_3^*-Theorem holds in all sections of G of 3-rank 2, then $x \in Z_3^*(G)$.

The proof of Theorem A relies upon local methods and arguments only. It follows the strategy of Rowley and Toborg, where the key is the connection between the 3-structure of the group and the 2-structure. Often we obtain structural information in this way, for example the isomorphism type of the Sylow 2-subgroups is known or we find a strongly embedded subgroup. In such a situation we refer to classification theorems that apply (see Theorem 4.4).

The article proceeds as follows: In Section 2 we prove preliminary results before, in Section 3, we concentrate on isolated elements of prime order p. Then, in Section 4, we start to investigate a minimal counterexample to the Z_p^*-Theorem. From Section 5 on we specialise to the prime 3. Sections 6, 7 and 8 deal with the local analysis of the 3- and 2-structure and this is where, mainly, our solubility hypotheses come into action. Then we finish our technical arguments in Section 9, prove Theorem A in Section 10 and conclude this article with a few remarks on future work and the difficulties that we expect.

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2. Preliminaries

Throughout, G denotes a finite group and p denotes a prime number. We mostly use standard group theory notation and introduce everything else when needed. For example, if $H \leq G$, then $H \max G$ means that H is a maximal subgroup of G. This appears in the first lemma of this section. Without further reference we will use that groups of odd order are soluble (see [11]). A proper subgroup H of G is **strongly p-embedded** if and only if $p \in \pi(H)$ but $p \notin \pi(H \cap H^g)$ for all $g \in G \setminus H$. Strongly 2-embedded subgroups are usually referred to as just **strongly embedded**.

Lemma 2.1. *Let A be a finite group acting coprimely on G. Then the following hold:*

(a) $[G, A] = [G, A, A]$.
(b) *$G = [G, A] \cdot C_G(A)$. If G is abelian, then the product is direct.*
(c) *If $N \trianglelefteq G$ is A-invariant, then $C_{G/N}(A) = C_G(A)N/N$.*
(d) *If A is elementary abelian, then $G = \langle C_G(B) \mid B \max A \rangle$.*
If additionally G is abelian, then $G = \prod\limits_{B \max A} C_G(B)$.
(e) *If A is elementary abelian, but not cyclic, then $G = \langle C_G(a) \mid a \in A^\# \rangle$.*
If moreover G is abelian, then $G = \prod\limits_{a \in A^\#} C_G(a)$.
(f) *There exist A-invariant Sylow p-subgroups of G and every A-invariant p-subgroup of G is contained in one.*
(g) *If G is an abelian p-group and A acts trivially on $\Omega_1(G)$, then $[G, A] = 1$.*
(h) *If A acts trivially on $G/\Phi(G)$, then $[G, A] = 1$.*

Proof. We look at Section 8 of [17]. Part (a) of our lemma is 8.2.7 (b), the first part of (b) is 8.2.7 (a) and the second part is 8.4.2. Moreover, (c) is 8.2.2, (d) follows from 8.3.4 (a), and (e) from 8.3.4 (b). Finally (f) is 8.2.3 (a) and (c), (g) is 8.4.3 and (h) is 8.2.9 (a). □

Lemma 2.2. *Let H be a group acting faithfully on the elementary abelian 2-group B. Suppose that $a \in H$ is an involution and that Q is an elementary abelian q-subgroup of H for some odd prime q such that a inverts Q. Moreover, let $k \geq 1$ be the smallest integer such that q divides $2^{2k} - 1$ and $|Q| = q^n$. Then the following hold:*

(a) *$|B : C_B(a)| \geq 2^{kn}$,*
(b) *if $q \geq 5$, then $|B : C_B(a)| \geq 2^{2n}$,*

(c) *if* $q > 5$*, then* $|B : C_B(a)| \geq 2^{3n}$ *and*
(d) *if* $C_B(Q) = 1$*, then* $|B| = |C_B(a)|^2$*.*

Proof. This is Lemma 1.2 of Appendix A1 in [13]. □

Lemma 2.3. *Let* H *be a group acting faithfully on the elementary abelian* 2*-group* B*. Suppose that* $h, a \in H$ *are such that* h *has odd order and* a *is an involution inverting* h*.*
If $D := [B, h]$*, then* $C_D(h) = 1$ *and all* h*-invariant* 2*-subgroups of* $C_B(a)$ *are centralised by* h*. If moreover* $A \leq C_H(\langle h, a\rangle)$ *and* $\langle A, a\rangle$ *acts quadratically on* B*, then* $[D, A] = 1$*.*

Proof. As B is abelian and h has odd order, Lemma 2.1 (b) implies that $B = [B, h] \times C_B(h) = D \times C_B(h)$ and therefore $C_D(h) = 1$. Now let $B_0 \leq C_B(a)$ be h-invariant. Then B_0 is $\langle h, a\rangle$-invariant and is centralized by $\langle a^{\langle a,h\rangle}\rangle = \langle a, h\rangle$, because a inverts h.
We finally suppose that $A \leq C_H(\langle a, h\rangle)$ and that $\langle A, a\rangle$ acts quadratically on B. Then $[D, A] \leq [B, \langle A, a\rangle]$ is centralised by $\langle A, a\rangle$. As $[D, A]$ is h-invariant, we conclude that $[D, A] \leq C_D(h) = 1$. □

The simple group $\mathrm{Sz}(8)$ will occur several times in our investigation.

Lemma 2.4. *Let* $K \cong \mathrm{Sz}(8)$*. Then* K *is a* $3'$*-group and the following holds:*

(a) *The outer automorphism group of* K *is cyclic of order* 3*.*
(b) *If* T *is a Sylow* 2*-subgroup of* K*, then* $\Omega_1(T) = Z(T)$ *is elementary abelian of order* 8*. Furthermore all involutions are conjugate in* K*.*
(c) *The Sylow* 5*-subgroups of* K *are cyclic of order* 5 *and they are centralised by some outer automorphism of* K *of order* 3*.*
(d) *The Sylow* 13*-subgroups of* K *are cyclic of order* 13 *and their normalisers are Frobenius groups of order* $4 \cdot 13$*. Moreover* $\mathrm{Aut}(K)$ *has exactly one class of elements of order* 13*.*
(e) K *does not have a proper subgroup of odd index and order divisible by* 5*.*

Proof. This follows from [23], see Theorem 7, Theorem 11, Proposition 8, Lemma 1, and Theorem 9. □

Lemma 2.5. *Let* K *be a component of* G *of order prime to* 3 *and let* $y \in G$ *be an element of order* 3 *such that* $C_G(y)$ *is soluble. Then* $K/Z(K)$ *is isomorphic to* $\mathrm{Sz}(8)$ *and* y *normalises* K*. If moreover* $O_2(G) = 1$*, then* K *is simple.*

Proof. The main result of [28] shows that $K/Z(K) =: \bar{K}$ is a Suzuki group $\mathrm{Sz}(2^{2n+1})$ for some positive integer n. In addition, Lemma 2.10 of [26] and the hypothesis on $C_G(y)$ yield that y induces a non-trivial automorphism on K. Theorem 11 of [23] implies that 3 divides $2n + 1$ and its proof shows that $C_{\bar{K}}(y) \cong \mathrm{Sz}(2^{\frac{1}{3}(2n+1)})$. As $C_G(y)$ is soluble, we deduce that $2n+1 = 3$ and so $\bar{K} \cong \mathrm{Sz}(8)$. Then Theorem 2 of [1] gives that K is simple if $O_2(G) = 1$. □

Lemma 2.6. *Suppose that* K*,* V *and* B *are groups and that the semi-direct product* $K \rtimes V$ *acts faithfully on* B*. Suppose further that* $K \cong \mathrm{Sz}(8)$*, that* V *is an elementary abelian* 3*-group of order at most* 27 *and that* B *is an elementary abelian* 2*-group, and suppose that* $x \in V$ *induces a non-trivial automorphism on* K *and that* $C_K(b)$ *is soluble for every* $b \in B^{\#}$*. Then the following hold:*

(a) *If* $1 \neq B_1 \leq B$ *is an* K*-invariant subgroup, then* $|B_1| \geq 2^{12}$*.*
(b) *If* $a \in K$ *is an involution, then* $|B : C_B(a)| \geq 2^6$*.*
(c) *There is a maximal subgroup* W *of* V *that centralises a subgroup* B_0 *of* B *of order at least* 2^4*.*

Proof. Let $1 \neq B_1$ be a K-invariant subgroup of B and let H be a Sylow 13-subgroup of K. Then $|H| = 13$ by Lemma 2.4 (d). Since K is non-abelian simple and $C_K(B_1)$ is soluble by assumption, we see that $C_K(B_1) = 1$. In particular, H acts non-trivially on B_1. Let $n \in \mathbb{N}$ be such that $|B_1| = 2^n$. As 13 does not divides $|\mathrm{GL}_{11}(2)|$, $|B_1| \geq 2^{12}$ and hence (a) holds. In addition H is inverted by some involution $a \in K$ by Lemma 2.4 (d), so Lemma 2.2 (a) implies that $|B : C_B(a)| \geq 2^6$. Since all involutions of K are conjugate in K by Lemma 2.4 (b), we see that (b) is true.

Furthermore Lemma 2.4 (a) and our hypothesis on $x \in V$ imply that $W := C_V(K)$ is a maximal subgroup of V. If W is not cyclic, then W has order 9 and Lemma 2.1 (e) gives that $B = \prod_{v \in W^\#} C_B(v)$. Thus there is an element $v \in W^\#$ such that $1 \neq C_B(v)$. If W is cyclic, then we set $v := 1$. Hence in both cases $B_1 := C_B(v) \neq 1$ is KV-invariant and so (a) implies that $|B_1| \geq 2^{12}$.

We choose $w \in W$ such that $W = \langle w, v \rangle$. Then $C_B(v) \cap C_B(w)$ is K-invariant. Again (a) forces $C_B(w) \cap C_B(v) = C_B(W)$ to have size at least 2^{12} or to be trivial.

In the first case (b) is true. In the second case $\langle w, x \rangle$ acts coprimely on B_1 and so Lemma 2.1 (e) yields that

$$\begin{aligned} 2^{12} \leq \quad & |B_1| = |C_B(\langle x, v \rangle) \cdot C_B(\langle xw, v \rangle) \cdot C_B(\langle xw^2, v \rangle) \cdot C_B(\langle w, v \rangle)| \\ \leq \quad & |C_B(\langle x, v \rangle)| \cdot |C_B(\langle xw, v \rangle)| \cdot |C_B(\langle xw^2, v \rangle)|. \end{aligned}$$

This provides some $i \in \{0, 1, 2\}$ such that $|C_B(\langle xw^i, v \rangle)| \geq 2^4$. Now (c) follows because $\langle xw^i, v \rangle$ is a maximal subgroup of V. □

3. Isolated elements

In this section, G denotes a finite group and p is a prime number. For our investigation of a minimal counterexample to the Z_p^*-Theorem we analyse properties of isolated elements.

Definition 3.1. *Let $H \leq G$.*
*We say that $x \in H \leq G$ is **isolated in** H if and only if $x^H \cap C_H(x) = \{x\}$.*
*A subgroup X of G is said to be **strongly closed in** H **with respect to** G if and only if, for all $x \in X$ and $g \in G$ such that $x^g \in H$, it follows that $x^g \in X$.*
*A subgroup X of G is said to be **weakly closed in** H **with respect to** G if and only if, for all $g \in G$ such that $X^g \leq H$, it follows that $X^g = X$.*
Moreover we denote by $Z_p^*(H)$ the full pre-image of $Z(H/O_{p'}(H))$ in H.

The following lemma clarifies the connection between the concepts just introduced.

Lemma 3.2. *Let $x \in G$ be a p-element, $x \in P \in Syl_p(G)$ and suppose that x is isolated in G. Then $x \in Z(P)$. Moreover $\langle x \rangle$ is strongly closed and weakly closed in P with respect to G.*

Proof. Suppose that x is isolated in G and let $N := N_P(C_P(x))$. Then

$$x^N \subseteq x^G \cap C_P(x) \subseteq x^G \cap C_G(x) = \{x\},$$

whence $N \leq C_G(x)$ and we conclude that $C_P(x) = P$. If x is isolated in G and $h \in G$ is such that $x^h \in P$, then it follows that $x^h \in x^G \cap C_G(x) = \{x\}$, because $P \leq C_G(x)$ and x is isolated. Therefore $x^h = x \in \langle x \rangle$, which means that $\langle x \rangle$ is strongly closed in P with respect to G. The same argument shows that $\langle x \rangle$ is weakly closed in P with respect to G. □

The group $\mathcal{A}_5$ shows that the "obvious" converse of this lemma is not true if p is odd.

Lemma 3.3. *Let $x \in G$ be a p-element. Then the following are equivalent:*

(a) *x is isolated in G.*

(b) *Whenever* $x \in P \in Syl_p(G)$*, then* $x^G \cap P = \{x\}$*.*
(c) *Whenever* $x \in P \in Syl_p(G)$*, then* $x^G \cap C_P(x) = \{x\}$*.*

Proof. Let $x \in P \in \mathrm{Syl}_p(G)$. If x is isolated in G, then the fact that $P \leq C_G(x)$ by Lemma 3.2 gives the assertion in (b).
Of course (b) implies (c), so now we suppose that (c) holds. Let $g \in G$ be such that $x^g \in C_G(x)$. Then $\langle x, x^g \rangle$ is a p-group and therefore we may choose $P \in \mathrm{Syl}_p(G)$ to be such that $\langle x, x^g \rangle \leq P$. Then $x^g \in x^G \cap C_P(x) = \{x\}$ and it follows that $x^g = x$. We conclude that (a) holds. □

Lemma 3.4. *Let* $H \leq G$ *and suppose that* $x \in G$ *is an isolated* p*-element in* G*.*

(a) *If* $x \in H$*, then* x *is isolated in* H*.*
(b) *If* R *is a* p*-subgroup of* G *containing* x*, then* $N_G(R) \leq C_G(x)$*. In particular* $x \in Z(R)$*.*
(c) *If* $x \in N \trianglelefteq H \leq G$*, then* $H = N \cdot C_H(x)$*.*
(d) *If* $N \trianglelefteq G$*, then* Nx *is isolated in* G/N*.*
(e) *If* G *has cyclic Sylow* p*-subgroups and* $x \neq 1$*, then* G *has a normal* p*-complement.*

Proof. (a) follows because $x^H \cap C_H(x) \subseteq x^G \cap C_G(x) = \{x\}$.
For (b) we let $R \subseteq P \in \mathrm{Syl}_p(G)$ and $g \in N_G(R)$. Then $x^g \in x^G \cap R \subseteq x^G \cap P \subseteq \{x\}$ by Lemma 3.3 (b) and therefore $x^g = x$.
(c) Let $Q \in \mathrm{Syl}_p(N)$ be such that $x \in Q$. Then $H = N \cdot N_H(Q)$ by a Frattini argument and (b) gives that $H = N \cdot C_H(x)$.
(d) Let $P \in \mathrm{Syl}_p(G)$ be such that $x \in P$. Then PN/N is a Sylow p-subgroup of G/N and $Nx \in PN/N$. Let $g \in G$ be such that $Nx^g \in PN/N$. Then $x^g \in PN$ and by Sylow's Theorem we find some $h \in N$ such that $x^{gh} \in P \cap x^G = \{x\}$. It follows that $Nx^g = Nx^{gh} = Nx$ and so Nx is isolated in G/N by Lemma 3.3.
(e) Let $P \in \mathrm{Syl}_p(G)$ be such that $1 \neq x \in P$. Then P is cyclic by hypothesis and $N_G(P) \leq N_G(\langle x \rangle) \leq C_G(x)$ since x is isolated. So $N_G(P)$ centralises $\langle x \rangle$ and $\Omega_1(P) \leq \langle x \rangle$. From Lemma 2.1 (g) we deduce that $N_G(P) = C_G(P)$ and then Burnside's p-Complement Theorem (see 7.2.1 of [17]) yields the assertion. □

Lemma 3.5. *Let* G *be a finite group and let* p *be an odd prime such that* $r_p(G) \leq 1$*. Suppose further that* $x \in G$ *is an isolated element of order* p*. Then* $x \in Z_p^*(G)$*.*

Proof. Our hypotheses and Proposition 1.4 of [6] imply that G has cyclic Sylow p-subgroups. Hence Lemma 3.4 (e) yields that G has a normal p-complement. Then $x \in \mathrm{Z}_p^*(G)$ by Lemma 3.2. □

4. A minimal counterexample to the Z_p^*-Theorem

We begin our analysis with a general minimality hypothesis, without restricting the prime p. That way the results can be cited in future work on the Z_p^*-Theorem. Throughout the remainder of this article, G is a finite group and p is a prime number. All other notation will be explicitly mentioned at the beginning of the section where it is used.

Hypothesis 4.1. *Suppose that* $x \in G$ *has order* p *and that* x *is isolated in* G*, but that* $x \notin Z_p^*(G)$*. Suppose further that* $x \in Z_p^*(H)$ *for all proper subgroups* $H \leq G$ *that contain* x *and that* $Nx \in Z_p^*(G/N)$ *for all* $1 \neq N \trianglelefteq G$*.*
Notation: *Let* $x \in P \in Syl_p(G)$ *and* $C := C_G(x)$*.*

Lemma 4.2. *Suppose that Hypothesis 4.1 holds.*

(a) *Whenever* $x \in H < G$*, then* $H = C_H(x) \cdot O_{p'}(H)$*.*

(b) *P is not cyclic.*
(c) *Let $g \in G$ and suppose that $S \subseteq C \cap C^g$. Then there exist $a \in C$ and $b \in C_G(S)$ such that $g = ab$.*
(d) *Suppose that $U \leq C$. Then $N_G(U) = N_C(U) \cdot C_G(U)$.*

Proof. (a) This follows from Lemma 2.1 (c) and the definition of $Z_p^*(H)$.
(b) If P is cyclic, then $x \in \mathrm{Z}_p^*(G)$ by Lemma 3.5. But this contradicts Hypothesis 4.1.
(c) The hypothesis implies that $x, x^g \in C_G(S)$. Let $x \in R \in \mathrm{Syl}_p(C_G(S))$. Then Sylow's Theorem and Lemma 3.3 (b) provide some $c \in C_G(S)$ such that $x^{gc} \in x^G \cap R \subseteq \{x\}$. Let $a := gc$ and $b := c^{-1}$. Then $a \in C_G(x)$, $b \in C_G(S)$ and $g = ab$.
(d) Let $g \in N_G(U)$. Then $U^g = U \leq C \cap C^g$ and, using (c), we may choose $a \in C$ and $b \in C_G(U)$ such that $g = ab$. This gives the statement, since $a = gb^{-1} \in C \cap N_G(U) = N_C(U)$. □

One of our first reduction results is that G is almost simple.

Lemma 4.3. *Suppose that Hypothesis 4.1 holds. Then G' is non-abelian simple and $G = G'\langle x \rangle$.*

Proof. Assume first that $N := O_{p'}(G) \neq 1$. Then Hypothesis 4.1 gives that $Nx \in \mathrm{Z}_p^*(G/N) = Z(G/N)$. But then $x \in \mathrm{Z}_p^*(G)$, contrary to our hypothesis. Thus $O_{p'}(G) = 1$.
If $x \in O_p(G)$, then Lemma 3.4 (b) implies the contradiction $G = N_G(O_p(G)) \leq C_G(x)$. So $x \notin O_p(G)$, but we know from Lemma 3.2 that x centralises $O_p(G)$. By the previous paragraph $F(G) = O_p(G)$, so $C_G(F^*(G)) \leq Z(F^*(G)) \leq F(G) = O_p(G)$, and we deduce that x does not centralise $E(G)$. In particular $E(G) \neq 1$.
For a last preparatory statement we consider $N := \langle x^G \rangle$. We note that $N \trianglelefteq G$, so $O_{p'}(N) \leq O_{p'}(G) = 1$, which means that $Z_p^*(N) = Z(N)$. If $N \neq G$, then by Lemma 4.2 (a) we see that $x \in Z(N)$ whence x commutes with all its conjugates. But x is isolated, which forces $x \in Z(G)$. This is a contradiction.
So we have: $O_{p'}(G) = 1, E(G) \neq 1$ and $\langle x^G \rangle = G$. Next we prove that $F(G) = 1$.
We assume otherwise and recall that $F(G) = O_p(G)$ is centralised by x. Then all conjugates of x centralise $F(G)$, so $G = \langle x^G \rangle$ centralises $F(G)$, whence $F(G) \leq Z(G)$. We consider the natural epimorphism $- : G \to G/O_p(G)$. Then $O_p(\bar{G}) = 1$ and $1 \neq \bar{x} \in Z_p^*(\bar{G})$. Let N denote the full pre-image of $O_{p'}(\bar{G})$ in G. Then $O_p(G)$ is a central Sylow p-subgroup of N, whence $N = O_p(G) \times O_{p'}(N)$ by Burnside's p-complement Theorem (7.2.1 of [17]). Now $O_{p'}(N) \leq O_{p'}(G) = 1$ and consequently $x \in Z_p^*(\bar{G}) = Z(\bar{G})$. But this implies that $x \in O_p(\bar{G}) = 1$, which is false. Therefore $F(G) = 1$.
Finally let E be a component of G that is not centralised by x. As $\langle E^G \rangle \trianglelefteq G$, this group has non-trivial Sylow p-subgroups, and x centralises one of them by Lemma 3.2. Then x centralises a Sylow p-subgroup of E. If $E \neq E^x$, then E has a central non-trivial Sylow p-subgroup. This is impossible because $E = E'$ is perfect. It follows that x normalises E. Moreover x is isolated in $E\langle x \rangle$ by Lemma 3.4 (a). If $E\langle x \rangle \neq G$, then Lemma 4.2 (a) implies that $x \in \mathrm{Z}_p^*(E\langle x \rangle)$ and hence x centralises E. This contradicts our choice of E. Altogether $F^*(G) = E(G) = E$ and $E \cdot \langle x \rangle = G$; in particular $E = G'$ is non-abelian and simple because $Z(E) \leq F(G) = 1$. □

Theorem 4.4. *Suppose that Hypothesis 4.1 holds. Then*
(a) *G has 2-rank at least 3,*
(b) *G does not have a strongly embedded subgroup,*
(c) *G is not $\mathcal{S}_4$-free,*
(d) *the Sylow 2-subgroups of G do not have any non-trivial strongly closed abelian subgroups, and*
(e) *a Sylow 2-subgroup of G is not isomorphic to one of $\mathrm{PSL}_3(2^n)$ for any $n \geq 2$.*

In particular G is not isomorphic to $\mathrm{PSL}_3(3)$*, to* $\mathrm{PSL}_2(r)$ *for some prime power r, to* $\mathrm{PSU}_3(q)$ *or to* $\mathrm{Sz}(q)$ *for some power q of* 2*.*

Proof. If G' is a $\mathcal{K}$–group in the sense of [15], then $\langle x \rangle$ is weakly closed in P with respect to G by Lemma 3.2. So Proposition 7.8.2 of [15] is applicable, and Lemma 4.2 (b) shows that P is not cyclic. If $G' \cong \mathrm{PSU}_3(p)$, then $G = G'$ by Theorem 2.5.12 of [15]. Lemma 4.3 and elementary computations show that P is extra-special of order p^3 and exponent p, and that $Z(P)$ is inverted in G. This is false because $x \in Z(P)$. Hence G' is one of the groups in Part (c) or (d) of Proposition 7.8.2 of [15], which we refer to as (*).

We note that G has even order and that $O(G) = 1$ by Lemma 4.3. If G has 2-rank 1, then it is soluble or, by the main result in [7], it has a central involution. Both cases cannot occur by Lemma 4.3. If G has 2-rank 2, then we apply the second main theorem of [2] to see that G' is isomorphic to one of $\mathcal{A}_7$, M_{11}, $\mathrm{PSU}_3(4)$, $\mathrm{PSL}_2(q)$, $\mathrm{PSL}_3(q)$ or $\mathrm{PSU}_3(q)$ for some odd prime power q. This contradicts (*).

If G has a strongly embedded subgroup, then we obtain a contradiction to (*) by applying the main result of [5], Burnside's p-Complement Theorem (see for example 7.2.1 in [17]), the Brauer-Suzuki Theorem [7], and Lemma 4.3.

The statements in (c) and (d) are equivalent by Theorem C on p. 47 in [13]. If G is $\mathcal{S}_4$-free or if the Sylow 2-subgroups of G have a non-trivial strongly closed abelian subgroup, then G' is a Goldschmidt group in the sense of [13]. This contradicts (*) again.

For (e) we apply Theorem A of [9]; it shows that $G' \cong \mathrm{PSL}_3(2^n)$ which we already know to be impossible. □

Remark 4.5. *We note that $\mathcal{A}_5$ and $\mathcal{A}_6$ do not have any isolated elements of order* 3*.*

Lemma 4.6. *Suppose that Hypothesis 4.1 holds and let q be a prime. Then the maximal x-invariant q-subgroups (with respect to inclusion) are trivial or Sylow q-subgroups of G. Moreover $C_G(x)$ acts transitively on the set of x-invariant Sylow q-subgroups of G.*

Proof. For $p = q$ the statement follows from 3.2 and Sylow's Theorem.

Let $q \neq p$ and let Q be an x-invariant q-subgroup of G that is maximal with respect to inclusion. Suppose that $Q \neq 1$. Then we have that $x \in N_G(Q) < G$ by Lemma 4.3. Now Lemma 4.2 (a) yields that $N_G(Q) = O_{p'}(N_G(Q)) \cdot C_{N_G(Q)}(x)$. By Lemma 2.1 (f) the group $O_{p'}(N_G(Q))$ has an x-invariant Sylow q-subgroup, so $N_G(Q)$ also has an x-invariant Sylow q-subgroup. Then the maximal choice of Q forces $Q \in \mathrm{Syl}_q(G)$.

Suppose that Q_1 is a further x-invariant Sylow q-subgroup of G. Then by Sylow's Theorem there first is some $g \in G$ such that $Q = Q_1^g$ and hence $x, x^g \in N_G(Q)$, and then there is some $h \in N_G(Q)$ such that $\langle x, x^{gh} \rangle$ is a p-subgroup of $N_G(Q)$. Now the fact that x is isolated in G forces $x = x^{gh}$ and hence $gh \in C_G(x)$. Moreover $Q_1^{gh} = Q^h = Q$, which proves the lemma. □

5. The special case $p = 3$

From now on we focus on the prime 3.

Hypothesis 5.1. *In addition to Hypothesis 4.1 we let $p = 3$.*

Lemma 5.2. *Suppose that Hypothesis 5.1 holds. Then C contains an element y of order* 3 *that is inverted by a* 2*-element. In particular G possesses an x-invariant Sylow* 2*-subgroup.*

Proof. G is not $\mathcal{S}_4$-free by Theorem 4.4 (c), in particular G is not $\mathcal{S}_3$-free and we let $N \trianglelefteq H \leq G$ be such that $H/N \cong \mathcal{S}_3$ and $R \in \mathrm{Syl}_3(H)$. By Sylow's Theorem we may suppose that $R \leq P$ and hence that x centralises R. With a Frattini argument we moreover see that $H = N_H(R) \cdot RN = N_H(R) \cdot N$ and hence $N_H(R)/N_N(R) \cong H/N \cong \mathcal{S}_3$. In particular there is some 2-element $a \in N_H(R)$ such that $[R, a] \neq 1$. Let $z \in R^{\#}$ be such that $1 \neq [z, a]$. Then $[z, a]^a = (z^{-1}z^a)^a = (z^{-1})^a z = [a, z] = [z, a]^{-1}$. Let $y \in \langle [z, a] \rangle$ have order 3. Then $y^a = y^{-1} \in R \leq C$ and Lemma 4.2 (c) provides some $c \in C$ such that $y^{-1} = y^a = y^c$. Moreover a suitable power of c is a 2-element of C that inverts y.
It follows that C has even order. In particular there exists a non-trivial x-invariant 2-subgroup of G, and then Lemma 4.6 gives that G has an x-invariant Sylow 2-subgroup. □

Definition 5.3. *Let q be prime.*

(a) *Let S be a Sylow q-subgroup of G. We denote by $\Gamma(S)$ the graph with vertex set $V_S := \{A \mid A \leq S$ is elementary abelian and $|A| \geq p^2\}$ and edge set $E_S := \{\{A, B\} \in V_S \times V_S \mid A \neq B$ and $A \leq C_G(B)\}$. Following the definition in Section 4.1 of Chapter II in* [19] *we say that G is* ***connected for the prime*** *q if and only if the graph $\Gamma(S)$ is connected. For brevity we also say that S is connected (omitting the prime).*

(b) *For all q-subgroups $U \leq G$ we define the group*

$$W_U := \langle O_{q'}(C_G(a)) \mid a \in U \text{ and } o(a) = q \rangle.$$

Lemma 5.4. *Suppose that Hypothesis 5.1 holds and that $r_3(G) \geq 3$, but that P is not connected. Then P is isomorphic to a Sylow 3-subgroup of $\mathcal{A}_9$.*

Proof. We proceed in several steps.

(I) *$Z(P)$ is cyclic and $r(P) = 3$. Moreover, there is a normal elementary abelian subgroup V of P of order 27 containing a normal subgroup of order 9 of P, and $x \in V$.*

Proof. The first assertions follow from the definition of connectedness and Corollary 10.22 (iii) of [14]. By a result of Konvisser (see for example Proposition 10.17 of [14]), there is an elementary abelian normal subgroup V of order 27 of P. The next statement is Lemma 1.4 of [6], and $x \in V$ because $x \in Z(P)$ and $r(P) = 3$. ■

We keep the subgroup V from the previous step.

(II) *There is an element $y \in P$ of order 3 such that $C_P(y) = \langle y \rangle \times K$, where $K \leq C_P([V, y])$ is cyclic and $\Omega_1(K) = \langle x \rangle$. Moreover $V\langle y \rangle$ is isomorphic to a Sylow 3-subgroup of $\mathcal{A}_9$. In particular $Q := [V, y]\langle y \rangle$ is extra-special of order 27 and exponent 3 and $\langle y, x \rangle = \Omega_1(C_P(y)) \leq Q$ is not normal in P.*

Proof. Lemma 10.21 (ii) and (iii) of [14] provide an element $y \in P$ such that $C_P(y) = \langle y \rangle \times K$ for some cyclic subgroup $K \leq P$. In particular $r(C_P(y)) = 2$. Moreover $x \in Z(P) \leq C_P(y)$ by Lemma 3.2 and $x \in V$ by (I).
As $r(C_P(y)) = 2$, it follows that $y \notin V$ and that $C_V(y)$ is cyclic. The automorphism group of V has exactly two conjugacy classes of elements of order 3 by [10] (see page 13). So there is just one class of elements of order 3 that do not centralise a subgroup of order 9. Hence the action of y on V is fully determined. It is equivalent to the conjugation action of $(147)(258)(369)$ on $\langle (123), (456), (789) \rangle$ in $\mathcal{A}_9$. Thus $V\langle y \rangle$ is isomorphic to a Sylow 3-subgroup of $\mathcal{A}_9$ and $Q = [V, y]\langle y \rangle$ is extra-special of order 27 and exponent 3 and contains $\Omega_1(Z(V\langle y \rangle)) = \langle x \rangle$.

The group $[V, y]$ is $C_P(y)$-invariant and has order 9. Thus we see that $C_P([V, y]) \cap C_P(y)$ is a maximal subgroup of $C_P(y) = \langle y\rangle \times K$. Since y does not centralise $[V, y]$, we can choose the notation such that $K = C_P([V, y]) \cap C_P(y)$. In particular $x \in K$ and so $\Omega_1(K) = \langle x\rangle$.
We finally conclude that $\Omega_1(C_P(y)) = \langle y, x\rangle \le Q \le V\langle y\rangle$, but $\langle y, x\rangle$ is not normal in $V\langle y\rangle$, and hence it is not normal in P. ■

(III) *$\Omega_1(P) = V\langle y\rangle$, $P = \Omega_1(P) \cdot K$ and Q is the unique subgroup of P of its isomorphism type.*

Proof. We set $R_0 := C_P(y)$ and for all integers $i \ge 1$ we define $R_i := N_P(R_{i-1})$. By (II) we have that $\langle x, y\rangle = \Omega_1(R_0)$ is characteristic in R_0. We deduce that $1 \ne |R_1 : R_0| \le |N_P(\Omega_1(R_0)) : R_0| \le 3$, as $|\Omega_1(R_0)| = 9$. On the other hand (II) implies that $\Omega_1(R_0)$ is a normal subgroup of Q. Altogether it follows that $R_1 = R_0Q = (\langle y\rangle \times K)([V, y]\langle y\rangle) = Q * K$ by (II). We deduce that $\Omega_1(R_1) = Q$, because $\Omega_1(K) \le \Omega_1(R_0) = \langle x, y\rangle \le Q$.
We remark that $V \cap R_1 = V \cap [V, y]R_0 = [V, y](V \cap R_0) = [V, y]\langle x\rangle = [V, y]$ is a maximal subgroup of V. Hence R_1 is a maximal subgroup of $V \cdot R_1$ and so $V \cdot R_1 \le R_2$. Moreover $R_2 = N_P(R_1) \le N_P(\Omega_1(R_1)) = N_P(Q)$ and $C_P(Q) \le C_P(y) \cap C_P([V, y]) = K \le C_P(Q)$.
This yields that $QC_P(Q) = \Omega_1(R_1) \cdot K = R_1$. As x is isolated in G, Theorem 1 of [29] together with Hilfssatz II 9.12 of [16] gives that $|N_P(Q)/QC_P(Q)|$ divides $|\mathrm{Sp}_2(3)| = |\mathrm{SL}_2(3)| = 3 \cdot 2^3$. In conclusion, $3 \ge |N_P(Q) : QC_P(Q)| \ge |R_2 : R_1| \ge 3$, and this implies that $R_2 = N_P(Q) = V \cdot R_1$.
By (II) we have that $V \cdot Q = V\langle y\rangle$ is isomorphic to a Sylow 3-subgroup of $\mathcal{A}_9$. In particular $\Omega_1(R_2) = V \cdot Q$ contains exactly one subgroup isomorphic to Q. Now we see that $R_3 = N_P(R_2) \le N_P(\Omega_1(R_2)) \le N_P(Q) = R_2$ and consequently $P = R_2$. ■

Assume for a contradiction that $K \ne \langle x\rangle$. Then we apply the generalised Thompson Transfer Lemma in the sense of Proposition 15.15 of [14] to $\Omega_1(P)$, P and G.
We have that $\Omega_1(P)$ is a proper normal subgroup of P with cyclic factor group and so Condition (a) of Proposition 15.15 of [14] holds. Let $u \in K$ be such that $u^3 = x$. Then u has order 9 and every G-conjugate of $u^3 = x$ in P is equal to $x \in \Omega_1(P)$, because x is isolated in G. This is Condition (b).
If $u^g \in P$ is an extremal conjugate of u, then u^g has order 9 and centralises a subgroup of P that is isomorphic to Q. From (III) we see that $u^g \in C_P(Q)$. This implies that $u^g \notin \Omega_1(P)$, as $C_{\Omega_1(P)}(Q) \le C_P(y) \cap V\langle y\rangle = \langle x, y\rangle$. Consequently 15.15 (i) of [14] is false.
As G does not have a normal subgroup N such that $u \notin N$ by Lemma 4.3, we see that the assertion of Proposition 15.15 in [14] does not apply. Thus we deduce that Condition (c) of the hypothesis is false. Hence there is some extremal conjugate $u^g \in P$ such that $u^g \notin \Omega_1(P)u$. Since $P/\Omega_1(P)$ is cyclic, we obtain an $i \in \mathbb{N}$ such that $i \equiv 2 \bmod 3$ and some $v \in \Omega_1(P)$ such that $u^g = u^i v$. Moreover u^g is extremal and therefore u^g centralises Q. In particular $u^g \in C_P(y) = \langle y\rangle \times K$ and hence $u^g \in C_{C_P(y)}(Q) = K$. Since K is cyclic and $u \in K$, we may suppose that $u^g = u^i$. Finally $x^g = (u^3)^g = (u^i)^3 = (u^3)^i = x^i = x^2$, which contradicts the fact that x is isolated. □

Lemma 5.5. *Suppose that Hypothesis 5.1 holds. If C is soluble, then $|G : C|$ is even.*

Proof. Let C be soluble. We know from Theorem 4.4 (a) that $r_2(G) \ge 3$. We assume for a contradiction that $|G : C|$ is odd, which means that C contains a Sylow 2-subgroup of G. Then our plan is to show that G has a strongly embedded subgroup (contrary to Theorem 4.4 (b)). Let $T \in \mathrm{Syl}_2(C) \subseteq \mathrm{Syl}_2(G)$ and let S be a Sylow 2-subgroup of $O_{2',2}(C)$ that is contained in T. Moreover we set $Z := \Omega_1(Z(S))$.

(I) *If $a \in T$ is an involution, then $C_G(a) = C_C(a)O(C_G(a))$. In particular $C_G(a)$ is soluble.*

Proof. Let $a \in T$ be an involution. Then $x \in C_G(a)$ and $C_G(a) = C_C(a)O_{3'}(C_G(a))$ by Lemma 4.2 (a). Thus Lemma 2.7 of Rowley ([20]), applied to $O_{3'}(C_G(a))$, yields that $C_G(a) = C_C(a)O(C_G(a))$ as stated. Then $C_G(a)$ is soluble because C is. ∎

(II) $|Z| \geq 4$.

Proof. We recall that $r_2(G) \geq 3$ and therefore T is not cyclic.
Next assume that $Z(S)$ is cyclic and let a denote the unique involution in $Z(S)$. Then $\langle a\rangle O(C)/O(C) = \Omega_1(Z(O_2(C/O(C)))) \trianglelefteq C/O(C)$ and hence $[a, C] \leq O(C)$. We know that $a \notin Z^*(G)$ by Lemma 4.3, so by Glauberman's Z^*-Theorem (see for example [12]) a is not isolated in G. Lemma 3.3 gives that $a^G \cap T \neq \{a\}$ and we choose $g \in G$ such that $a \neq a^g \in T$. Then $a^g \in C \cap C^g$ and Lemma 4.2 (c) gives elements $c \in C$ and $b \in C_G(a^g)$ such that $g = cb$. In particular $a^g = a^c$ which implies that $a^{-1} \cdot a^c \in T \cap [a, C] \leq T \cap O(C) = 1$. Therefore $a^c = a$. This is a contradiction, showing that $Z(S)$ is not cyclic and hence $|Z| \geq 4$. ∎

(III) *Z is contained in an elementary abelian subgroup of order at least* 8 *of T.*

Proof. If $|Z| \geq 8$, then the assertion follows. Hence suppose that $|Z| = 4$. We recall that $r_2(G) \geq 3$, so we let $A \leq T$ be an elementary abelian subgroup of order at least 8. Now $Z \trianglelefteq T$, so A normalises Z and $A/C_A(Z)$ is isomorphic to a subgroup of $\mathcal{S}_3$. If possible, then we choose $a \in C_A(Z) \setminus Z$ and we see that $\langle Z, a\rangle$ is elementary abelian of order 8. Otherwise $C_A(Z) \leq Z$, then $C_A(Z) = Z$ and therefore Z is contained in the elementary abelian subgroup A of order at least 8 of T. ∎

(IV) *$W := W_Z$ has odd order and is normalised by $N_G(S)$ and by C.*

Proof. Using (I) and (III), we see from Lemma 1.4 of [27] that W has odd order. As Z is a 2-group, we do not need any hypothesis on known simple groups to apply the lemma. As Z is a characteristic subgroup of S, we also know that $N_G(S)$ normalises W.
A Frattini argument gives that $C = O_{2',2}(C) \cdot N_C(S) = O(C) \cdot N_C(W)$. We apply Lemma 2.1 (e) and the fact that Z is elementary abelian, and not cyclic, to see that $O(C) = \langle C_{O(C)}(c) \mid c \in Z^\#\rangle$.
Let $c \in Z^\#$. Then $C_{O(C)}(c)O(C_G(c))$ is a normal subgroup of odd order of $C_C(c)O(C_G(c)) = C_G(c)$ by (I). It follows that $O(C) \leq W_Z = W$ and then $C \leq N_G(W)$ as stated. ∎

(V) *Let $a \in T$ be an involution. Then $C_G(a) \leq N_G(W)$ and $O(C_G(a)) \leq W$.*

Proof. Suppose first that $C_T(a)$ has an elementary abelian, non-cyclic subgroup that, in the graph $\Gamma(T)$ (see Definition 5.3), lies in the connected component of Z. Then we apply again Lemma 1.4 of [27] to see that $O(C_G(a)) \leq W$. Thus the fact that $C \leq N_G(W)$ by (IV) implies that $C_G(a) = C_C(a) \cdot O(C_G(a)) \leq N_G(W)$ by (I).
So we may suppose that G is not connected for the prime 2 and that $a \in T \setminus C_T(Z)$. If $c \in C$ is such that $a^c \in C_T(Z)$, then we have that $C_G(a) = C_G(a^c)^{c^{-1}} \leq N_G(W)^{c^{-1}} = N_G(W)$ and hence $O(C_G(a)) = O(C_G(a^c))^{c^{-1}} \leq W^{c^{-1}} = W$ because $C \leq N_G(W)$. So we are left with the case where $a \notin C_T(Z)$ and a is not C-conjugate into $C_T(Z)$. Then Thompson's Transfer Lemma implies that $C \neq O^2(C)$. Now Lemma 4.2 (c), Lemma 4.3 and the Focal Subgroup Theorem (e.g. Theorem 15.7 in [14]) give that
$T = T \cap G' = \langle a^{-1}a^g \mid g \in G \text{ and } a, a^g \in T\rangle = \langle a^{-1}a^g \mid a, a^g \in T \text{ and } g \in C_G(a)C\rangle$
$= \langle a^{-1}a^g \mid g \in C \text{ and } a, a^g \in T\rangle = T \cap C'$, which is impossible. ∎

Let $t \in N_G(W)$ be an involution. By Sylow's Theorem there is some $g \in N_G(W)$ such that $t^g \in T$. We conclude that $(C_G(t))^g = C_G(t^g) \leq N_G(W)$ and hence $C_G(t) \leq N_G(W)$ from (V). This shows that $N_G(W)$ is a strongly embedded subgroup of G or that $O^{2'}(G) \leq N_G(W)$. The first case cannot occur by Theorem 4.4 (b). In the second case we deduce from Lemma 4.3 that $W = 1$ and so (I) and (V) imply that $C_G(a) \le C$ for all involutions $a \in T$. Thus we see, again using Sylow's Theorem, that C is strongly embedded in G, again contrary to Theorem 4.4 (b). $\Box$

6. The 3-structure of G

The last lemma of the previous section motivates further investigation of the case where C is soluble. Under mild additional conditions, we will find a strongly 3-embedded subgroup of G that contains C. Such a subgroup will be key in our further analysis of the 2-structure and the 3-structure of G.

Hypothesis 6.1. *Suppose that Hypothesis 5.1 holds. We keep the notation from Hypothesis 4.1 and we further suppose that C is soluble.*

Lemma 6.2. *Suppose that Hypothesis 6.1 holds and that $r_3(G) \ge 3$. Then P is connected.*

Proof. Assume for a contradiction that P is not connected. Then Lemma 5.4 implies that P is isomorphic to a Sylow 3-subgroup of $\mathcal{A}_9$. We deduce that $\langle x \rangle = Z(P)$, from Lemma 3.2, and so $x \in P' \le G'$. In particular $G = G'$ is simple by Lemma 4.3. A theorem of Grün (see 7.1.8 of [17]) yields that $C = O^3(C)$, because $\langle x \rangle$ is weakly closed in P with respect to G by Lemma 3.2. Let $- : C \to C/O_{3'}(C)$ be the natural epimorphism. Then $\bar{P} \cong P$, $\bar{C} = O^3(\bar{C})$ and, as C is soluble by Hypothesis 6.1, we see that $F^*(\bar{C}) = O_3(\bar{C})$. We set $\bar{Q} := O_3(\bar{C})$. Then $C_{\bar{P}}(\bar{Q}) \le \bar{Q}$ and $\bar{Q} \trianglelefteq \bar{P}$. Consequently the structure of P implies that $\bar{Q} = \bar{P}$ or that $\bar{Q}$ is a maximal subgroup of $\bar{P}$.
The maximal subgroups of $\bar{P}$ have order 27. One is elementary abelian and three are extra-special. If $\bar{R}$ is an extra-special maximal subgroup of $\bar{P}$, then $N_{\bar{C}}(\bar{R})$ centralises $Z(\bar{R}) = \langle \bar{x} \rangle$. Thus we deduce from Theorem 1 of [29] the structure of $\tilde{D} := N_{\bar{C}}(\bar{R})/\bar{R}C_{\bar{C}}(\bar{R})$. If $\bar{R}$ has exponent 9, then $\tilde{D}$ has order 3. If $\bar{R}$ has exponent 3, then the theorem yields that $\tilde{D}$ is isomorphic to a subgroup of $\mathrm{Sp}_2(3)$. With Hilfssatz II 9.12 in [16] we moreover see that $\mathrm{Sp}_2(3) \cong \mathrm{SL}_2(3)$ and so $\tilde{D}$ has a normal 3-complement.
In both cases $O^3(N_{\bar{C}}(\bar{R})) \neq N_{\bar{C}}(\bar{R})$ and hence $\bar{C} \neq N_{\bar{C}}(\bar{R})$. Thus $\bar{Q}$ is not extra-special. Since $\bar{P}$ has a unique maximal subgroup $\bar{R}_0$ that is extra-special of exponent 3 (and hence R_0 is characteristic in $\bar{P}$), we moreover conclude that $N_{\bar{C}}(\bar{P}) \le N_{\bar{C}}(\bar{R}_0)$. Consequently $\bar{Q} \neq \bar{P}$, because otherwise $\bar{C} = N_{\bar{C}}(\bar{P})$ implies that $\bar{C} = O^3(\bar{C}) = O^3(N_{\bar{C}}(\bar{P})) = O^3(N_{\bar{C}}(\bar{R}_0)) \lneq N_{\bar{C}}(\bar{R}_0) \le \bar{C}$. This is a contradiction.
In conclusion $\bar{Q}$ is elementary abelian of order 27 and so $\bar{C}/\bar{Q}$ is isomorphic to a subgroup $\hat{C}$ of $\mathrm{GL}_3(3)$ fixing a non-zero vector $v \in V$, where V is a vector space of dimension 3 over $\mathrm{GF}(3)$. Moreover 3 divides the order of $\hat{C}$, but 9 does not, $O_3(\hat{C}) = 1$ and $O^3(\hat{C}) = \hat{C}$. Let $\hat{C}_0 := \hat{C} \cap \mathrm{SL}_3(3)$. Then $\hat{C}_0$ is a normal subgroup of $\hat{C}$ of index at most 2. Hence $O_3(\hat{C}_0) = 1$ and so page 13 of [10] yields that $\hat{C}_0$ is a subgroup of $2\mathcal{S}_4$. Again 3 divides the order of $\hat{C}$ and therefore of $\hat{C}_0$. We recall that $O_3(\hat{C}_0) = 1$ and hence $\hat{C}_0$ has a subgroup $\hat{A}$ isomorphic to $\mathcal{A}_4$. Furthermore we obtain from [10] that $\mathrm{SL}_3(3)$ has only one class of involutions. Therefore all involutions centralise a subspace of dimension 1 of V and invert a subspace of dimension 2 of V. As $\hat{A}$ fixes v, all involutions of $\hat{A}$ fix the same 1-dimensional subspace of V. Let $a, b \in \hat{A}$ be two distinct involutions and let V_a and V_b be the subspaces that are inverted by a and b, respectively. Then ab centralises $V_a \cap V_b$, which contradicts our statement above. $\Box$

Lemma 6.3. *Suppose that Hypothesis 6.1 holds and that $r_3(G) \ge 3$. Then G has a strongly 3-embedded subgroup that contains C.*

Proof. Let $y \in G$ be an element of order 3. By Sylow's Theorem we may suppose that $y \in P$. We recall that $x \in Z(P)$ by Lemma 3.2, so $x \in C_G(y)$ and therefore $C_G(y) = C_C(y)O_{3'}(C_G(y))$ by Lemma 4.2 (a). As C is soluble by Hypothesis 6.1, we deduce that $F^*(C_G(y)/O_{3'}(C_G(y)))$ is a 3-group. In addition Corollary A1 of [20] provides some $y \in P$ of order 3 such that $O_{3'}(C_G(y)) \neq 1$. Now we may apply Theorem A of [27], which gives our assertion. $\square$

Lemma 6.4. *Suppose that Hypothesis 6.1 holds and that M is a strongly 3-embedded subgroup of G such that $C \leq M$. The the following is true:*

(a) *For all 3-subgroups $1 \neq R$ of M we have that $N_G(R) \leq M$.*
(b) *The group M is 3-soluble and a maximal subgroup of G.*
(c) *Let q be a prime. Then there is a Sylow q-subgroup Q of M such that PQ is a subgroup of M and $[Q, x]$ is P-invariant.*
(d) *Suppose that $H \leq G$ is 3-soluble and that $H \cap C$ has non-cyclic Sylow 3-subgroups. Then $H \leq M$.*
(e) *If $x \in U \leq G$ is such that U has non-cyclic Sylow 3-subgroups, then M is the unique maximal subgroup of G containing U.*

Proof. For (a) let $1 \neq R$ be a 3-subgroup of M and let $g \in N_G(R)$. Then $R \leq M \cap M^g$ and so 3 divides $M \cap M^g$. This implies that $g \in M$.
For (b) we see that $x \in M < G$. Hence Lemma 4.2 (a) yields that $M = C \cdot O_{3'}(M)$. Then M is 3-soluble because C is soluble. Let L be a maximal subgroup of G such that $M \leq L$. As $x \in M \leq L$ we see that $L = C_L(x) \cdot O_{3'}(L)$ by Lemma 4.2 (a). Using Lemma 4.2 (b) let $V \leq P$ be an elementary abelian subgroup of order 9. Then Lemma 2.1 (e) implies that $O_{3'}(L) = \langle C_{O_{3'}(L)}(v) \mid v \in V^\# \rangle \leq M$, because M is strongly 3-embedded. Thus $L \leq M$ by (a).
The statement in (c) is clear if $q = 3$ or if $q \notin \pi(M)$, so we suppose that $q \neq 3$ and $q \in \pi(M)$. By Lemma 2.1 (f) there exists a P-invariant Sylow q-subgroup Q_0 of $O_{3'}(M)$. Then a Frattini argument and Lemma 4.2 (a) yield that $M = N_M(Q_0) \cdot O_{3'}(M) = N_C(Q_0) \cdot O_{3'}(N_M(Q_0)) \cdot O_{3'}(M) = N_C(Q_0) \cdot O_{3'}(M)$.
As $N_C(Q_0)$ is soluble and contains P, we find a Sylow q-subgroup U of $N_C(Q_0)$ such that PU is a subgroup of $N_C(Q_0)$. Now $Q := UQ_0$ is a Sylow q-subgroup of M and $PQ = (PU)Q_0$ is a subgroup of M. Lemma 4.2 (a) implies that $[Q, x] \leq [M, x] \cap Q \leq O_{3'}(M) \cap Q = Q_0$. Now we let $y \in P$. First we observe that $[Q, x]^y \leq Q_0^y = Q_0 \leq Q$ because Q_0 is P-invariant. Then Lemma 2.1 (a) and the fact that $x \in Z(P)$ give that $[Q, x]^y = [Q, x, x]^y = [[Q, x]^y, x^y] \leq [Q, x^y] = [Q, x]$.
For (d) let P_0 be a Sylow 3-subgroup of $H \cap C$ and $P_0 \leq R \in \mathrm{Syl}_3(H)$. Then for all subgroups U of P_0 we have that $N_G(U) \leq M$ by (a), and in particular the 3-group $N_R(P_0)$ lies in M. Then (a) forces $R \leq M$. We also know that R is not cyclic by hypothesis, so we let $Y \leq R$ be an elementary abelian subgroup of order 9. Lemma 2.1 (e) yields that $O_{3'}(H) = \langle O_{3'}(H) \cap C_G(y) \mid y \in Y^\# \rangle \leq M$ by (a).Next we let $R_0 := O_{3',3}(H) \cap R$. Then $N_G(R_0) \leq M$ by (a) since $R \leq M$, and a Frattini argument gives that $H = N_H(R_0) \cdot O_{3'}(H)$, because H is 3-soluble. Consequently $H = N_H(R_0) \cdot O_{3'}(H) \leq M$ as stated.
Finally for (e) we suppose that $U \leq G$ is such that the Sylow 3-subgroups of U are not cyclic and $x \in U$. Let L be a maximal subgroup of G containing U. Then Lemma 4.2 (a) implies that $L = C_L(x) \cdot O_{3'}(L)$. In particular L is 3-soluble and $L \cap C$ has non-cyclic Sylow 3-subgroups. Thus (d) implies that $L \leq M$ and hence $L = M$. $\square$

Lemma 6.5. *Suppose that Hypothesis 6.1 holds and that $r_3(G) \geq 3$. Then $|M : C|$ is even.*

Proof. Assume for a contradiction that $|M : C|$ is odd. By Theorem 4.4 (c) we know that G has a section isomorphic to $\mathcal{S}_4$. We know follow the ideas of Lemma 5.2 in [20]. There is a non-trivial 2-subgroup T such that $N_G(T)$ involves $\mathcal{S}_4$ by Lemma 2.3 of [20], and we choose T of maximal order with that property. Let $H := O^{2'}(N_G(T))$. By conjugation we may choose T such that $N_P(T)$ is a Sylow 3-subgroup of

$N_G(T)$.

From Lemma 6.3 we obtain a strongly 3-embedded subgroup M of G such that $C \subseteq M$.

(I) *$T = O_2(N_G(T))$ and $O_{2',2}(H) = O(H)O_2(H)$.*

Proof. The maximal choice of T implies that $T = O_2(N_G(T)) = O_2(H)$. If $S_0 \in \mathrm{Syl}_2(O_{2',2}(H))$, then $T = O_2(H)$ is contained in S_0. A Frattini argument yields that $H = N_H(S_0) \cdot O_{2',2}(H) = N_H(S_0) \cdot O(H)$ and so $N_H(S_0)/N_{O(H)}(S_0) \cong H/O(H)$. Since $N_G(T)$ involves the group $\mathcal{S}_4$, H does as well. In particular $H/O(H)$ is not $\mathcal{S}_4$-free. Altogether $N_G(S_0)$ is not $\mathcal{S}_4$-free and the maximal choice of T forces $S_0 = T$. In particular $O_{2',2}(H) = O(H)O_2(H)$. ∎

(II) *A Sylow 2-subgroup S_0 of C has rank 1 and $C_G(S_0) \nleq M$. In addtion $C_G(y)$ has odd order for every $y \in P \setminus \langle x \rangle$ of order 3.*

Proof. As C is soluble by Hypothesis 6.1, there is a Sylow 2-subgroup S_0 of C such that PS_0 is a subgroup of C. By our assumption S_0 is a Sylow 2-subgroup of M and so Lemma 5.5 implies that $N_G(S_0) \nleq M$. Consequently, we see from Lemma 4.2 (d) that $C_G(S_0) \lneq M$, as $N_G(S_0) = N_C(S_0)C_G(S_0)$, and Lemma 6.4 (e) forces $C_G(a)$ to have cyclic Sylow 3-subgroups for every $a \in S_0 \leq C$. In particular $C_G(y)$ has odd order for every $y \in P \setminus \langle x \rangle$ of order 3.

From the hypothesis $r(P) \geq 3$ we moreover obtain an elementary abelian subgroup V of P of order 9 such that $V \cap \langle x \rangle = 1$. Then for all $v \in V^\#$ the investigation above says that $C_G(v)$ has odd order. Hence we deduce from Lemma 2.1 (e) that $O_2(S_0P) = \langle C_{O_2(S_0P)}(v) \mid v \in V^\# \rangle = 1$. This implies that $F^*(S_0P) = O_3(S_0P)$.

Assume for a contradiction that S_0 has an elementary abelian subgroup A that is not cyclic. Then for all $a \in A^\#$ our assumption implies that $\Omega_1(C_P(a)) \leq \langle x \rangle$. By Corollary 14.4 of [6] there is a characteristic subgroup R of $O_3(S_0P)$ of exponent 3 such that every non-trivial $3'$-automorphism of $O_3(S_0P)$ induces a non-trivial automorphism on R. Then Lemma 2.1 (e) forces $R = \langle C_R(a) \mid a \in A^\# \rangle \leq \langle x \rangle$. Thus A centralises R and hence it centralises $O_3(S_0P) = F^*(S_0P)$. But this is a contradiction. ∎

(III) *The group $H/O(H)$ has elements of order 3, but no elements of order 6.*

Proof. Since H involves $\mathcal{S}_4$, the group $H/O(H)$ contains an element of order 3.

Assume for a contradiction that $H/O(H)$ has an element of order 6. Then there exists a 2-element $b \in H$ such that $C_H(b)$ is divisible by 3. This gives an element $y \in H$ of order 3 such that $C_G(y)$ has even order. Since $N_P(T)$ is a Sylow 3-subgroup of $N_G(T)$, we may suppose that $y \in P$. Hence (II) implies that $y \in \langle x \rangle$ and in particular $x \in H \leq N_G(T)$. As $N_G(T)$ does not have a normal 3-complement, Lemma 3.4 (e) and 6.4 (e) show that $N_G(T) \leq M$. Let $S_0 \in \mathrm{Syl}_2(M)$ be such that $T \leq S_0$. Then $C_G(S_0) \leq C_G(T) \leq N_G(T) \leq M$ contradicts (II). ∎

We see by (I) that G and $N_G(T)$ satisfy Voraussetzung I of [22]. Furthermore $H/O(H)$ contains an element of order 3, but no element of order 6 by (III). We apply Satz A of [22] and let S be a Sylow 2-subgroup of G. With regard to Lemma 5.2 we may suppose that S is x-invariant.

By Theorem 4.4 (a) and (e) the group S is neither dihedral, semi-dihedral, a wreath product $C_{2^n} \wr C_2$ for some $n \geq 1$ nor isomorphic to a Sylow 2-subgroup of $\mathrm{PSL}_3(2^n)$ for some $n \geq 2$. Hence (a), (b) and (c) of Satz A are false. Additionally Part (d) of Theorem 4.4 implies that Part (f) of Satz A does not hold.

Assume for a contradiction that Satz A (d) of [22] holds. Then $Z(S)$ is cyclic and hence it is centralised by x. If $x \in O(C_G(Z(S)))$, then $[S, x] \leq O(C_G(Z(S))) \cap S = 1$, contrary to Lemma 5.5. Thus Satz A (d) of [22] implies that x embeds into $\mathcal{A}_5$. But this is impossible by Remark 4.5.

We conclude that Part (f) of Satz A in [22] is true. Then $J(S)$ is isomorphic to a Sylow 2-subgroup of $\mathrm{PSL}_3(4)$, in particular it has exactly two, and it is x-invariant. Now x normalises both elementary abelian subgroups of $J(S)$ of order 2^4, so we let A be one of them and set $N := N_G(A)$. Then Satz A (e) of [22] yields that $N/C_G(A) \cong A_6 \cong \mathrm{PSL}_2(9)$, which by Remark 4.5 does not contain an isolated element of order 3. Thus x centralises A. But A is elementary abelian of order 2^4, contrary to (II). $\square$

7. The 2-structure of G

We prove first that G is connected for the prime 2. Then we extend our hypothesis and show that G has local characteristic 2, and we use this information in order to restrict the structure of a strongly 3-embedded subgroup of G containing C.

Theorem 7.1. *Suppose that Hypothesis 6.1 holds and that $r_3(G) \geq 3$. Then G is connected for the prime* 2.

Proof. By Lemma 6.3 there is a strongly 3-embedded subgroup M of G containing C. Then Lemma 6.4 (c) provides a Sylow 2-subgroup T of M such that $[T, x]$ is normalised by P and $TP \leq M$, and T is contained in an x-invariant Sylow 2-subgroup S of G by Lemma 4.6.
We assume for a contradiction that S is not connected. Let $c \in Z(S)$ be an involution and set $S_0 := [S, x]$.

(I) *$Z(S)$ is cyclic and S does not have any normal elementary abelian subgroups of rank* 3 *or more. In particular c is the unique involution in $Z(S)$.*

Proof. It follows directly from our assumption and Definition 5.3 (a) that $\Omega_1(Z(S))$ is cyclic. Therefore $Z(S)$ is cyclic. The lemma in Section 4.1 of Chapter II in [19] yields the second assertion. ∎

(II) *S has a unique elementary abelian normal subgroup A of order* 4*, and then $c \in A \leq C$. Moreover $C_S(A)$ is a maximal subgroup of S.*

Proof. $S \in \mathrm{Syl}_2(G')$ by Lemma 4.3 and $S \not\cong D_8$ by Theorem 4.4 (a). Then Lemma 1 of [18] gives the first statement. It follows from (I) that $c \in A$, and the uniqueness property gives that $A^x = A$ because S is x-invariant. Moreover, x normalises $Z(S)$ and in particular $c^x = c$. This forces $[A, x] = 1$. Then $C_S(A)$ is a maximal subgroup of S because $A \not\leq Z(S)$ by (I). ∎

We keep the subgroup A from (II) and we let $a \in A$ be such that $A = \langle a, c\rangle$.

(III) *Let $Q \leq S_0$ be x-invariant and such that $[Q, x] = Q \cong Q_8$. Then $N_S(Q) = C_S(Q)Q \neq S$.*

Proof. We note that $x \in N_G(Q)$ and $\mathrm{Aut}(Q) \cong \mathcal{S}_4$. As x is isolated, this implies that $N_G(Q) = \langle x\rangle Q C_G(Q)$. In particular $N_S(Q) = QC_S(Q)$. Assume for a contradiction that $S = C_S(Q)Q$. Then $Q \trianglelefteq S$ and so $c \in Q$. Since A is normal in S, we have that $[A, S] \leq A \leq C$. This implies that $[[A, S], x] = 1 = [[x, A], S]$. Thus the Three Subgroups Lemma (see 1.5.6 of [17]) yields that $1 = [[S, x], A] = [S_0, A] \geq [Q, A]$.
Using (I) we see that $A \not\leq Z(S)$. Hence there is some $d \in S = C_S(Q)Q = C_S(Q)C_S(A)$ such that $a^d = ac$. We may choose $d \in C_S(Q)$. Then we let B be a maximal abelian normal subgroup of $C_S(Q)$ that contains A and we show that $B = A$:
In order to do this we let $t \in Q$ be of order 4. Then $t^2 = c$, $\langle t\rangle \trianglelefteq Q$ and $t \notin C_S(Q)$. Now t centralises $B \leq C_S(Q)$ and hence $\langle B, t\rangle$ is abelian. Moreover $\langle B, t\rangle$ is normal in $C_S(Q)Q = S$. From (I) it follows that $\langle B, t\rangle$ has rank 2 and then $\Omega_1(\langle B, t\rangle) = A$ by (II).
If B has an element u of order 4 that squares to c, then t and u are commuting elements of order 4 of $\langle B, t\rangle$ that square to c, so their product tu is an involution. In particular $tu \in \Omega_1(\langle B, t\rangle) = A \leq B$ and so $t = (tu)u^{-1} \in B \leq C_S(Q)$. This is a contradiction. Thus c is not a square in B.

If there is some $b \in B$ of order 4 such that $b^2 = a$, then $b^d \in B \trianglelefteq C_S(S_0)$ and $(bb^d)^2 = b^2(b^2)^d = aa^d = c$. So bb^d is an element of order 4 in B that squares to c, which is impossible by the previous paragraph. Hence B does not have any elements that square to a or c. A similar argument shows that B does not have any elements that square to ac.

This implies that $B = A$ and so A is a maximal abelian normal subgroup of $C_S(Q)$. We deduce from 5.1.7 of [17] that $C_S(A) \cap C_S(Q) = A$. This forces $C_S(a) \cap C_S(Q) = A$ and hence 5.3.10 of [17] yields that $C_S(Q)$ is dihedral or semi-dihedral. Together with the fact that $A \trianglelefteq C_S(Q)$ it follows that $C_S(Q) = A$ or $C_S(Q) \cong D_8$. But $d \in C_S(Q)$ and hence $C_S(Q) \cong D_8$.

Let A_0 be a fours group in $C_S(Q)$ that is distinct from A. Then $A_0 \trianglelefteq C_S(Q)$ and $[A_0, Q] = 1$ and therefore $A_0 \trianglelefteq S = C_S(Q) \cdot Q$, contrary to (II). This is our final contradiction. ∎

We let $- : S\langle x \rangle \to S\langle x \rangle / \Phi(S_0)$ denote the natural epimorphism.

(IV) *$A = Z(S_0) = \Phi(S_0) = S_0' = C_{S_0}(x)$. In particular $\bar{x}$ acts fixed-point-freely on $\bar{S}_0$.*

Proof. Lemmas 2.1 (b) and 5.5 give that $S_0 \neq 1$. We intend to use Satz III 13.6 of [16] for the group $S_0\langle x \rangle$. Therefore we take a characteristic subgroup B of S_0 that is abelian. Now B is x-invariant because S_0 is. If B is cyclic, then x centralises it. Otherwise we note that $S = N_S(S_0)$ normalises $\Omega_1(B)$, which is not cyclic, and then it follows from the uniqueness of A and (I) that $\Omega_1(B) = A$. Hence x centralises $\Omega_1(B)$ and then it centralises the abelian group B by Lemma 2.1 (g). As $[[S_0, x], x] = [S_0, x] = S_0$ by Lemma 2.1 (a), Satz III 13.6 of [16] gives that S_0 is special. We conclude that $Z(S_0) = \Phi(S_0) = S_0'$ is an elementary abelian normal subgroup of S. Thus (I) and (II) yield that $Z(S_0) = A$ or $Z(S_0) = \langle c \rangle$. In both cases $Z(S_0) \leq C_G(x)$.

Considering the action of $\overline{x}$ on the elementary abelian group $\overline{S_0}$, we see that $[\overline{S_0}, \overline{x}] = \overline{[S_0, x]} = \overline{S_0}$ and hence $\overline{S_0} \cap C_{\overline{S_0}}(\overline{x}) = 1$. This implies that $\overline{C_{S_0}(x)} = C_{\bar{S}_0}(\bar{x}) = 1$ by Lemma 2.1 (c). Hence $C_{S_0}(x) \leq \Phi(S_0) = Z(S_0)$ implies that $C_{S_0}(x) = Z(S_0) = \Phi(S_0) = S_0'$ and that $\bar{x}$ acts fixed-point-freely on $\bar{S}_0$.

Assume for a contradiction that $\Phi(S_0) = Z(S_0) = S_0' = C_{S_0}(x) = \langle c \rangle$. Then S_0 is extra-special and $A \not\leq S_0$, as $A \leq C$ by (II). In particular S_0 does not contain any normal fours subgroup of S. With Lemma 1.4 of [6] we obtain that S_0 is dihedral or quaternion of order 8. Since S_0 admits an automorphism of order 3 induced by x, we conclude that $S_0 \trianglelefteq S$ is a quaternion group of order 8, contrary to Lemma 2.1 (a) and (III). ∎

(V) *Let $U \leq S_0$ be $C_S(x)$-invariant such that $A \leq U$. Then $U \trianglelefteq S$ and we have that $U = S_0$, $U = A$, or $U = [U, x] \times \langle a \rangle \cong Q_8 \times C_2$ and $C_S(x) \leq C_S([U, x])$. In particular $|\bar{S}_0| = 16$.*

Proof. The group $\bar{S}_0 = S_0/\Phi(S_0)$ is elementary abelian. Part (I), the Four Generator Theorem of [18] and Burnside's Basis Theorem (see III 3.15 of [16]) yield that $\overline{S_0}$ has order at most 16.

Furthermore $\bar{x}$ acts fixed-point-freely on $\bar{S}_0$ by (IV). Thus $|\bar{U}| - 1$ is divisible by 3. This implies that $|\bar{U}| \in \{1, 4, 16\}$. If $\bar{U} = 1$, then $U = A$. If $|\bar{U}| = 16$, then $U = S_0$.

If $\bar{U}$ has order 4, then $|U| = 16$ and $U \trianglelefteq S_0$ by (IV). Hence $U \trianglelefteq S_0 C_S(x) = S$ by our assumption. It follows from (I) that $r(Z(U)) \leq 2$. Moreover $A \leq Z(U)$ by (IV) and so x centralises $\Omega_1(Z(U))$. Then Lemma 2.1 (g) implies that x centralises $Z(U)$. Since x acts fixed-point-freely on $\bar{U}$ by (IV), we deduce that U is not abelian.

As $U' \leq A$ by (IV), we conclude that $1 \neq U' \leq A$. Furthermore Proposition 1.6 of [6] implies that $|U : U'| \neq 4$, as $A \leq Z(U)$. Altogether U' is a cyclic characteristic subgroup of $U \trianglelefteq S$. This implies, together with (I), that $U' = \langle c \rangle \leq C$.

Lemma 2.1 (b) and (c) yield that $U/U' = [U/U', x] \times C_{U/U'}(x) = [U, x]/U' \times C_U(x)/U'$. We conclude that $U = [U, x]\langle c \rangle \times \langle a \rangle$ from $C_U(x) = \langle a, x \rangle$, and that $[U, x]\langle c \rangle$ is a non-abelian x-invariant subgroup

of order 8 of U. Since $[U,x]$ is not centralised by x, it follows that $[U,x]\langle c\rangle = [U,x] \cong Q_8$. Thus we have that $U = [U,x] \times \langle a\rangle \cong Q_8 \times C_2$.
As U is $C_S(x)$-invariant, we see that $C_S(x) \leq N_S([U,x]) \cap C = [U,x]C_S([U,x]) \cap C = C_S([U,x])$ by (III). Finally $[U,x] \lneq U$ and $[S_0,x] = S_0$ by Lemma 2.1 (a), and consequently $U \neq S_0$. ∎

(VI) $S = T$.

Proof. We first notice that $S \cap M = T$ and $C_S(x) = C_T(x)$. Moreover Lemma 2.1 (b) shows that $M \cap S_0 = [M \cap S_0, x] \cdot (C \cap M \cap S_0) = [T,x] \cdot C_{S_0}(x)$. This implies that $M \cap S_0$ is a $C_S(x)$-invariant subgroup of S_0. As $A \leq M \cap S_0$ by (IV), it follows that $M \cap S_0$ is normal in S_0 and that $M \cap S_0 \trianglelefteq S$. We may apply (V) and see that our assertion follows if $|M \cap S_0| > 16$.
If $M \cap S_0 = [T,x] \cdot \langle a\rangle \cong Q_8 \times \langle a\rangle$, then $1 \neq (M \cap S_0)'$ is characteristic in $M \cap S_0$ and, therefore, normal in S. Furthermore $(M \cap S_0)' = Z([T,x])$ is normalised by P. Thus Lemma 6.4 (e) shows that $S \leq N_G(Z([T,x])) \leq M$.
We assume for a contradiction that $M \cap S_0 = A = C \cap S_0$. Then $[T,x] = 1$ and so $S \neq T \leq C$ contradicts Lemma 6.5. ∎

(VII) $C_S(S_0) = A$.

Proof. As $C_S(S_0)$ is x-invariant, Lemma 2.1 (b) and (IV) yield that
$C_S(S_0) = (C_S(S_0) \cap C)[C_S(S_0), x] = (C_S(S_0) \cap C)(C_S(S_0) \cap S_0) = (C_S(S_0) \cap C)A \leq C$.
Let $t \in C_S(S_0)$ be such that $t^2 \in A$. Then by (IV) the group $\overline{S_0\langle t\rangle}$ is abelian and $t^2 \in A = \Phi(S_0)$. It follows that $\overline{S_0\langle t\rangle}$ is elementary abelian and so $\Phi(S_0\langle t\rangle) = \Phi(S_0)$. Then the Four Generator Theorem of [18], Burnside's Basis Theorem (see III 3.15 of [16]) and (V) imply that $|\overline{S_0\langle t\rangle}| \leq 16 = |\bar{S}_0|$. Thus $t \in S_0 \cap C_S(S_0) = A$. This implies that $A = C_S(S_0)$. ∎

(VIII) *There is some $y \in P \setminus \langle x\rangle$ of order 3 that centralises S_0 and there is some $C_S(x)$-invariant normal subgroup of S_0 of order 16 containing A.*

Proof. Let V be an elementary abelian subgroup of order 27 of P. Then (VI) implies that V normalises $[T,x] = S_0$ and so $\Phi(S_0) = C \cap S_0$. We conclude that the elementary abelian group $\bar{S}_0$ of order 16 is V-invariant. From $|\mathrm{GL}_4(2)|_3 = 3^2$ it follows that $|C_V(\bar{S}_0)| \neq 3$. Thus Lemma 2.1 (h) gives the first assertion.
Considering the action of $\bar{x}$ on $\overline{S_0}$ by conjugation, we see that there are five $\bar{x}$-orbits. A straight forward computation shows that every orbit generates a subgroup of order 4 of $\overline{S_0}$. In particular the five $\bar{x}$-invariant subgroups of $\bar{S}_0$ are permuted by $\overline{C_S(x)}$. Since $\overline{C_S(x)}$ is a 2-group, at least one of the five x-invariant fours groups in $\overline{S_0}$ is normalised by $\overline{C_S(x)}$. So its full pre-image is a $C_S(x)$-invariant subgroup of S_0 of order 16 containing A. By (IV) we see that the full pre-image is normal in S_0. ∎

(IX) *There is an involution $d \in C_S(x)$ such that $\langle d\rangle S_0 = S$.*

Proof. According to (VIII) let E be a $C_S(x)$-invariant normal subgroup of S_0 such that $A \leq E$ and E has order 16. Then (V) implies that $E = [E,x] \times \langle a\rangle \cong Q_8 \times \langle a\rangle$ and $C_S(x) \leq C_S([E,x]) \leq N_S([E,x])$. Now (III) and the fact that $S = S_0C_S(x)$ show that $N_{S_0}([E,x])$ is a proper $C_S(x)$-invariant subgroup of S_0. As $A \leq E \leq N_{S_0}([E,x])$, we deduce from (V) that $E = N_{S_0}([E,x])$. In particular $C_{S_0}(E) = Z(E) = A$ (**).
We set $D := C_{C_S(x)}(E)$. Then $A \leq D \leq C_{C_S(x)}(A)$, and from $C_S(x) \leq C_S([E,x])$ and $E = [E,x] \cdot A$ we obtain that $D = C_{C_S(x)}(A)$. It follows from Lemma 2.1 (b) that $C_S(E) = (C_S(E) \cap C)[C_S(E), x] \leq DS_0$. Then Dedekind's modular law (see 1.1.11 of [17]) and (**) imply that $C_S(E) = D(C_S(E) \cap S_0) = DA = D$.

As E is a normal subgroup of S, we see that S normalises first $C_S(E) = D$ and then $C_G(D)$. Now $x \in C_G(D)$ and therefore $S_0 = [S, x] \leq S \cap C_G(D) = C_S(D)$, which together with (VII) yields that $D \leq C_S(S_0) = A$. This implies that $D = A$, i.e. $C_{C_S(x)}(A) = D = A$. Then it follows from Lemma 2.1 (b) that $C_S(A) = (C_S(A) \cap C)[C_S(A), x] \leq AS_0 = S_0$ and so that $C_S(A) = S_0$ from (IV).
Finally $|C_S(x) : A| = |C_S(x) : C \cap S_0| = |S : S_0| = |S : C_S(A)| = 2$ by (II). Thus $C_S(x)$ is a non-abelian group of order 8 containing A. More precisely $C_S(x)$ is a dihedral group of order 8. This provides an involution $d \in C_S(x)$ such that $S = C_S(x)S_0 = \langle d\rangle AS_0 = \langle d\rangle S_0$. ∎

As $O^2(G) = G$, we obtain from Thompson's Transfer Lemma (see 12.1.1 of [17]) an element $g \in G$ such that $d^g \in S_0$. Consequently y centralises d^g by (VIII) and x^g centralises d^g. But y is not conjugate to x^g in G, because x is isolated in G. Thus $C_G(b^g)$ has non-cyclic Sylow 3-subgroups and Lemma 6.4 (e) yields that $y \in C_G(b^g) \leq M^g$. Now $g \in M$ by Part (b) of the same lemma and then $d \in S_0^{g^{-1}} \leq [x, M]^{g^{-1}} = [x, C \cdot O_{3'}(M)]^{g^{-1}} = [x, O_{3'}(M)]^{g^{-1}} \leq O_{3'}(M)^{g^{-1}} = O_{3'}(M)$ by Lemma 4.2 (a).
This implies that $S = \langle d\rangle S_0 \leq O_{3'}(M)$. In particular $[v, b] \in O_{3'}(M)$ for all 3-elements $v \in M$ and all 2-elements $b \in M$. This contradicts Lemma 5.2. □

Here is what we work with in the remainder of this section:

Hypothesis 7.2. *In addition to Hypothesis 6.1, suppose the following:*

(a) *The centraliser of every involution of every section of G is soluble.*
(b) $r_3(G) \geq 3$.

Corollary 7.3. *Suppose that Hypothesis 7.2 holds. Then G has local characteristic 2.*

Proof. We check the hypothesis of Theorem A of [27] with the prime 2. Theorem 7.1 shows that G is connected for the prime 2. By Hypothesis 7.2 (a) we have that $F^*(C_G(s)/O(C_G(s)))$ is a 2-subgroup for every involution $s \in G$. Finally $r_2(G) \geq 3$ by Theorem 4.4 (a). So the theorem yields that $O(C_G(s)) = 1$ for every involution $s \in G$ because, by Theorem 4.4 (b), G does not have a strongly embedded subgroup. Then Hypothesis 7.2 (a) and 12.1.2 of [17] yield the assertion. □

In the next section we will analyse the structure of M in much more detail. A special case will be treated now:

Lemma 7.4. *Suppose that Hypothesis 7.2 holds and let M be a strongly 3-embedded subgroup of G containing C. Then $F^*(M) = O_2(M)$.*

Proof. By Lemma 6.4 (c) there is a Sylow 2-subgroup S of M such that $S_0 := [S, x]$ is normalised by P and $SP \leq M$.

(I) *If $R \leq P$ is not cyclic, then $r_2(C_G(R)) \leq 1$ or $O(M) = 1$.*

Proof. Let $A \leq C_G(R)$ be elementary abelian, but not cyclic. Then $R \leq C_G(a)$ for all $a \in A^\#$. Since $C_G(a)$ is soluble by Hypothesis 7.2 (a), we deduce from Lemma 6.4 (d) that $C_G(a) \leq M$ for all $a \in A^\#$. Then for every $a \in A^\#$ we have that $C_G(a) \cap O(M) \leq O(C_G(a)) = 1$, as G has local characteristic 2 by Corollary 7.3. Finally, Lemma 2.1 (e) implies that $O(M) = \langle C_{O(M)}(a) \mid a \in A^\# \rangle = 1$. ∎

(II) $E(M) = 1$.

Proof. Assume for a contradiction that $E(M) \neq 1$. Then Hypothesis 7.2 (a) implies that $E(M)$ is simple and $O_2(M) = 1$. Since $x \in M$ and C is soluble by Hypothesis 6.1, we deduce from Lemma 2.5 that $E(M) \cong \mathrm{Sz}(8)$. Moreover, P normalises $E(M)$ and $r_3(G) \geq 3$ by Hypothesis 7.2 (b), so Lemma 2.4 (a) gives an elementary abelian subgroup W of P of order 9 that centralises $E(M)$. Hence Lemma

2.4 (b) implies that $r_2(C_G(W)) \geq 3$. In particular (I) yields that $O(M) = 1$ and so $F(M) = 1$. But now $W \leq C_M(E(M)) = C_M(F^*(M)) \leq F^*(M) = E(M) \cong \mathrm{Sz}(8)$, which contradicts Lemma 2.4. ∎

We assume for a contradiction that $O(M) \neq 1$.

(III) *Let T be a P-invariant 2-subgroup of M. Then $Z(T)$ is cyclic and $[P, Z(T)] = 1$.*

Proof. Let V be an elementary abelian subgroup of P of order 27. Then the abelian group $Z(T)$ is normalised by V and so Lemma 2.1 (d) yields that $Z(T) = \prod_{W \max V} C_{Z(T)}(W)$. Our assumption and (I) imply that $C_{Z(T)}(W)$ is cyclic for all $W \max V$. Then, as V normalises $C_{Z(T)}(W)$, it follows that V centralises $C_{Z(T)}(W)$ for all $W \max V$. Altogether $Z(T) \leq C_G(V)$ and so $Z(T)$ is cyclic. Since a cyclic 2-group does not admit an automorphism of order 3, we finally obtain that $[P, Z(T)] = 1$. ∎

(IV) *$O(M) = F(M) = F^*(M)$.*

Proof. Otherwise (II) implies that $O_2(M) \neq 1$. Thus, by (III), there is an involution $a \in Z(O_2(M))$ such that $P \leq C_G(a)$. Then $C_G(a) \leq M$ by Lemma 6.4 (e) and $O(M) \leq C_G(O_2(M)) \leq C_G(a)$. Then Corollary 7.3 gives that $O(M) \leq O(C_G(a)) = 1$, which is a contradiction. ∎

(V) *S is a Sylow 2-subgroup of G.*

Proof. From Lemma 6.5 and Lemma 2.1 (b) we have that $S_0 = [S, x] \neq 1$ and so $1 \neq Z(S) \cap S_0 \leq Z(S_0)$. Consequently $Z(S) \cap S_0$ is centralised by P by (III). In particular $C_G(Z(S)) \leq C_G(Z(S) \cap S) \leq M$ by Lemma 6.4 (e) and $Z(S) \cap S_0$ is centralised by x. Now Lemma 2.1 (b) shows that $Z(S) = [Z(S), x] \cdot C_{Z(S)}(x) \subseteq (Z(S) \cap S_0) \cdot C \subseteq C$ and Lemma 4.2 (d) yields that $N_G(S) \subseteq N_G(Z(S)) = N_C(Z(S)) \cdot C_G(Z(S)) \leq M$. ∎

Let $c \in Z(S_0)$ be an involution.

(VI) *If $a \in S$ is an involution and $a \neq c$, then $C_P(a) = 1$.*

Proof. As S_0 is a normal subgroup of S and $Z(S_0)$ is cyclic by (III), we see that $c \in \Omega_1(Z(S_0)) \leq Z(S)$. In addition $P \leq C_G(c)$ by (III) and so Lemma 6.4 (e) yields that $C_G(c) \leq M$. Then it follows from Corollary 7.3 that $O(M) \cap C_G(c) \leq O(C_G(c)) = 1$.
Let $a \in S$ be an involution distinct from c. Assume for a contradiction that $O(M) \cap C_G(a) = 1$. Then $O(M) = \langle O(M) \cap C_G(a), O(M) \cap C_G(c), O(M) \cap C_G(ac) \rangle \leq C_G(ac)$ by Lemma 2.1 (e). In particular $ac \in C_G(O(M)) \leq O(M)$ by (IV), which is a contradiction.
As S is a Sylow 2-subgroup of G by (V), there is some $g \in G$ such that $a^g \in O_2(C_G(a))^g \leq S$. Then $c \in Z(S) \subseteq C_G(a^g) \cap C_G(O_2(C_G(a^g)))$. Moreover $O_2(C_G(a^g)) = F^*(C_G(a^g))$ by Corollary 7.3, so we conclude that $c \in O_2(C_G(a^g))$.
Now we see that $[O(M) \cap C_G(a^g), c] \leq O(M) \cap O_2(C_G(a^g)) = 1$ and hence $O(M) \cap C_G(a^g) \leq O(M) \cap C_G(c) = 1$.
Thus we deduce that $a^g = c$ by the investigation above.
Assume for a contradiction that $1 \neq C_P(a)$. Then $1 \neq C_P(a) \leq C_G(a) \leq M^g$. But M is strongly 3-embedded. This implies that $g \in M$. Therefore $C_G(a) \cap O(M) = C_G(c^{g^{-1}}) \cap O(M) = (C_G(c) \cap O(M))^{g^{-1}} = 1$, which is false. ∎

We now work towards a final contradiction.
Let $A \leq S$ be elementary abelian of order 4 and such that $c \notin A$. This choice is possible by Theorem 4.4 (a).
Then Lemma 2.1 (e) and (VI) imply that $O_3(PS) = \langle C_G(a) \cap O_3(PS) \mid a \in A^\# \rangle \leq \langle C_P(a) \mid a \in A^\# \rangle = 1$.
Let $V \leq P$ be elementary abelian of order 27 and set $U := O_2(PS)$. Again Lemma 2.1 (e) yields that

$U = \langle C_U(v) \mid v \in V^\# \rangle$. Let $v \in V^\#$ be such that $C_U(v)$ has maximal order. Then it follows from (VI) that $r(C_U(v)) \leq 1$ and so $C_U(v)$ is cyclic or a quaternion group. Moreover $C_U(v)$ is normalised by V. As cyclic 2-groups and quaternion groups do not admit a non-cyclic 3-group of automorphisms, V has a maximal subgroup W that centralises $C_U(v)$. In particular $C_U(v) \leq C_U(w)$ for all $w \in W^\#$. The choice of v implies that $C_U(v) = C_U(w)$ for all $w \in W^\#$ and hence $U = \langle C_U(w) \mid w \in W^\# \rangle = C_U(v)$. This gives the contradiction $v \in C_{PS}(U) = C_{PS}(F^*(PS)) \leq F^*(PS) = U$.
Now $O(M) = 1$, and together with (II) the statement follows. □

8. The structure of M

In [20] and [26] it becomes apparent that the connection between the 2-structure and the 3-structure is crucial in the analysis of a minimal counterexample. Therefore one of our main ideas is to prove that M contains a Sylow 2-subgroup of G. The way we approach this is very much inspired by Thompson's work on N-groups, see [24] and in particular [25].

Definition 8.1. *Let $U \trianglelefteq M \leq G$. Then U is **p-reduced (in M)** if and only if $O_p(M/C_M(U)) = 1$.*

Hypothesis 8.2. *Suppose that Hypothesis 7.2 holds.*
*We use the following **notation**:*
Let M be a strongly 3-embedded subgroup that contains C and let $T \in \mathrm{Syl}_2(M)$ be such that $PT = TP$. Moreover we set $R(M) := \langle U \leq M \mid U \trianglelefteq M$ and U is 2-reduced in $M\rangle$ and $B := \Omega_1(R(M))$, and whenever $H \leq G$, then $I^*(H) := \{h \in H \mid o(h) = 2$ and $C_G(h) \leq H\}$. We let $- : M \to M/C_M(B)$ denote the natural epimorphism.
Finally $T_0 := \langle B^g \mid g \in G$ and $B^g \leq T\rangle$.

Remark 8.3. *The notation makes sense in light of Lemma 6.3 and Lemma 6.4 (c).*
The factor group $\bar{M}$ acts on B via $b^{\bar{g}} := b^g$ for all $b \in B$ and $\bar{g} \in \bar{M}$. In particular $C_B(\bar{g}) = C_B(g)$ and $[B, \bar{g}] = [B, g]$ for all $g \in M$. In the following we therefore do not distinguish between a calculation in M and a calculation in $B \rtimes \bar{M}$.

Lemma 8.4. *Suppose that Hypothesis 8.2 holds. Then the following is true:*

(a) *$R(M)$ and B are abelian 2-reduced subgroups of M and $N_G(B) = M$.*
(b) *$F(\bar{M})$ has odd order.*
(c) *If $x \in C_M(B)$, then M is soluble.*
(d) *If A is an elementary abelian normal 2-subgroup of M that is 2-reduced in M, then $A \leq B$.*
(e) *Suppose that A is an elementary abelian 2-subgroup of G such that for some $g \in G$ the group $C_{M^g}(A)$ has non-cyclic Sylow 3-subgroups. Then $A^\# \subseteq I^*(M^g)$.*
(f) *If $C_P(B)$ is not cyclic, then $B^\# \subseteq I^*(M)$.*
(g) *$\bar{C} = C_{\bar{M}}(\bar{x})$.*

Proof. For (a) we apply Lemma 5.9 of [24]. Since $O_2(M) \neq 1$, the lemma says that $R(M)$ is a non-trivial, abelian and 2-reduced subgroup of M. Hence $B = \Omega_1(R(M))$ is non-trivial and abelian. Lemma 2.1 (g) implies that B is also 2-reduced. Moreover $R(M)$ is generated by normal 2-subgroups of M, so both $R(M)$ and B are normal 2-subgroups of M. Since M is a maximal subgroup of G by Lemma 6.4 (b), we deduce from Lemma 4.3 that $N_G(B) = M$.
Then (b) follows from (a) and the definition of 2-reducibility.

We turn to (c) and suppose that $x \in C_M(B)$. We note that $C_M(B) \trianglelefteq M$ and that $C_M(B)$ is soluble by Hypothesis 7.2 (a). Lemma 3.4 (c) implies that $M = C_M(B) \cdot C$ and then $\bar{M} = \bar{C}$ is soluble by Hypothesis 6.1. As $C_M(B)$ is also soluble, the assertion follows.
Let A be an elementary abelian normal 2-subgroup of M that is 2-reduced in M. Then $A \leq R(M)$ by the definition of $R(M)$ and, as A is elementary abelian, we see that $A \leq \Omega_1(R(M)) = B$. This is (d).
For (e) we let $R \in \mathrm{Syl}_3(C_{M^g}(A))$. Then by Sylow's Theorem there is some $h \in M^g$ such that $R^h \leq P^g$. Let $a \in A^\#$. Then $R^h \leq C_G(a^h)$ and we know that $C_G(a)$ is soluble by Hypothesis 7.2 (a). Therefore $C_G(a)^h$ is soluble. We apply Lemma 6.4 (d) to P^g and M^g and deduce that $C_G(a)^h \leq M^g$. This implies that $C_G(a) \leq M^g$ because $h \in M^g$. In addition, (f) is a consequence of (e).
Finally for (g) we observe that $\bar{C} \leq C_{\bar{M}}(\bar{x})$. Let $\bar{c} \in C_{\bar{M}}(\bar{x})$ and let $c \in M$ be a pre-image of $\bar{c}$. Then $x^c \in \langle x \rangle C_M(B)$ and Sylow's Theorem provides some $d \in C_M(B)$ such that $x^{cd} \in \langle x \rangle C_P(B)$. As x is isolated in G, Lemma 3.3 shows that $cd \in C$ and so $\bar{c} = \overline{cd} \in \bar{C}$. □

Lemma 8.5. *Suppose that Hypothesis 8.2 holds, that $\bar{y} \in \bar{P}$ is such that $\bar{y}^3 = 1$, that $b \in M$ is an involution and $\bar{y}^{\bar{b}} = \bar{y}^{-1}$. If $[B, \bar{y}]$ has order 4, then $[B, \bar{y}]^\# \subseteq I^*(M)$.*

Proof. We set $D := [B, \bar{y}]$. If $C_M(D)$ has non-cyclic Sylow 3-subgroups, then Lemma 8.4 (e) yields the assertion. We assume for a contradiction that $C_M(D)$ has cyclic Sylow 3-subgroups.
From Lemma 3.4 (d) we see that $\bar{y} \neq \bar{x}$. Let $y \in P$ be a pre-image of $\bar{y}$ of minimal order and let $H := C_M(B)\langle y, b \rangle$. Then $\mathcal{S}_3 \cong H/C_M(B)$. Now let $R \in \mathrm{Syl}_3(C_M(B))$ be such that $\langle y \rangle R \in \mathrm{Syl}_3(H)$. Then R is cyclic by assumption and $y \notin R$, so if $\langle y \rangle R$ is cyclic, then $R = 1$ and $o(y) = 3$. If $\langle y \rangle R$ is not cyclic, then Theorem 1.2 (a) of [6] and our choice of y as a pre-image of $\bar{y}$ of minimal order gives that $o(y) = 3$.
We know that $B \trianglelefteq M$ by Lemma 8.4 (a), and together with Lemma 6.4 (a) we deduce that $N_G(\langle y \rangle) = N_M(\langle y \rangle) \leq N_M([B, y]) = N_M(D)$. Moreover D has order 4 and so $N_M(D)/C_M(D)$ is isomorphic to a subgroup of $\mathcal{S}_3$. This implies that the Sylow 3-subgroups of $N_G(\langle y \rangle)$ have a maximal subgroup that is cyclic. Let $R_0 \in \mathrm{Syl}_3(N_G(\langle y \rangle)$ be such that $x \in C_P(y) \leq R_0$. Then Theorem 1.2 (a) of [6] shows that $\Omega_1(R_0) = \langle x, y \rangle$. We recall that $r(P) \geq 3$ and that P is connected by Hypothesis 7.2 and Lemma 6.2. So we find an elementary abelian non-cyclic subgroup W of P different from $\langle x, y \rangle$ such that $[\langle x, y \rangle, W] = 1$. In particular $W \leq C_P(y)$ and so $W \leq \Omega_1(C_P(y)) = \langle x, y \rangle$, which is a contradiction. □

Lemma 8.6. *Suppose that Hypothesis 8.2 holds, that V is an elementary abelian subgroup of order 27 of M and that $D \leq B$ is not cyclic and V-invariant. Then the following is true:*

(a) *For all maximal subgroups W of V the group $C_{[D,V]}(W)$ is trivial or of order at least 4.*
(b) *There exists a non-cyclic subgroup $A \leq D$ such that all involutions in A lie in $I^*(M)$.*
(c) *If B_0 is a maximal subgroup of D, then $C_G(B_0) \leq M$. In particular $C_G(D) \leq M$.*

Proof. By Lemma 2.1 (b) we know that $D = C_D(V) \times [V, D]$. As V is abelian, we deduce for all maximal subgroups W of V that $C_{[D,V]}(W)$ is normalised, but not centralised by all $v \in V \setminus W$. A cyclic 2-group does not admit an automorphism of order 3 and therefore (a) is true. Moreover there is some $W \max V$ such that $C_{[D,V]}(W)$ has order at least 4, or otherwise $D = C_D(V)$. In both cases we find a maximal subgroup W of V such that $C_D(W)$ is not cyclic. We keep this subgroup W and notice that $|C_D(W)| \geq 4$. Using Lemma 8.4 (e), we see that all involutions of $C_D(W)$ are contained in $I^*(M)$, which is (b). If $B_0 \leq D$ is a maximal subgroup, then $C_{B_0}(W) = B_0 \cap C_D(W) \neq 1$ because $|C_D(W)| \geq 4$ and $|D : B_0| = 2$. If $b \in C_{B_0}(W)$ is an involution, then as above it follows that $C_G(b) \leq M$ and therefore $C_G(B_0) \leq M$. As $C_G(D) \leq C_G(B_0)$, we obtain (c). □

Lemma 8.7. *If $|B| \geq 8$ and B_0 is a subgroup of B of index at most 4, then $C_G(B_0) \leq M$.*

Proof. We assume for a contradiction that there is some subgroup B_0 of B of index at most 4 such that $C_G(B_0) \nleq M$. Hypothesis 7.2 (b) gives an elementary abelian subgroup V of P that has order 27. Hence, as $P \leq M$, we deduce from Lemma 8.6 (c) that $|B : B_0| = 4$.
Let W be a maximal subgroup of W. If $|C_B(W)| \geq 8$, then $B_0 \cap C_B(W) \neq 1$ and so we see that $C_G(B_0) \leq C_G(B_0 \cap C_B(W)) \leq M$, because Lemma 8.4 (e) yields that $C_B(W)^\# \subseteq I^*(M)$. This is a contradiction. Thus 8.6 (a) implies that $C_{[B,V]}(W) = 1$ or that $|C_{[B,V]}(W)| = 4$ and $C_B(V) = 1$.
Lemma 2.1 (d) shows that $B = \langle C_B(W) \mid W \max V\rangle$. From $|B| \geq 8$ and the investigation above, we obtain maximal subgroups $W_1 \neq W_2$ of V such that $B_1 := C_{[B,V]}(W_1)$ and $B_2 := C_{[B,V]}(W_2)$ have order 4. As $C_{[B,V]}(V) = 1$ by Lemma 2.1 (b), it follows that $B_1B_2 = B_1 \times B_2$ is an V-invariant subgroup of order 16. Let $w_i \in W_i \setminus W_{3-i}$ for all $i \in \{1, 2\}$. Then w_i centralises B_i and acts non-trivially on B_{3-i}. In particular $B_{3-i}\langle w_i\rangle \cong \mathcal{A}_4$ and $B_1B_2\langle w_1, w_2\rangle \cong \mathcal{A}_4 \times \mathcal{A}_4$. We recall that $|B_1B_2| = 16$ and $|B : B_0| = 4$ and we see that $D := B_0 \cap (B_1B_2)$ has order at least 4, and since $B_0 \cap B_1 = 1$ and $|B_1| = 4$, we also have that $|D| = 4$.
We apply Lemma 5.31 of [24]. If $i \in \{1, 2\}$, then $B_i \cap D \leq C_B(W_i) \cap B_0 = 1$, so we find an element $w \in \langle w_1, w_2\rangle$ of order 3 that normalises D. Now $\langle w\rangle(W_1 \cap W_2)$ is an elementary abelian subgroup of order 9 of $N_G(B)$ and so Lemma 6.4 (e) implies that $N_G(D) \leq M$. Altogether we obtain the contradiction $C_G(B_0) \leq C_G(D) \leq N_G(D) \leq M$. □

Lemma 8.8. *Suppose that Hypothesis 8.2 holds and let $\bar{A}$ be an elementary abelian 2-subgroup of $\bar{M}$.*

(a) *If $\bar{U} \leq C_{\bar{A}}(O_3(\bar{M}))$, then $|B : C_B(\bar{U})| \geq |\bar{U}|^2$ and $|[B, \bar{U}]| \geq |\bar{U}|^2$.*
(b) *There is a subgroup $\bar{A}_1$ of $\bar{A}$ such that $\bar{A} = \bar{A}_1 \times C_{\bar{A}}(O_3(\bar{M}))$ and there exist some integer $k \geq 0$ and elements $\bar{a}_1, ..., \bar{a}_k \in \bar{A}_1$, $\bar{y}_1, ..., \bar{y}_k \in O_3(\bar{M})$ of order 3 such that for all $i \in \{1, ..., k\}$ we have: $\langle \bar{y}_i, \bar{a}_i\rangle \cong S_3$ and $\langle \bar{y}_1, ..., \bar{y}_k\rangle\bar{A}_1 = \bigtimes_{i=1}^{k} \langle \bar{y}_i, \bar{a}_i\rangle$.*
(c) *If $E(\bar{M}) = 1$, then there exist some integer $l \geq 0$ and elements $\bar{d}_1, ..., \bar{d}_l \in \bar{A}$ such that for every $i \in \{1, ..., l\}$ there is an odd prime q_i and an element $\bar{g}_i$ of order q_i of $F(\bar{M})$ such that $\bar{d}_i$ inverts $\bar{g}_i$ and $\langle \bar{g}_1, ..., \bar{g}_l\rangle\bar{A} = \bigtimes_{i=1}^{l} \langle \bar{g}_i, \bar{d}_i\rangle$.*

Proof. We first remark that $F(\bar{M})$ has odd order by Lemma 8.4 (b).
We set $\bar{K} := O_{3'}(F^*(\bar{M})) \cdot C_{\bar{A}}(O_3(\bar{M}))$. Since M is 3-soluble by Lemma 6.4 (b), the group $C_{\bar{A}}(O_3(\bar{M}))$ acts faithfully on $O_{3'}(F^*(\bar{M}))$. This implies that $O_2(\bar{K}) = 1$ and that $\bar{K}$ is a $3'$-group, so we may apply Theorem A of [13] to $\bar{K}$ and deduce that (a) holds.
In addition let $\bar{A}_1$ be a complement of $C_{\bar{A}}(O_3(\bar{M}))$ in $\bar{A}$. Then $\bar{A}_1$ acts faithfully on $O_3(\bar{M})$ and if $E(\bar{M}) = 1$, then $\bar{A}$ acts faithfully on $F(\bar{M})$. Thus (b) and (c) follow from Thompson's Dihedral Lemma (see for example Lemma 24.1 in [14]). □

Lemma 8.9. *Suppose that Hypothesis 8.2 holds. If B_0 is a maximal subgroup of B and $b \in M$ is an involution such that $C_B(b) = B_0$, then $[B, b]$ has order 2 and $C_M([B, b])$ has non-cyclic Sylow 3-subgroups. In particular $[B, b]^\# \subseteq I^*(M)$.*

Proof. As $C_B(b) = B_0 \neq B$, we see that $\bar{b} \neq 1$, and then 8.4.1 of [17] implies that $[B, b]$ has order 2.
We apply Lemma 8.8 to the elementary abelian 2-group $\langle \bar{b}\rangle$. If $\bar{b}$ centralises $O_3(\bar{M})$, then Part (a) of the lemma implies that $2 = |B : C_B(\bar{b})| \geq |\langle \bar{b}\rangle|^2 = 2^2$, which is false. So there is some element $\bar{y} \in O_3(\bar{M})$ of order 3 that is inverted by $\bar{b}$. Then Lemma 2.1 (b) yields that $B = [B, \bar{y}] \times C_B(\bar{y})$ and so $2 = |B : C_B(b)| \geq |[B, \bar{y}] : C_{[B,\bar{y}]}(b)|$. Moreover we see from Lemma 2.2 (d) that

$$|C_{[B,\bar{y}]}(b)|^2 = |[B,\bar{y}]| = |C_{[B,\bar{y}]}(b)| \cdot |[B,\bar{y}] : C_{[B,\bar{y}]}(b)| \leq |C_{[B,\bar{y}]}(b)| \cdot 2.$$

We conclude that $|C_{[B,\bar{y}]}(b)| \leq 2$ and so $|[B,\bar{y}]| \leq 4$. Since $\bar{y}$ has order 3 and acts non-trivially on B, it follows that $[B,\bar{y}]$ has order 4. Then Lemma 8.5 yields that $[B,y]^\# \subseteq I^*(M)$.

Furthermore $2 = |[B,\bar{y}] : C_{[B,\bar{y}]}(b)|$ and hence $1 \neq [[B,\bar{y}],\bar{b}] \leq [B,\bar{b}] \cap [B,\bar{y}]$. As $[B,b]$ has order 2, we deduce that $[B,b] = [B,\bar{b}] \leq [B,\bar{y}]$. Altogether it follows that $[B,b]^\# \subseteq [B,y]^\# \subseteq I^*(M)$. □

Lemma 8.10. *Suppose that Hypothesis 8.2 holds and let $g \in G$ be such that $[B,B^g] \leq B \cap B^g$. Then $[B,B^g] = 1$.*

Proof. Assume for a contradiction that $[B,B^g] \neq 1$. Then by hypothesis $1 \neq [B,B^g] \leq B \cap B^g$. Therefore $B \leq N_G(B^g) = M^g$ and also $B^g \leq N_G(B) = M$ by Lemma 8.4 (a).By symmetry we may suppose that $|B : C_B(B^g)| \leq |B^g : C_{B^g}(B)| \neq 1$.

We set $\bar{B}_2 := C_{\bar{B}^g}(O_3(\bar{M}))$. Then Lemma 8.8 (a) implies that $|\bar{B}_2|^2 \leq |B : C_B(\bar{B}_2)| \leq |B : C_B(\bar{B}^g)| = |B : C_B(B^g)| \leq |B^g : C_{B^g}(B)| = |\bar{B}^g|$. This shows that $\bar{B}_2 \neq \bar{B}^g$.

We apply Part (b) of Lemma 8.8 to obtain a non-trivial subgroup $\bar{B}_1$ of $\bar{B}^g$ and an integer $k \geq 1$ such that $\bar{B}^g = \bar{B}_1 \times C_{\bar{B}^g}(O_3(\bar{M}))$ and $|\bar{B}_1| = 2^k$. Moreover there are $\bar{y}_1, ..., \bar{y}_k \leq O_3(\bar{M})$ of order 3 and such that $\langle \bar{y}_1, ..., \bar{y}_k \rangle \cdot \bar{B}_1$ is a direct product of groups isomorphic to $\mathcal{S}_3$.

For all $i \in \{1,..,k\}$ we see that $D_i := [B,\bar{y}_i]$ is non-trivial because $\bar{y}_i \neq 1$. Additionally $\bar{y}_i$ centralises $\bar{B}_2 = C_{\bar{B}^g}(O_3(\bar{M}))$ and a maximal subgroup of $\bar{B}_1$. We set $\bar{A}_i := C_{\bar{B}^g}(\bar{y}_i)$ and let A_i denote the full pre-image of $\bar{A}_i$ in B^g. Then A_i is a maximal subgroup of B^g for every $i \in \{1,..,k\}$.

Now we fix some $i \in \{1,..,k\}$. By Lemma 2.3 the y_i-invariant group D_i is not centralised by B^g. In particular there is an element $d_i \in D_i \setminus C_{D_i}(B^g)$. We further notice that $[B,\bar{B}^g] = [B,B^g] \leq B \cap B^g$ and that B^g is abelian, so it follows that $\bar{B}^g$ acts quadratically on B. Thus $[d_i, A_i] \leq [D_i, A_i] = [D_i, \bar{A}_i] = 1$ by Lemma 2.3. This implies that $A_i = C_{B^g}(d_i) = C_{B^g}(D_i)$ and Lemma 8.9 yields that $\varnothing \neq [B^g, d_i]^\# \subseteq I^*(M^g)$ and that $C_{M^g}([B^g, d_i])$ has non-cyclic Sylow 3-subgroups.

Assume for a contradiction that there is some $j \in \{1,...,k\} \setminus \{i\}$. Then Lemma 8.8 (b) provides an element $\bar{a}_j$ of $\bar{A}_i$ that inverts $\bar{y}_j$. Moreover $[\langle \bar{y}_j \rangle, \bar{B}^g] = \langle \bar{y}_j \rangle$ and $\bar{y}_j \in C_{\bar{M}}(\bar{y}_i)$. This implies that $D_i = [B,\bar{y}_i]$ is $\bar{y}_j$-invariant. As $D_i \leq C_B(\bar{A}_i) \leq C_B(\bar{a}_j)$, Lemma 2.3 yields that $[D_i, \bar{y}_j] = 1$. In particular $[d_i, \bar{y}_j] = 1$.

We apply the Three Subgroups Lemma (see 1.5.6 of [17]). From $[[d_i, \langle \bar{y}_j \rangle], \bar{B}^g] = 1$ and $[d_i, [\langle \bar{y}_j \rangle, \bar{B}^g]] = [d_i, \langle \bar{y}_j \rangle] = 1$ it follows that $[B^g, d_i] = [d_i, \bar{B}^g]$ is centralised by $\langle \bar{y}_j \rangle$. Let y be a pre-image of $\bar{y}_j$ in M. As $\bar{y}_j^3 = 1$, we may choose y as a 3-element. Then $y \in C_G([B^g, d_i]) \cap M \leq M^g \cap M$ and it follows that $g \in M$, as M is strongly 3-embedded. Now, Lemma 8.4 (a) implies the contradiction $B^g = B$.

This shows that $k = 1$ and $|\bar{B}_1| = 2^k = 2$. Moreover $\bar{B}_2 = \bar{A}_1$ and therefore $D_1 \leq C_B(\bar{A}_1) = C_B(\bar{B}_2)$. As $D_1 \nleq C_B(B^g) = C_B(\bar{B}^g)$, we deduce that $C_B(\bar{B}_2) \neq C_B(\bar{B}^g)$.

We recall that $\bar{B}_2 = C_{\bar{B}^g}(O_3(\bar{M}))$ and apply Lemma 8.8 (d) once more to see that

$$2 \cdot |\bar{B}_2| = |\bar{B}_1| \cdot |\bar{B}_2| = |\bar{B}^g| = |B : C_B(\bar{B}^g)| \gneq |B : C_B(\bar{B}_2)| \geq |\bar{B}_2|^2$$

and hence $\bar{B}_2 = 1$. Altogether we have proven that $2 = |\bar{B}^g| = |B^g : C_{B^g}(B)| \geq |B : C_B(B^g)| \neq 1$. This implies that $2 = |B^g : C_{B^g}(B)| = |B : C_B(B^g)|$ and the situation is symmetric in B and B^g.

Consequently we find some $d \in B^g$ such that $1 \neq [B,d]^\# \subseteq I^*(M)$. We recall that $2 = |B^g : C_{B^g}(B)|$, whence $B^g = \langle d \rangle C_{B^g}(B)$. It follows that $[B,B^g] = [B,d]$ and by a similar argument that $[B,B^g] = [B^g, d_1]$.

Finally $C_G([B^g,d_1]) = C_G([B,d]) \leq M$, but $C_{M^g}([B^g,d_i])$ has non-cyclic Sylow 3-subgroups. Since M is strongly 3-embedded it follows that $g \in M$ and so $B^g = B$ by Lemma 8.4 (a). This is a contradiction. □

Lemma 8.11. *Suppose that Hypothesis 8.2 holds and that $E(\bar{M}) \neq 1$. Then we have that $|B| \geq 2^{12}$. If in addition $g \in G$ is such that $|B^g : B^g \cap M| \leq 2$, then $[B,B^g] = 1$.*

Proof. If $x \in C_M(B)$, then M is soluble by Lemma 8.4 (c), contrary to our hypothesis. Therefore $\bar{x} \neq 1$. In addition Hypothesis 7.2 (a) implies that $E(\bar{M})$ simple. We recall that M is 3-soluble by Lemma 6.4 (b), so $E(\bar{M})$ is a $3'$-group. As $\bar{C} = C_{\bar{M}}(\bar{x})$ is soluble by Lemma 8.4 (g) and Hypothesis 6.1, Lemma 2.5 implies that $E(\bar{M}) \cong$Sz(8) is normalised by $\bar{x}$ and that $\bar{x}$ induces non-trivial automorphisms on $E(\bar{M})$.

Additionally Hypothesis 7.2 (b) provides an elementary abelian 3-subgroup V of order 27 of P such that $x \in V$. Now we may apply Lemma 2.6 to the semi-direct product $E(\bar{M}) \rtimes \bar{V}$ acting on B. Part (a) implies that $|B| \geq 2^{12}$.

Moreover, from Lemma 2.4 (a) and Hypothesis 7.2 (a) we deduce that $|\bar{M} : E(\bar{M})|$ is odd.

We now prove the following statement:

(*) If $g \in G$ is such that $|B^g : B^g \cap M| \leq 2$, then $B \leq M^g$.

Proof. Part (c) of Lemma 2.6 provides a subgroup B_0 of B of order 16 and with the property that B_0 is centralised by some non-cyclic 3-subgroup of M. Thus $B_0^\# \subseteq I^*(M)$ by Lemma 8.4 (e). If $B_0^g \cap C_{B^g}(B) \neq 1$, then it follows that $B \leq C_G(C_{B^g}(B)) \leq C_G(B_0^g \cap C_{B^g}(B)) \leq M^g$.

So we may assume for a contradiction that $B_0^g \cap C_{B^g}(B) = 1$. In particular $|B^g : C_{B^g}(B)| \geq |B_0| = 2^4$. We set $A := B^g \cap M$. Then, as $|\bar{M} : E(\bar{M})|$ is odd, we see that $\bar{A} \leq E(\bar{M}) \cong$ Sz(8). In particular, $|\bar{A}| \leq 2^3$ by Lemma 2.4 (b). Now $2^4 \geq |B^g : C_{B^g}(B)| = |B^g : A| \cdot |\bar{A}|$ implies that $|\bar{A}| = 2^3$ and $B^g \neq A$. Then $\bar{A} = \Omega_1(\bar{S}_0)$ for some Sylow 2-subgroup $\bar{S}_0$ of $E(\bar{M})$ by Lemma 2.4 (b).

Again $\bar{V}$ acts coprimely on $E(\bar{M})$ and hence Lemma 2.1 (f) implies that $\bar{V}$ normalises a Sylow 2-subgroup of $E(\bar{M})$. By Sylow's Theorem we may suppose that $\bar{S}_0$ is $\bar{V}$-invariant. In particular $\bar{A} = \Omega_1(\bar{A})$ is also $\bar{V}$-invariant.

Then $D := C_B(\bar{A}) = C_B(A)$ is V-invariant and Lemma 8.6 (c) implies that $C_G(D) \leq M$. Moreover Lemma 8.7, applied to the maximal subgroup A of B^g, yields that $D \leq C_G(A) \leq M^g$.

Since $|\bar{M} : E(\bar{M})|$ and so $|M^g/C_G(B^g) : E(M^g/C_G(B^g))|$ is odd, we obtain that $DC_G(B^g)/C_G(B^g) \leq E(M^g/C_G(B^g)) \cong$ Sz(8). Furthermore, Part (b) of Lemma 2.6 yields that $D \leq C_G(B^g)$, because every element of D centralises at least the maximal subgroup A of B^g.

Consequently we obtain the contradiction $B^g \subseteq C_G(D) \leq M$, as $|B^g : B^g \cap M| = |B^g : A| \neq 1$. ■

Finally let $g \in G$ be such that $|B^g : B^g \cap M| \leq 2$. Then (*) gives that $B \leq M^g$ and so $|B^{g^{-1}} : B^{g^{-1}} \cap M| = |B : B \cap M^g| = 1 \leq 2$. Again (*) implies that $B \leq M^{g^{-1}}$. In particular, $B^g \leq M$ and so $[B, B^g] \leq B \cap B^g$. It follows that $[B, B^g] = 1$ by Lemma 8.10. □

Lemma 8.12. *Suppose that Hypothesis 8.2 holds. If $g \in G$ is such that $B^g \leq M$, then $[B, B^g] = 1$. In particular $B \leq Z(T_0)$.*

Proof. Assume for a contradiction that there is some $g \in G$ such that $B^g \leq M$ and $[B^g, B] \neq 1$. Then Lemma 8.11 yields that $E(\bar{M}) = 1$. In addition B is not cyclic and by Sylow's Theorem we may suppose that $B^g \leq T$.

We deduce from Lemma 8.10 that B does not normalise B^g, and then we apply Lemma 8.8 (c) to $\bar{A} := \bar{B}^g$. It provides an element $\bar{d}_1 \in \bar{A}$ and an element $\bar{g}_1 \in \bar{M}$ of odd order that is inverted by $\bar{d}_1$ and that centralises a complement $\bar{A}_1$ of $\langle \bar{d}_1 \rangle$ in $\bar{A}$.

Let A_1 be the full pre-image of $\bar{A}_1$ in B^g. Then $|B^g : A_1| = |\bar{B}^g : \bar{A}_1| = |\langle \bar{d}_1 \rangle| = 2$. In addition $[B, \bar{g}_1]$ is invariant under $C_{\bar{B}^g}(\bar{g}_1) = \bar{A}_1$. We set $B_1 := C_{[B,\bar{g}_1]}(A_1)$. Then $\bar{g}_1$ normalises B_1 and $B_1 \neq 1$, since A_1 and $[B, \bar{g}_1]$ are 2-groups. Now Lemma 2.3 shows that $C_{B_1}(\bar{g}_1) \leq C_{[B,\bar{g}_1]}(\bar{g}_i) = 1$ and that $B_1 \nleq C_B(\bar{d}_1)$. In particular $[B^g, B_1] \geq [\bar{d}, B_1] \neq 1$.

Lemma 8.7 yields that $C_G(A_1) \leq M^g$, because A_1 is a maximal subgroup of B^g. Let $b_1 \in B_1 \setminus C_M(B^g)$. Then $b_1 \in C_G(A_1) \leq M^g$ and $C_{B^g}(b_1) = A_1$ is a maximal subgroup of B^g. Therefore, Lemma 8.9 implies that $C_G([B^g, b_1]) \leq M^g$. As $[B^g, b_1] \leq B$ and B is abelian, we obtain the contradiction $B \leq M^g$. $\square$

Lemma 8.13. *Suppose that Hypothesis 8.2 holds. If $U \leq G$ is such that $T \in \mathrm{Syl}_2(U)$ and $T_0 \leq N \trianglelefteq U$, then $U = N \cdot N_U(T_0)$. In particular $M = C_G(B) \cdot N_M(T_0)$. Moreover one of the following is true:*

(a) *$C_P(B)$ is not cyclic and $B^\# \subseteq I^*(M)$.*

(b) *M is the unique maximal subgroup that contains $N_G(T_0)$.*

Proof. We recall that $T_0 := \langle B^g \mid g \in G \text{ and } B^g \leq T\rangle$. First let $U \leq G$ be such that $T \in \mathrm{Syl}_2(U)$ and $T_0 \leq N \trianglelefteq U$. Then a Frattini argument shows that $U = N \cdot N_U(N \cap T)$. Let $h \in N_U(N \cap T)$ and $g \in G$ be such that $B^g \leq T$. Then $B^g \leq T_0 \leq N \cap T$ and so $B^{gh} \leq N \cap T \leq T$. The definition of T_0 yields that $B^{gh} \leq T_0$. It follows that $N_U(N \cap T)$ normalises T_0 and then $U = N \cdot N_U(T_0)$.
By Lemma 8.12 we see that T_0 centralises B and so $T_0 \leq C_T(B)$. Therefore $M = C_G(B) \cdot N_M(T_0)$.
If $C_P(B)$ is not cyclic, then (a) holds by Lemma 8.4 (f).
If $C_P(B)$ is cyclic, then we recall that $C_P(B) \trianglelefteq P$ and therefore $N_P(T_0)$ is not cyclic because $r(P) \geq 3$ by Hypothesis 7.2 (b). Then we note that $x \in N_M(C_T(B)) \leq N_G(T_0)$, so Lemma 6.4 (e) implies that M is the unique maximal subgroup of G containing $N_G(T_0)$. This is (b). $\square$

Corollary 8.14. *Suppose that Hypothesis 8.2 holds. Then $N_G(T) \leq M$, which means that T is a Sylow 2-subgroup of G.*

Proof. We first notice that $M/C_M(\Omega_1(Z(T)))$ is a $2'$-group. Hence $\Omega_1(Z(T)) \leq B$ by Lemma 8.4 (d). If Lemma 8.13 (a) holds, then $C_P(\Omega_1(Z(T))) \leq N_P(\Omega_1(Z(T)))$ is not cyclic. We recall that $\Omega_1(Z(T))$ is x-invariant by Hypothesis 8.2. So Lemma 6.4 (e) shows that $N_G(T) \leq N_G(\Omega_1(Z(T))) \leq M$ as stated.
If Lemma 8.13 (b) holds, then $N_G(T) \leq N_G(T_0) \leq M$ by definition of T_0. $\square$

9. A uniqueness result

Lemma 9.1. *Suppose that Hypothesis 8.2 holds and that A is an elementary abelian subgroup of M of order 4. Suppose that $A \cong \bar{A}$ and that $C_B(A)$ has index at most 4 in B. Then for each element $b \in B \setminus C_B(A)$ there is an element $a \in A$ such that $C_G([b, a]) \leq M$.*

Proof. Let $\bar{A}_1$ be as in Lemma 8.8 (b) and set $\bar{A}_2 := C_{\bar{A}}(O_3(\bar{M}))$. Then Part (a) of the lemma and our hypotheses imply that $|\bar{A}_2|^2 \leq |B : C_B(\bar{A}_2)| \leq |B : C_B(\bar{A})| = |B : C_B(A)| \leq 4$. We conclude that $|\bar{A}_2| \leq 2$ and therefore $\bar{A}_1 \neq 1$, because $|\bar{A}| = |A| = 4$. Furthermore we see that $C_B(\bar{A}_2) = C_B(\bar{A})$, if $\bar{A}_2 \neq 1$.
Now let $\bar{a}_1 \in \bar{A}_1$ be an involution and let $\bar{y}_1 \in O_3(\bar{M})$ be of order 3 such that $\bar{a}_1$ inverts $\bar{y}_1$, as in Lemma 8.8 (b). If $\bar{A}_2 \neq 1$, then we let $\bar{a}_2 \in \bar{A}_2^\#$, and otherwise we refer to Lemma 8.8 (b) again and let $\bar{y}_2 \in O_3(\bar{M})$ be of order 3 and $\bar{a}_2 \in \bar{A}_1$ be of order 2 such that $\bar{a}_2$ inverts $\bar{y}_1$ and $\bar{A}_1 = \langle \bar{a}_1 \rangle \times \langle \bar{a}_2 \rangle$. In both cases $[\bar{a}_2, \bar{y}_1] = 1$.
Let $D_1 := [B, \bar{y}_1]$. Then $C_{D_1}(\bar{y}_1) = 1$, but $1 \neq C_{D_1}(\bar{a}_2)$ is $\bar{y}_1$-invariant and therefore Lemma 2.3 tells us that it is not centralised by $\bar{a}_1$. This shows that $C_{D_1}(\bar{a}_2) \not\leq C_B(A)$. In particular $C_B(\bar{a}_2) \neq C_B(A)$ and we conclude that $\bar{A}_2 = 1$.
Furthermore for all $i \in \{1, 2\}$, it is true that $C_B(A) \neq C_B(\bar{a}_i)$. From $C_B(A) = C_B(\bar{A}) \leq C_B(\bar{a}_i)$, $|B : C_B(A)| \leq 4$ and $1 \neq \bar{a}_i$, it follows that $C_B(\bar{a}_i)$ is a maximal subgroup of B. We apply Lemma 8.9 to see that $[B, a_i]^\# \subseteq I^*(M)$ for all $i \in \{1, 2\}$.
Finally let $b \in B \setminus C_B(A)$. Then there is some $i \in \{1, 2\}$ such that $1 \neq [b, a_i] \in I^*(M)$. $\square$

Lemma 9.2. *Suppose that Hypothesis 8.2 holds and that $|B| \geq 8$. If $g \in G$ and $|B^g : B^g \cap M| \leq 2$, then $[B, B^g] = 1$.*

Proof. Assume for a contradiction that $g \in G$ is such that $[B, B^g] \neq 1$ and $|B^g : B^g \cap M| \leq 2$. Then Lemmas 8.11 and 8.12 imply that $E(\bar{M}) = 1$ and that $B^g \not\leq M$, so $|B^g : B^g \cap M| = 2$. As $|B^g| = |B| \geq 8$, we know that $|B^g \cap M| \geq 4$. We set $A := B^g \cap M$ and $A_0 := C_{B^g}(B) = C_A(B)$ (see Lemma 8.4 (a)).

(I) $B \not\leq M^g$, $|\bar{A}| \geq 4$ *and* $A_0 \cap I^*(M^g) = \varnothing$.

Proof. First, assume for a contradiction that $B \leq M^g$. Then $B \leq T_0^g$ and Lemma 8.12, applied to B^g, yields that $B^g \leq Z(T_0^g) \leq C_G(B)$. This gives the contradiction $B^g \leq C_G(B) \leq N_G(B) = M$, by Lemma 8.4 (a).
Now assume that $|\bar{A}| \leq 2$. We note that $\bar{A} \cong A/A_0$ and therefore $|A : A_0| \leq 2$. Then $1 \neq A_0$ has index at most 4 in B^g and we apply Lemma 8.7 to B^g and M^g. It yields the contradiction $B \leq C_G(A_0) \leq M^g$. If $a \in A_0 \cap I^*(M^g)$ is an involution, then $B \leq C_G(A_0) \leq C_G(a) \leq M^g$, which is another contradiction. ∎

Let $k \in \mathbb{N}$ be such that $2^k = |\bar{A}| \geq 4$ and, with Lemma 8.8, let $a_1, ..., a_k \in A$ and for all $i \in \{1, ..., k\}$ let $\bar{y}_i \in F(\bar{M})$ be of odd prime order q_i and such that $\bar{a}_i \in \bar{A}$ inverts $\bar{y}_i$. In addition $[\bar{y}_i, \bar{a}_j] = 1$ for all $j \in \{1, ..., k\} \setminus \{i\}$ and $\bar{A}_E \leq C_{\bar{M}}(F(\bar{M}))$.
We fix $i \in \{1, ..., k\}$, we let A_i denote the full pre-image of $C_{\bar{A}}(\bar{y}_i)$ in A and set $D_i := [B, \bar{y}_i]$ and $B_i := C_{D_i}(\bar{A}_i)$, and we keep all this notation for the remainder of the proof.

(II) $C_G(A_i) \leq M^g$, $q_i = 3$, $|B_i| = 4$ *and* $\bar{y}_i$ *acts transitively on* B_i. *Moreover there is some involution* $b_i \in D_i \setminus I^*(M)$ *such that* $\langle b_i \rangle = C_{D_i}(A) = [B_i, a_i] \leq B \cap B^g \cap [B_i, A]$.

Proof. First we note that $A = A_i \times \langle a_i \rangle$ and so A_i is a maximal subgroup of A. Then $C_G(A_i) \leq M^g$ by Lemma 8.7. Next we see that $D_i \neq 1$, because $\bar{y}_i \neq 1$. Since D_i and A_i are 2-groups, it follows that $B_i \neq 1$. Then $A_i \leq C_G(B_i)$ and $A = A_i \times \langle a_i \rangle$, so we see that $C_{B_i}(A) = C_{B_i}(a_i)$. Moreover Lemma 2.3 implies that $\bar{a}_i$, and hence a_i, does not centralise the $\bar{y}_i$-invariant group B_i.
Let E be a complement of $C_{B_i}(a_i)$ in B_i. Then $E \leq B_i \leq C_G(A_i) \leq M^g$, as we have seen above, and thus $C_G(B^g) \cap E \leq C_G(a_i) \cap E = 1$. Assume for a contradiction that $|E| \geq 4$. Then we may apply Lemma 9.1 to a fours group E_0 in E and we obtain, for every $a \in A \setminus A_i$, some $b \in E_0$ such that $C_G([a, b]) \leq M^g$. For such elements b and a of G we see that $[a, b] \in [A, E] \leq [M, B] \leq B$ and then $B \leq C_G([a, b]) \leq M^g$, because B is abelian. This contradicts the first statement of (I).
We conclude that $|B_i : C_{B_i}(a_i)| = |E| = 2$. Thus Lemma 2.2 (a) yields that q_i divides $2^{2 \cdot 1} - 1 = 3$. Then $q_i = 3$. Moreover Part (d) of the same lemma gives that

$$|C_{B_i}(a_i)|^2 = |B_i| = |B_i : C_{B_i}(a_i)| \cdot |C_{B_i}(a_i)| = 2 \cdot |C_{B_i}(a_i)|.$$

Altogether we have that $|C_{B_i}(a_i)| = 2$ and so $|B_i| = |C_{B_i}(a_i)|^2 = 4$. The group $B_i = C_{D_i}(\bar{A}_i)$ is $\bar{y}_i$-invariant, as $[\bar{A}_i, \bar{y}_i] = 1$. Since $\bar{y}_i$ acts fixed-point freely on D_i by Lemma 2.1 (b), we see that $\bar{y}_i$ acts transitively on B_i.
Now $1 \neq [B_i, a_i] \leq C_{B_i}(a_i)$ is cyclic and hence

$$[B_i, a_i] = C_{B_i}(a_i) = C_{D_i}(a_i) \cap B_i = C_{D_i}(a_i) \cap C_{D_i}(A_i) = C_{D_i}(A).$$

Let b_i be a generator of $[B_i, a_i]$. Then $b_i \in D_i$ and $B_i \leq C_G(A_i) \leq M^g$ implies that

$$b_i \in [B_i, a_i] \leq [B, a_i] \cap [B_i, B^g] \cap [B_i, A] \leq B \cap B^g \cap [B_i, A].$$

Since B^g is abelian, we deduce that $B^g \le C_G(b_i)$ and hence $C_G(b_i) \nleq M$ because of our assumption that $B^g \nleq M$. Therefore $b_i \notin I^*(M)$. ■

We deduce from (I) that $\langle \bar{y}_1, ..., \bar{y}_k \rangle \le O_3(\bar{M}) \le \bar{P}$. In particular $\langle \bar{y}_1, ..., \bar{y}_k, \bar{x} \rangle$ is a 3-group and the normaliser of $\langle \bar{y}_1, ..., \bar{y}_k \rangle$ in $\bar{M}$ has an $\bar{x}$-invariant Sylow 2-subgroup. We may suppose that this 2-group contains $\bar{A}$.

(III) *$B_i \le D_i \cap M^g$ and $|D_i \cap M^g| \le 8$.*

Proof. First $B_i = D_i \cap C_G(A_i) \le D_i \cap M^g$ by (II). Then we set $E_i := D_i \cap M^g = N_{D_i}(B^g)$, and we assume for a contradiction that $|E_i| \ge 16$.
Recall that $\langle b_i \rangle = C_{D_i}(A)$ by (II). As $C_{E_i}(B^g) \le C_{E_i}(A) \le C_{D_i}(A) = \langle b_i \rangle$, we find a subgroup E of order 8 of E_i such that $C_E(B^g) = 1$ and $[E, A] \le [E_i, A] \le D_i \cap B^g \le C_{D_i}(A) = \langle b_i \rangle$.
Thus, for every element e of E_i, we see that $[A, e] \le \langle b_i \rangle$ is cyclic. Then 8.4.1 of [17] implies that $|A : C_A(e)| = |[A, e]| \le 2$. Now we recall that $|B^g : A| = 2$ and we deduce that

$$|B^g : C_{B^g}(e)| \le |B^g : C_A(e)| = |B^g : A| \cdot |A : C_A(e)| \le 2 \cdot |A : C_A(e)| \le 4.$$

Again 8.4.1 of [17] shows that $16 \le |E_i| \le |B| = |B^g| = |B^g : C_{B^g}(e)| \cdot |C_{B^g}(e)| = |B^g : C_{B^g}(e)|^2 \le 16$. Altogether $B = E_i \le M^g$, which contradicts (I). ■

(IV) *$|D_i| = 16$.*

Proof. If $D_i = B_i$, then (II) and Lemma 8.5 imply that $D_i^\# \subseteq I^*(M)$. This contradicts the statement $b_1 \notin I^*(M)$ in (II). Hence $D_i \ne B_i$. We recall that $|B_i| = 4$ by (II) and therefore $|D_i| \ge 8$. At the same time $D_i = [B, \bar{y}_i]$ admits a fixed-point-free automorphism of order 3 by (II), which means that $|D_i| \ge 16$.
We set $E_i := N_{D_i}(B^g) = D_i \cap M^g$. Then $b_i \in B_i \le E_i$ by (III) and $[E_i, A] \le D_i \cap B^g \le D_i \cap C_G(A) = \langle b_i \rangle$ by (II). Hypothesis 7.2 (b) and Lemma 8.4 (a) imply that B^g is normalised by some elementary abelian subgroup of G of order 27. Then it follows from Lemma 8.6 (c) and the fact that $|B^g : A| = 2$ that there is an involution $a \in A$ such that $C_G(a) \le M^g$. This means that $C_{D_i}(a) \le E_i$. Assume for a contradiction that $|C_{D_i}(a)| \ge 8$.
If $\bar{a}$ is centralised by $\bar{y}_i$, then $\bar{y}_i$ induces a fixed-point-free automorphism of order 3 on $C_{D_i}(a)$ and hence $|E_i| \ge |C_{D_i}(a)| \ge 16$. This contradicts (III). Therefore $\bar{a}$ does not centralise $\bar{y}_i$. It follows that $a \notin A_i$ and so $A = \langle a \rangle \times A_i$. Since $[B_i, A_i] = 1$, but $[B, A_i] \ne 1$ by (II), we conclude that a does not centralise B_i. In particular $|E_i| \ge |B_i C_{D_i}(a)| \gneq |C_{D_i}(a)| \ge 8$, which also contradicts (III).
Consequently $|C_{D_i}(a)| \le 4$. The quadratic action of a on D_i implies that $|D_i| \le |C_{D_i}(a)|^2 \le 16$. ■

(V) *$C_{D_1}(\bar{y}_2) = D_1$ or $D_1 = D_2$.*

Proof. Assume for a contradiction that y_2 does neither centralise D_1 nor act fixed-point-freely on D_1. Then $[D_1, \bar{y}_2]$ and $C_{D_1}(\bar{y}_2)$ are both elementary abelian of order 4 and normalised by A, because D_1 and $\bar{y}_2$ are $\bar{A}$-invariant. Thus both groups contain a non-trivial element that is centralised by A. But $C_{D_1}(A) = \langle b_1 \rangle$ by (II), so we obtain a contradiction. Therefore, we either have that $C_{D_1}(\bar{y}_2) = D_1$ or that $D_1 \le [D_1, y_2] \le D_2$. But in the second case (IV) implies that $D_1 = D_2$. ■

(VI) *The following hold:*

(a) $x \in C_G(D_1)$ and $C_G(D_1)$ has cyclic Sylow 3-subgroups.

(b) $D_1 = D_2$ and $\bar{y}_1$ and $\bar{y}_2$ act fixed-point freely on D_1.

(c) $k = 2$, in particular $A = A_1 A_2$, $A_1 \cap A_2 = A_0$ and $|A : A_0| = 4$.

(d) $\langle \bar{y}_1, \bar{y}_2 \rangle$ induces a Sylow 3-subgroup of $\mathrm{Aut}(D_1)$ on D_1.

In particular there is some $\bar{z} \in \langle \bar{y}_1, \bar{y}_2 \rangle$ such that $|C_{D_1}(\bar{z})| = 4$.

(e) For every prime $q \geq 5$ we have that $O_q(\bar{M}) \leq C_{\bar{M}}(D_1)$.

(f) There is some $b_0 \in D_1 \setminus M^g$ such that $|N_M(D_1) : C_{N_M(D_1)}(b_0)|_2 \leq 4$.

Proof. Let $H := N_{\bar{M}}(D_1)/C_{\bar{M}}(D_1)$. Then H is isomorphic to a subgroup of $\mathrm{Aut}(D_1) \cong \mathrm{GL}_4(2) \cong \mathcal{A}_8$ by (IV).

Let R be a Sylow 3-subgroup of the full pre-image of $C_{\bar{M}}(\bar{y}_1)$ in M such that $\langle \bar{y}_1, ..., \bar{y}_k, \bar{x} \rangle \leq \bar{R}$. Then R normalises D_1. By (II) we have that $C_G(D_1) \leq C_G(b_1) \nleq M$. Since $C_G(b_1)$ is 3-soluble by Hypothesis 8.2, we conclude with Lemma 6.4 (d) that the Sylow 3-subgroups of $C_G(D_1)$, and hence of $C_R(D_1)$, are cyclic. This is one of the statements in (a).

Furthermore D_1 is $\bar{A}$-invariant and $D_1 \neq C_{D_1}(\bar{A}_1) = B_1 \neq \langle b_1 \rangle = C_{D_1}(\bar{a}_1)$. It follows that $\bar{A}$ induces a group of automorphism of order at least 4 on D_1.

We recall that $\bar{y}_1$ acts fixed-point-freely on $D_1^{\#}$ and refer to Page 22 of [10]. Then we see that the group $N_{\bar{M}}(\langle \bar{y}_1 \rangle)/C_{N_{\bar{M}}(\langle \bar{y}_1 \rangle)}(D_1)$ is isomorphic to a subgroup of $(\mathcal{A}_5 \times C_3) \rtimes C_2$ of order divisible by 12. Hence $N_{\bar{M}}(\langle \bar{y}_1 \rangle)/\langle \bar{y}_1 \rangle C_{N_{\bar{M}}(\langle \bar{y}_1 \rangle)}(D_1)$ is isomorphic to a subgroup of $\mathcal{S}_5$ of order divisible by 4. Keeping in mind that $\bar{a}_1$ inverts $\bar{y}_1$ and does not centralise D_1, we improve the previous statement: $N_{\bar{M}}(\langle \bar{y}_1 \rangle)/\langle \bar{y}_1 \rangle C_{N_{\bar{M}}(\langle \bar{y}_1 \rangle)}(D_1)$ is isomorphic to a subgroup of $\mathcal{S}_5$ that has order divisible by 4 and a subgroup of index 2.

For the second statement in (a), assume for a contradiction that $x \notin C_G(D_1)$. As x is isolated in G, the image of x in $N_{\bar{M}}(\bar{y}_1)/\langle \bar{y}_1 \rangle C_{N_{\bar{M}}(\bar{y}_1)}(D_1)$ is isolated. The only subgroups of $\mathcal{S}_5$ of order divisible by 4 that have an isolated element of order 3 are isomorphic to $\mathcal{A}_4$. But $\mathcal{A}_4$ does not have a subgroup of index 2. This contradicts the previous paragraph and so (a) holds.

We also know that $AC_G(D_1)$ has cyclic Sylow 3-subgroups and that $\langle x \rangle = \Omega_1(C_R(D_1))$. As x is not inverted in G, this implies that $AC_G(D_1)$ has a normal 3-complement. Moreover, every element of $\langle \bar{y}_1, ..., \bar{y}_k \rangle$ is inverted by $\bar{a}_1, ..., \bar{a}_k \in A$. So we conclude that $\langle \bar{y}_1, ..., \bar{y}_k \rangle \cap C_{\bar{M}}(D_1) = 1$.

It follows that $C_{D_1}(\bar{y}_2) \neq D_1$ and hence $D_1 = D_2 = [B, y_2]$ by (V). Now $\bar{y}_1$ and $\bar{y}_2$ act fixed-point freely on D_1, so (b) is true. Furthermore, the fact that $r_3((\mathcal{A}_5 \times C_3) \rtimes C_2) = 2$ implies that $k = 2$. In particular $4 = 2^k = |\bar{A}| = |A : A_0|$ and $\bar{A} = \bar{A}_1 \times \bar{A}_2$, so $A_1 \cap A_2 = A_0$ and $A_1 A_2 = A$. This is (c).

We recall that $H := N_{\bar{M}}(D_1)/C_{\bar{M}}(D_1)$ and it follows from this that $\mathcal{S}_3 \times \mathcal{S}_3 \cong \langle \bar{y}_1, \bar{y}_2 \rangle \bar{A}$ is isomorphic to a subgroup of H. As $\langle \bar{y}_1, \bar{y}_2 \rangle C_{\bar{M}}(D_1)/C_{\bar{M}}(D_1)$ is a 3-subgroup of H, it induces a Sylow 3-subgroup of $\mathrm{Aut}(D_1) \cong \mathrm{GL}_4(2)$ on D_1. So we find an element $\bar{z} \in \langle \bar{y}_1, \bar{y}_2 \rangle$ such that $|C_{D_1}(\bar{z})| = 4$, which means that (d) holds.

Moreover $\langle \bar{y}_1, \bar{y}_2 \rangle \leq O_3(\bar{M})$ by (II), so it follows that $O_3(H)$ is a Sylow 3-subgroup of H.

Having in mind that H is isomorphic to a subgroup of $\mathrm{GL}_2(4) \cong \mathcal{A}_8$ and that a 3-Sylow subgroup of $\mathcal{A}_8$ is self-centralising, we see that $O_3(\bar{H}) = F^*(\bar{H})$. This implies that a Sylow 2-subgroup of H is isomorphic to a subgroup of the normaliser in $\mathcal{A}_8$ of a Sylow 3-subgroup. This normaliser has order $2^8 \cdot 3^2$. As $O_q(\bar{M})$ centralises $\bar{y}_1$ and therefore normalises $D_1 = [B, \bar{y}_1]$, we conclude, for every prime $q \geq 5$, that $O_q(\bar{M}) \leq C_{\bar{M}}(D_1)$. This is (e).

Finally assume for a contradiction that every element $b \in D_1 \setminus M^g$ is centralised by a Sylow 2-subgroup of H. Since $\bar{A} \leq N_{\bar{M}}(D_1)$ and $C_{D_1}(\bar{A}) = \langle b_1 \rangle$, a Sylow 2-subgroup of H centralises exactly one involution of D_1. Consequently we see from Sylow's Theorem that $D_1 \setminus M^g$ is in the same orbit as b_1. Furthermore $b_1 \in B^g \leq M^g$ and all involutions of B_1 are conjugate under $\bar{y}_1$ by (II).

Hence $11 = 3 + (16 - 8) \leq |B_1^{\#}| + |D_1 \setminus M^g| = |B_1^{\#} \cup D_1 \setminus M^g| \leq |b_1^H| = |H : C_H(b_1)| \leq |H|_3 = 9$.

This contradiction proves (f) because $|H|_2 = 2^3$. ∎

(VII) $B \cap M^g \cap I^*(M) \neq \varnothing$.

Proof. We know from (VI) (a) that x centralises D_1. First we show that $C_{D_1}(\bar{z}) \cap M^g \neq 1$ for some $\bar{z} \in \langle \bar{y}_1, \bar{y}_2 \rangle$.
For this we use (III) and (VI) (b) and we deduce that $B_1B_2 \leq D_1 \cap M^g$. Again by (VI) (b), both $\bar{y}_1$ and $\bar{y}_2$ act fixed-point-freely on $D_1^{\#}$, so Lemma 2.1 (e) yields that $D_1 = C_{D_1}(\bar{y}_1\bar{y}_2) \times C_{D_1}(\bar{y}_1\bar{y}_2^2)$. Assume that $C_{D_1}(\bar{z}) \cap B_1B_2 = 1$ for all $\bar{z} \in \{\bar{y}_1\bar{y}_2, \bar{y}_1\bar{y}_2^2\}$. Then B_1B_2 has order at most 4 and (II) implies that $B_1 = B_2$ is centralised by A_1 and A_2. But $A = A_1A_2$ by (VI) (c) and $1 \neq [B_1, a_1]$ by (II), which gives a contradiction.
So we have some element $\bar{z} \in \langle \bar{y}_1, \bar{y}_2 \rangle$ such that $C_{D_1}(\bar{z}) \cap M^g \neq 1$, which means that there is some $d \in C_{D_1}(\bar{z}) \cap B \cap M^g \leq B \cap M^g$ such that $\langle x, z \rangle \leq C_G(d)$. Then Lemma 8.4 (e) yields the assertion. ■

(VIII) *If $B_0 \leq B$ is such that $B_0^{\#} \subseteq I^*(M)$, then $|B_0| \leq 8$.*

Proof. Let B_0 be a subgroup of B of order 2^4. Here we need to recall that $A_0 = C_A(B)$, and then $|B^g : A_0| = |B^g : A| \cdot |A : A_0| = 2 \cdot |\bar{A}| = 2 \cdot 2^k = 8$ by (VI) (c). As $|B_0^g| = 16$, this means that $B_0^g \cap A_0 \neq 1$. Moreover $A_0 \cap I^*(M^g) = \varnothing$ by (I), which implies that $(B_0^g)^{\#} \nsubseteq I^*(M^g)$ and then $B_0^{\#} \nsubseteq I^*(M)$. ■

(IX) $D_1 = [B, \langle \bar{y}_1, \bar{y}_2 \rangle]$ *and* $|B| \leq 2^7$.

Proof. We set $D := D_1 (= D_2$ by (VI) (b)). Then $D = [B, \bar{y}_1] = [B, \bar{y}_2]$ and so $D = \langle D^{\langle \bar{y}_1, \bar{y}_2 \rangle} \rangle = [B, \langle \bar{y}_1, \bar{y}_2 \rangle]$.
As $C_B(\langle \bar{y}_1, \bar{y}_2 \rangle) \cdot \langle \bar{y}_1, \bar{y}_2 \rangle$ has non-cyclic Sylow 3-subgroups, we may apply Lemma 8.4 (e) and then (VIII). This yields that $|C_B(\langle \bar{y}_1, \bar{y}_2 \rangle)| \leq 8$. Finally Lemma 2.1 (b) and (IV) give that
$|B| = |C_B(\langle \bar{y}_1, \bar{y}_2 \rangle) \times [B, \langle \bar{y}_1, \bar{y}_2 \rangle]| = |C_B(\langle \bar{y}_1, \bar{y}_2 \rangle)| \cdot |D| \leq 8 \cdot 16 = 2^7$. ■

(X) *If $1 \neq \bar{N}$ is an abelian normal subgroup of $\bar{M}$ of odd order, then $C_B(\bar{N}) = 1$.*

Proof. Assume for a contradiction that there is a non-trivial $\bar{M}$-invariant subgroup B_0 of B of order at most 8. Then B_0 does not admit a group of automorphisms of order 9, because if $m \leq 3$, then $|\mathrm{GL}_m(2)|$ divides $(2^3 - 1) \cdot (2^3 - 2) \cdot (2^3 - 2^2) = 7 \cdot 3 \cdot 2^3$. Hence Lemma 8.4 (e) implies, together with Hypothesis 7.2, that $B_0^{\#} \subseteq I^*(M)$.
Furthermore there is an involution $b \in C_{B_0}(A)$ because the 2-group A normalises the 2-group B_0. Then Lemma 8.7 and the fact that $|B^g : A| = 2$ give that $b \in C_B(A) \leq B \cap M^g$. As $B \cap M^g$ normalises B_0^g, there is an involution $c \in B_0^g$ such that $B \cap M^g$ centralises c. In particular $c \in C_{B^g}(b) \leq M \cap B^g = A$ because $b \in B_0^{\#} \subseteq I^*(M)$. Now $c \in (B_0^{\#})^g \subseteq I^*(M^g)$, so we deduce from (I) that $c \notin A_0$ and $\bar{c} \neq 1$. Hence (VI) (c) provides some $i \in \{1, 2\}$ such that $c \notin A_i$ and such that $A = \langle c \rangle A_i$. Then (III) implies that $B_1B_2 \leq D_1 \cap M^g \leq B \cap M^g$ centralises c. This shows that $B_i = C_{D_1}(A_i)$ is centralised by $\langle c \rangle A_i = A$. From (II) we obtain the contradiction $4 = |B_i| \leq |C_{D_1}(A)| = |\langle b_i \rangle| = 2$.
Finally let $1 \neq \bar{N} \trianglelefteq \bar{M}$ be abelian and of odd order. Then Lemma 2.1 (b) implies that $B = [B, \bar{N}] \times C_B(\bar{N})$. Since $\bar{N}$ is normal in $\bar{M}$, the groups $[B, \bar{N}]$ and $C_B(\bar{N})$ are $\bar{M}$-invariant. But $|B| \leq 2^7 < 2^4 \cdot 2^4$ by (IX). As $[B, \bar{N}] \neq 1$, the size restriction from the previous paragraph yields that $C_B(\bar{N}) = 1$. ■

(XI) $B = D_1$ *and (therefore)* $N_M(D_1) = M$.

Proof. Let $q \geq 5$ be a prime number. Then (VI) (e) and (X) imply first that $O_q(\bar{M})$ centralises D_1 and then that $Z(O_q(\bar{M})) = 1$, because $Z(O_q(\bar{M}))$ is an $\bar{M}$-invariant subgroup of B. We deduce that $F^*(\bar{M}) = O_3(\bar{M})$ and therefore $\bar{x} \in C_{\bar{M}}(O_3(\bar{M})) \leq O_3(\bar{M})$. As $\bar{x}$ is isolated in $\bar{M}$, this means that $\langle \bar{x} \rangle$ is an abelian normal subgroup of $\bar{M}$. Moreover $1 \neq D_1 \leq C_B(\langle \bar{x} \rangle)$ by (VI) (a). Then (X) forces $\langle \bar{x} \rangle = 1$, i.e. $x \in C_M(B)$.

Using (VI) (d) we choose an element $\bar{z} \in \langle \bar{y}_1, \bar{y}_2 \rangle^\#$ that centralises a subgroup of order 4 in D_1. Since $\bar{z}$ is inverted by some element of $\bar{A}$ and x is isolated in M, we conclude that a Sylow 3-subgroup of the full pre-image of $\langle \bar{z} \rangle$ in M is not cyclic. Then (VIII) and Lemma 8.4 (e) yield that $C_B(\bar{z})$ has order at most 8. Recalling that $4 \leq |D_1 \cap C_B(\bar{z})|$ we deduce, together with Lemma 2.1 (b), that $C_B(\langle \bar{y}_1, \bar{y}_2 \rangle) \subseteq C_B(\bar{z}) \setminus [B, \bar{y}_1]^\# = C_B(\bar{z}) \setminus D_1^\#$, so $C_B(\langle \bar{y}_1, \bar{y}_2 \rangle)$ is cyclic.
Moreover $\bar{H} := N_{O_3(\bar{M})}(\langle \bar{y}_1, \bar{y}_2 \rangle)$ normalises $C_B(\langle \bar{y}_1, \bar{y}_2 \rangle)$. Then it centralises $C_B(\langle \bar{y}_1, \bar{y}_2 \rangle)$ because this is a cyclic 2-group. Furthermore, (VI) (d) shows that $N_{\bar{H}}(D_1)/C_{\bar{H}}(D_1) = \langle \bar{y}_1, \bar{y}_2 \rangle C_{\bar{H}}(D_1)/C_{\bar{H}}(D_1)$.
Now we have the following: $D_1 = D_2 = [B, \langle \bar{y}_1, \bar{y}_2 \rangle]$ by (VI) (b) and (IX), and $B = D_1 \times C_B(\langle \bar{y}_1, \bar{y}_2 \rangle)$ by Lemma 2.1 (b). Thus we conclude that $\bar{H} = \langle \bar{y}_1, \bar{y}_2 \rangle$ and that, therefore, $\bar{H} = O_3(\bar{M})$ is an abelian normal subgroup of $\bar{M}$ of odd order. Then (X) gives that $C_B(\bar{H}) = 1$ and finally $B = [B, \bar{H}] \times C_B(\bar{H}) = D_1 \times C_B(\bar{H}) = D_1$. ■

(XII) *$B^g \leq C_G(b_1)$ and b_1 is 2-central in M and in M^g.*

Proof. First (II) yields that $b_1 \in B \cap B^g$. We recall that B is abelian, hence $B, B^g \leq C_G(b_1)$.
By Sylow's Theorem there is a Sylow 2-subgroup S_0 of M such that $AB \leq S_0$. Then (II) and (XI) show that $1 \neq Z(S_0) \cap B \leq C_B(A) = C_D(A) = \langle b_1 \rangle$. This means that b_1 is a 2-central involution of M.
We apply (IV) and (XI): Then $|B^g| = 16$ and therefore A has order 8. Moreover $b_1 \in B \cap B^g$ by (II), so in particular (VI) (c) shows that $A_0 = \langle b_1 \rangle$. In addition (II) yields that A does not centralise any of the groups $B_1 = C_{D_1}(\bar{A}_1)$ or $B_2 = C_{D_2}(\bar{A}_2)$. Then we deduce from (VI) (c) that $A_1 \cap A_2 = A_0$ and so $C_A(B_1) \cap C_A(B_2) \leq A_1 \cap A_2 = A_0 = \langle b_1 \rangle$.
Let S be a Sylow 2-subgroup of M^g such that $B^g(B \cap M^g) \leq S$. According to (VII) we let $b \in B \cap M^g \cap I^*(M)$. Then $Z(S) \leq C_G(b) \leq M$ and so we see that $b_1 \in Z(S)$ as follows:
$1 \neq Z(S) \cap B^g \leq C_G(B \cap M^g) \cap A = C_A(B \cap M^g) \leq C_A(B_1) \cap C_A(B_2) = \langle b_1 \rangle$ by (III). ■

For our final contradiction we investigate $C_G(b_1)$. By (XII) we know that $B^g \leq C_G(b_1)$ and hence $C_G(b_1) \nleq M$.
Moreover (VI) (a) gives that $x \in C_G(D_1) \leq C_G(b_1)$. Together with Lemma 6.4 (e) this implies that $C_G(b_1)$ has cyclic Sylow 3-subgroups and then Lemma 3.4 (e) yields that $C_G(b_1)$ has a normal 3-complement.
We set $L := O_{3'}(C_G(b_1))$ and let $\wedge : C_G(b_1) \to C_G(b_1)/O_2(L)$ denote the natural epimorphism.
Then $O_2(L) = O_2(C_G(b_1))$ and Corollary 7.3 implies that $\hat{L}$ acts faithfully on $O_2(L)$ and $O_2(\hat{L}) = 1$. Consequently $\hat{L}$ acts faithfully on the elementary abelian 2-group $O_2(L)/\Phi(O_2(L)) =: E$.
Furthermore, (XII) and Corollary 8.14 show that $M \cap L$ and $M^g \cap L$ contain Sylow 2-subgroups of L. We conclude that $O_2(L) \leq M^g \cap M$. Therefore, if $b_0 \in D_1 \setminus M^g$ is an element as in (VI) (f), then $b_0 \notin O_2(L)$. Then (VI) (e) and (XI) imply that $|M : C_M(b_0)|_2 = |N_M(D_1) : C_{N_M(D_1)}(b_0)|_2 \leq 4$.
We recall that $O_2(\hat{L}) = 1$ and that $L \leq C_G(b_1)$ is soluble by Hypothesis 8.2. It follows that $[F(\hat{L}), \hat{b}_0] \neq 1$ and so there is a prime q such that $[O_q(\hat{L}), \hat{b}_0] \neq 1$. Let $\hat{Q}$ be an elementary abelian q-subgroup of $[F(\hat{L}), \hat{b}_0]$ that is inverted by $\hat{b}_0$. Then $q \geq 5$ and we compute that

$$\begin{array}{rcccl} |E : C_E(\hat{b}_0)| & \leq & |E : C_{O_2(L)}(\hat{b}_0)\phi(O_2(L))/\phi(O_2(L))| & = & |O_2(L) : C_{O_2(L)}(\hat{b}_0)\phi(O_2(L))| \\ & \leq & |O_2(L) : C_{O_2(L)}(\hat{b}_0)| & \leq & |O_2(L) : C_{O_2(L)}(b_0)| \\ & \leq & |M : C_M(b_0)|_2 & \leq & 4. \end{array}$$

Therefore, Lemma 2.2 (c) implies that $q \leq 5$. Now $q = 5$ and $|E : C_E(\hat{b}_0)| = 4$. Moreover, Part (b) of the same lemma shows that $\hat{Q}$ is cyclic.

Assume for a contradiction that $\hat{Q}$ is $\hat{x}$-invariant. Then it follows by Lemma 2.1 (c) that $\hat{Q} \leq C_{\hat{L}}(\hat{x}) = \widehat{C \cap L}$, because a group of order 5 does not admit an automorphism of order 3. This implies the contradiction $1 \neq \hat{Q} \leq [\widehat{C \cap L}, \hat{b}_0] \leq [\widehat{C \cap L, B}] \leq \hat{B}$.
We have that $|[E, \hat{Q}] : C_{[E,\hat{Q}]}(\hat{b}_0)| \leq |E : C_E(\hat{b}_0)| = 4$ which, together with Part (d) of Lemma 2.2, gives that $|C_{[E,\hat{Q}]}(\hat{b}_0)|^2 = |[E, \hat{Q}]| = |[E, \hat{Q}] : C_{[E,\hat{Q}]}(\hat{b}_0)| \cdot |C_{[E,\hat{Q}]}(\hat{b}_0)|$.
Then $4 \geq |[E, \hat{Q}] : C_{[E,\hat{Q}]}(\hat{b}_0)| = |C_{[E,\hat{Q}]}(\hat{b}_0)|$. In particular we see that $|[E, \hat{Q}]| = |C_{[E,\hat{Q}]}(\hat{b}_0)|^2 = 16$.
Now let $\hat{Z} := Z([O_q(\hat{L}), \hat{b}_0])$. Then $\hat{Z}$ is $\hat{x}$-invariant because $\hat{L}$ is $\hat{x}$-invariant. Thus $\hat{b}_0$ centralises $\hat{Z}$. We deduce that $[E, \hat{Q}]$ and $C_{[E,\hat{Q}]}(\hat{b}_0)$ are both $\hat{Z}$-invariant. Moreover $|C_{[E,\hat{Q}]}(\hat{b}_0)| = 4$, whence we see that $\hat{Z}$ centralises $C_{[E,\hat{Q}]}(\hat{b}_0)$.
We calculated above that $|[E, \hat{Q}]| = 16$ and therefore $C_{[E,\hat{Q}]}(\hat{b}_0) \leq [E, \hat{Q}]$ implies that $\hat{Z}$ centralises $[E, \hat{Q}]$. On the other hand $\hat{Q}$ normalises the abelian group $[E, \hat{Z}]$. So Lemma 2.1 (b) yields that $[[E, \hat{Z}], \hat{Q}] \leq C_E(\hat{Z}) \cap [E, \hat{Z}] = 1$ and it follows that $[E, \hat{Z}] = C_{[E,\hat{Z}]}(\hat{Q}) \times [[E, \hat{Z}], \hat{Q}] = C_{[E,\hat{Z}]}(\hat{Q})$.
We apply 8.1.8 of [17]: Then for every $\hat{h} \in O_q(\hat{L})$ the commutator $[\hat{h}, \hat{b}_0]$ is inverted by $\hat{b}_0$. Thus our investigation above shows that $[E, \hat{Z}] \leq C_E(\langle [\hat{h}, \hat{b}_0] \mid \hat{h} \in O_q(\hat{L}) \rangle) = C_E([O_q(\hat{L}), \hat{b}_0]) \leq C_E(\hat{Z})$.
We obtain our final contradiction: $E = [E, \hat{Z}]C_E(\hat{Z}) = C_E(\hat{Z})$. □

For the next lemma we recall that $T_0 := \langle B^g \mid g \in G \text{ and } B^g \leq T \rangle$.

Lemma 9.3. *Suppose that Hypothesis 8.2 holds and that $|B| \geq 8$.*
Let $T_1 := \langle A^g \mid A \max B,\ g \in G \text{ and } A^g \leq T \rangle$. Then T_0 is contained in T_1, and $N_G(T_0) \leq M$ or $N_G(T_1) \leq M$.

Proof. Let $g \in G$ be such that $B^g \leq T$ and let B_1 and B_2 be different maximal subgroups of B. Then $B_i^g \leq T$ and so $B_i^g \leq T_1$ for all $i \in \{1, 2\}$. This implies that $B^g = B_1^g B_2^g \leq T_1$ and in particular $T_0 \leq T_1$.
Suppose now that $T_0 \leq O_2(TP)$. Then we see that $TP = O_2(TP) \cdot N_{TP}(T_0)$ by Lemma 8.13. Thus $N_G(T_0)$ has non-cyclic Sylow 3-subgroups. We recall that $N_G(T)$ normalises T_0 by definition of T_0 and hence $x \in N_G(T_0)$. Then $N_G(T_0) \leq M$ by Lemma 6.4 (e).
We suppose that $N_G(T_0) \nleq M$. Then $T_0 \nleq O_2(TP)$ and Part (b) of Lemma 8.13 does not hold. So (a) of the lemma is true and this means that $C_G(b) \leq M$ for all $b \in B^\#$. We will refer to this by $(*)$.
Now there is some $g \in G$ such that $B^g \leq T$, but $B^g \nleq O_2(TP)$. The elementary abelian group $B^g O_2(TP)/O_2(TP)$ acts faithfully on $O_3(TP/O_2(TP))$. Thus Thompson's Dihedral Lemma (see Lemma 24.1 of [14]) provides an element $y \in P \cap O_{2,3}(TP)$ of order 3 and some $b \in B^g$ such that $y^b = y^{-1}$ and $[y, B^g] \leq \langle y \rangle O_2(PT)$. In particular y is inverted in G and therefore it is not isolated in G. We conclude that y is not conjugate to an element of $\langle x \rangle$.
We will show that y normalises $Z(T_1)$. Then, as T is x-invariant, we see that x normalises T_1 and hence it normalises $Z(T_1)$. So Lemma 6.4 (e) gives our statement $\langle x, y \rangle \leq N_G(T_1) \leq M$.
We set $L := O_2(TP)\langle y \rangle B^g$ and $B_0 := O_2(L) \cap B^g$. Then we have that $F(L/O_2(L)) = O_3(L/O_2(L)) = \langle y \rangle O_2(L)/O_2(L)$ is cyclic of order 3. As y is inverted by $b \in B^g \subseteq L$, we see that $L/O_2(L) \cong \mathcal{S}_3$ and $B^g = B_0\langle b \rangle$. Hence $|B^g : B_0| = 2$ and $|B^{gy} : B_0^y| = 2$. Furthermore we recall that $B^g \leq T$, so it follows that $T \cap L \in \mathrm{Syl}_2(L)$ and then that $O_2(L) \leq T$. We moreover define $L_1 := \langle B_0^L \rangle \leq O_2(L) \leq T$ and $Z := C_{PT}(L_1)$. Then for all $h \in L$ we have that $|B^{gh} : B_0^h| \leq 2$, and the definition of T_1 implies that $L_1 \leq T_1$ and so $Z(T_1) \leq C_T(L_1) \leq Z$.

We see that Z centralises B_0^h for every $h \in L$. Thus $Z \le C_G(B_0) \cap C_G(B_0^y)$ and we deduce from Lemma 8.6 (c) and the fact that $B_0 \max B^g$ that $Z \le M^g \cap M^{gy}$. In particular Z is a 2-group, since M is strongly 3-embedded and, by definition, Z normalises B^g and B^{gy}. Moreover $L \le N_G(L_1) \cap PT \le N_G(Z)$.
If b centralises Z, then Lemma 2.3 shows that y centralises $Z/\Phi(Z)$ and hence $[Z, y] = 1$ by Lemma 2.1 (h). In particular $Z(T_1)$ is y-invariant in this case.
Recall that $B^g = B_0\langle b\rangle$ and assume for a contradiction that $[b, Z] \ne 1$. Then $1 \ne [b, Z] \le [B^g, Z] \le B^g \cap Z$ and so $B^{gy} \cap Z = (B^g \cap Z)^y \ne 1$. Consequently there is some non-trivial element $b_0 \in B^{gy} \cap Z$. We conclude that $B_0 \le C_G(Z) \le C_G(B^{gy} \cap Z) \le C_G(b_0)$ and $(*)$ yields that $B_0 \le C_G(b_0) \le M^{gy}$. Using Lemma 9.2 and the fact that $|B^g : B_0| = 2$, we deduce that $[B^g, B^{gy}] = 1$. This implies the contradiction $1 = [b, b^y] = by^{-1}byby^{-1}by = y^4 = y$. It follows that b centralises Z. $\square$

Lemma 9.4. *Suppose that Hypothesis 8.2 holds. Let H be a proper subgroup of G such that $H \cap M$ contains x and some x-invariant Sylow 2-subgroup S of H. Suppose that every proper subgroup of H containing $\langle x\rangle \cdot S$ is a subgroup of M. Then one of the following holds:*
(a) $H \le M$ or
(b) H is soluble, it has a normal 3-complement and the Sylow 3-subgroups of H have order 3.

Proof. Suppose that $H \not\le M$. Then H has cyclic Sylow 3-subgroups containing x by Lemma 6.4 (e). Hence Lemma 3.4 (e) yields that H has a normal 3-complement. Since $H \cap C \le H \cap M$ contains a Sylow 3-subgroup of H by Lemma 3.2 and $H \not\le M$, the hypothesis of our lemma yields that $H = O_{3'}(H) \cdot \langle x\rangle$. In particular we see that $\langle x\rangle \in \mathrm{Syl}_3(H)$.
Assume for a contradiction that H is not soluble and let N be the largest normal soluble subgroup of H. Then $N \cdot S \cdot \langle x\rangle$ is a proper subgroup of H and so our hypothesis implies that $N \le M$.
If $x \in N$, then Lemma 3.4 (c) gives that $H = N \cdot C_H(x) \subseteq M \cdot C \subseteq M$. This is a contradiction. Thus $x \notin N$ and it follows that N is a $3'$-group.
Let $\hat{}: H \to H/N$ denote the natural epimorphism. Then $F(\hat{H}) = 1$, so $F^*(\hat{H}) = E(\hat{H})$ by our choice of N. Then Hypothesis 8.2 shows that $E(\hat{H})$ is simple. We apply Lemma 2.5 to the 3-nilpotent group $\hat{H}$: As C is soluble, we deduce that $E(\hat{H})$ is isomorphic to $\mathrm{Sz}(8)$ and that $\langle\hat{x}\rangle$ induces the full outer automorphism group on $E(\hat{H})$ by Lemma 2.4 (a).
Let E be the full pre-image of $E(\hat{H})$ in H. Then Lemma 2.4 (c) provides some $Q \in \mathrm{Syl}_5(E)$ such that $\hat{Q}$ is centralised by $\hat{x}$. Hence $\widehat{C_H(x)} = C_{\hat{H}}(\hat{x})$ (by Lemma 2.1 (c)) has order divisible by 5. As $\widehat{C_H(x)} \le \widehat{M \cap H}$, it follows that $\widehat{M \cap H}$ contains an element of order 5 of $\hat{E}$.
In addition $M \cap H$ contains S, which is a Sylow 2-subgroup of H, and therefore $|\hat{H} : \widehat{H \cap M}|$ is odd. In particular $|E : E \cap M|$ is odd. But $\hat{E}$ is isomorphic to $\mathrm{Sz}(8)$, so it does not have a proper subgroup of odd index containing an element of order 5, by Lemma 2.4 (e). Then we conclude that $\hat{E} \le \widehat{E \cap M}$.
This implies that $E \le H \cap M$. From $F^*(\hat{H}) = E(\hat{H}) = \hat{E}$ and the fact that $\langle\hat{x}\rangle$ induces the full outer automorphism group on $\hat{E}$, we see that $E \cdot \langle x\rangle$ is a normal subgroup of H in which $\langle x\rangle$ is a Sylow 3-subgroup. Thus a Frattini argument shows that $H = (E \cdot \langle x\rangle) \cdot N_H(\langle x\rangle) = E \cdot N_H(\langle x\rangle) \subseteq E \cdot C \subseteq M$. This is a contradiction. It follows that H is soluble with the properties stated in (b). $\square$

Lemma 9.5. *Suppose that Hypothesis 8.2 holds. Then*
(a) $|B| \le 4$, *or*
(b) *whenever H is a proper subgroup of G that contains $T\langle x\rangle$, then $H \le M$.*

Proof. Assume that $|B| > 4$ and that $T\langle x\rangle \le H \not\le M$ for some proper subgroup H of G. In addition we choose H of minimal order.

(I) *H is soluble and has a normal 3-complement. Furthermore $O(H) = 1$ and $\langle T, x\rangle$ is a Hall $\{2,3\}$-subgroup of H.*

Proof. Lemma 9.4 shows that H is soluble and has a normal 3-complement and cyclic Sylow 3-subgroups of order 3. This implies that $\langle T, x\rangle$ is a Hall $\{2,3\}$-subgroup of H, because $x \in N_G(T)$ and $T \in \mathrm{Syl}_2(G)$ by Corollary 8.14. Moreover B is not cyclic and therefore Lemma 8.6 (b) and Hypothesis 7.2 (b) give a subgroup $A \le B$ of order at least 4 such that $A^\# \subseteq I^*(M)$. Now $A \le T \le H$ whence A acts coprimely on $O(H)$. It follows that $O(H) = \langle C_{O(H)}(a) \mid a \in A^\#\rangle \le M$ by Lemma 2.1 (e) and then $[O(H), O_2(M)] \le O(H) \cap O_2(M) = 1$, because $O_2(M) \le T \le H$. Moreover $O_2(M) = F^*(M)$ by Lemma 7.4, which yields that $O(H) = 1$. ∎

(II) *There are a prime q and an x-invariant Sylow q-subgroup Q of H such that $H = Q \cdot T \cdot \langle x\rangle$, $Q \cdot O_2(H) = O_{2,q}(H) \trianglelefteq H$ and $\Phi(Q) = M \cap Q$.*

Proof. As $H \not\le M$ and $\langle T, x\rangle \le M$ is a Hall $\{2,3\}$-subgroup of H by (I), there is a prime $q \ge 5$ such that q divides $|H : H \cap M|$. We recall that H is soluble and has a normal 3-complement. Hence Lemma 2.1 (f) yields an x-invariant Sylow q-subgroup Q of H that is contained in $O_{3'}(H)$, and then the solubility of H implies that Q can be chosen such that QT is a Hall-subgroup of H. Then $Q \not\le M$ and by the minimal choice of H we get $H = Q \cdot T \cdot \langle x\rangle$. In particular $O_{2,q}(H) \ne O_2(H)$, since H has a normal 3-complement by (I).
We set $L := T \cap O_{2,q,2}(H) \in \mathrm{Syl}_2(O_{2,q,2}(H))$. A Frattini argument gives that $H = O_{2,q,2}(H) \cdot N_H(L) = O_{2,q}(H) \cdot N_H(L)$. Since $L \trianglelefteq T$ and L is normalised by x, it follows from the minimal choice of H and the fact that $\langle x, T\rangle \le N_H(L)$ that $O_{2,q}(H) \not\le M$ or $N_H(L) = H$. In the second case we conclude that $L = O_2(H)$ and from $T \cap O_{2,q,2}(H) = L$ we deduce that $T = L$. Hence $O_{2,q}(H) = O_{3'}(H) \not\le M$, too.
Again the minimal choice of H implies that $Q \in \mathrm{Syl}_q(O_{2,q}(H))$. Therefore $QO_2(H) = O_{2,q}(H) \trianglelefteq H$ and so $\Phi(Q)O_2(H) \trianglelefteq H$. We deduce, once more from the minimal choice of H, that $\Phi(Q) \le M$. Hence $M \cap Q$ is normal in Q and it is also normalised by $\langle x, T\rangle$. Furthermore Maschke's Theorem (for example 8.4.6 of [17]) provides a subgroup Q_0 of Q such that $Q_0O_2(H)$ is $\langle x, T\rangle$-invariant and such that $Q = (Q \cap M) \cdot Q_0$ and $(Q \cap M) \cap Q_0 = \Phi(Q)$. Finally the minimal choice of H shows that $H = \langle T, x\rangle Q_0$ and so $Q_0 = Q$ implies that $\Phi(Q) = M \cap Q$. ∎

(III) *Let $A_0 \le T$ be elementary abelian. Suppose that, whenever A is a maximal subgroup of A_0 and $h \in G$ is such that $A^h \le M$, then $A_0^h \in M$. Then $A_0 \le O_2(H)$.*

Proof. We assume for a contradiction that $A_0 \not\le O_2(H)$ and let $\sim\colon H \to H/O_2(H)$ denote the natural epimorphism. Then $1 \ne \tilde{A}_0$ is elementary abelian and so Lemma 2.1 (d) implies that $\tilde{Q} = \langle C_{\tilde{Q}}(\tilde{A}) \mid \tilde{A} \max \tilde{Q}\rangle$. Let A be a maximal subgroup of A_0 and let $Q_0 \le Q$ be such that $\tilde{Q}_0 = C_{\tilde{Q}}(\tilde{A})$. Then $\tilde{Q}_0$ is $\tilde{A}_0$-invariant and we see that $L := Q_0A_0O_2(H)$ is a subgroup of H. Moreover, $A_0O_2(H)$ is a Sylow 2-subgroup of L and so $O_2(L) \le A_0O_2(H) \le T \le M$. We recall that $\tilde{Q}_0 = C_{\tilde{Q}}(\tilde{A})$, so it follows that $A \le O_2(L)$. Let $h \in Q_0$. Then $\tilde{A}^h \le O_2(L) \le M$ and our hypothesis yields that $A_0^h \le M$. In particular $[Q_0, A_0] \le \langle A_0^{Q_0}\rangle \le M$ and so $[\tilde{Q}_0, \tilde{A}_0] \le \widetilde{[Q, A_0]} \le \widetilde{(Q \cap M)}$.
We now apply Lemma 2.1 (b), which together with (II) gives that

$$C_{\tilde{Q}}(\tilde{A}) = \tilde{Q}_0 = C_{Q_0}(A_0)[\tilde{Q}_0, \tilde{A}_0] \le C_{\tilde{Q}}(\tilde{A}_0)\widetilde{(Q \cap M)} = C_{\tilde{Q}}(\tilde{A}_0)\widetilde{\Phi(Q)} = C_{\tilde{Q}}(\tilde{A}_0)\Phi(\tilde{Q}).$$

It follows that $\tilde{Q} = \langle C_{\tilde{Q}}(\tilde{A}) \mid \tilde{A} \max \tilde{Q}\rangle \le C_{\tilde{Q}}(\tilde{A}_0)\Phi(\tilde{Q})$ and so $\tilde{Q} = C_{\tilde{Q}}(\tilde{A}_0)$. Moreover, we deduce from (II) that $\tilde{Q} = F^*(\widetilde{QT})$. This implies that $\tilde{A}_0 \le C_{\widetilde{QT}}(F^*(\widetilde{QT})) = F^*(\widetilde{QT}) = \tilde{Q}$. This is a contradiction, because $\tilde{Q}$ is a q-group and $\tilde{A}_0$ is a non-trivial 2-group. ∎

(IV) *$N_G(T_0) \not\le M$ and $B^\# \subseteq I^*(M)$.*

Proof. Let $g \in G$ be such that $B^g \leq T$. Then for every maximal subgroup A of B^g and every $h \in G$ such that $A^h \leq M$, we see that $A^h \leq B^{gh} \cap M$ and so $|B^{gh} : B^{gh} \cap M| \leq 2$. Thus Lemma 8.4 (a) and Lemma 9.2 yield that $B^{gh} \leq C_G(B) \leq M$. In particular (III) shows that $B^g \leq O_2(H)$.
We conclude that $T_0 \leq O_2(H)$. Moreover Lemma 8.13 yields that $H = O_2(H) \cdot N_H(T_0)$ and then the fact that $O_2(H) \leq T \leq N_G(T_0)$ implies that $T_0 \trianglelefteq H$.
In particular $H \leq N_G(T_0)$, whence $N_G(T_0) \nleq M$. Consequently Part (a) of Lemma 8.13 holds. ■

(V) *If $g \in G$ and $|B^g : B^g \cap M| \leq 4$, then $B^g \leq M$.*

Proof. Assume for a contradiction that $B^g \nleq M$ and set $A := B^g \cap M$. Then $A \neq 1$ because $|B| \geq 8$. If $B \cap B^g \neq 1$, then (IV) implies that $B^g \leq C_G(b) \leq M$ for all $b \in B^\# \cap B^g$. This is a contradiction. If $C_A(B) \neq 1$, then (IV) shows that $B \leq C_G(a) \leq M^g$ for all $a \in C_A(B)^\#$. We then deduce from Lemma 8.12 that $[B, B^g] = 1$, which gives the contradiction $B^g \leq N_G(B) = M$. Consequently $B^g \cap B = C_A(B) = 1$.
Now let $a \in A^\#$. Then $1 \neq C_B(a) \leq M^g$ by (IV) and for all $b \in C_B(a)^\#$ we see by 9.1.1 (b) of [17] that $|C_{B^g}(b)|^2 \geq |B^g| \geq 8$. As $C_{B^g}(b) \leq B^g \cap M = A$, we deduce that A is not cyclic. In particular the group $\bar{A} = A/C_M(B) \cong A$ is not cyclic. For all $c \in A^\#$ we moreover have that $[A, C_B(c)] \leq [A, M^g] \cap B \leq B^g \cap B = 1$ and hence $C_B(c) \leq C_B(A)$. This implies that $C_B(c) = C_B(A)$ for all $c \in A^\#$. We recall that $a \in A^\# \subseteq M$. Thus $\bar{a} \neq 1$, as $C_A(B) = 1$, and so 6.7.7 in [17] provides some $\bar{h} \in \bar{M}$ of odd order that is inverted by $\bar{a}$.
Assume for a contradiction that $C_{\bar{A}}(\bar{h}) \neq 1$ and let $c \in A$ be such that $1 \neq \bar{c} \in C_{\bar{A}}(\bar{h})$. Then $C_{[B,\bar{h}]}(\bar{c})$ is $\bar{h}$-invariant. We have seen that $C_{[B,\bar{h}]}(\bar{c}) = C_{[B,\bar{h}]}(c) = C_{[B,\bar{h}]}(a) = C_{[B,\bar{h}]}(\bar{a})$. Consequently Lemma 2.3 implies that $C_{[B,\bar{h}]}(\bar{c}) \leq C_{[B,\bar{h}]}(\bar{h}) = 1$. But $[B, \bar{h}]$ is a 2-group and $\bar{c}$ has order 2, which leads to the contradiction $[B, \bar{h}] = 1$.
Thus $C_{\bar{A}}(\bar{h}) = 1$. We view this statement in light of Lemma 8.8: For all $i \in \{1, ..., k\}$ and all $j \in \{1, ..., l\}$, the elements $\bar{y}_i$ and $\bar{g}_j$ are inverted by some element of $\bar{A}$ and it is true that $C_{\bar{A}}(\bar{y}_i)$ respectively $C_{\bar{A}}(\bar{g}_j)$ is a maximal subgroup of $\bar{A}$. As $\bar{A}$ is not cyclic, we deduce that $k = 0 = l$ and thus $E(\bar{M}) \neq 1$, which forces $[\bar{A}, O_3(\bar{M})] = 1$. In particular $|B| \geq 2^{12}$ by Lemma 8.11 and Lemma 8.8 (a) shows that $|B : C_B(\bar{A})| \geq |\bar{A}|^2 = |A|^2 = (|B|/|B : A|)^2$. It follows that $2^{12} \leq |B| \leq |C_B(\bar{A})| \cdot |B| \leq |B : A|^2 \leq 4^2 = 16$. This is a contradiction. ■

We now work towards a final contradiction.
Let $T_1 := \langle A^g \mid A \max B,\ g \in G \text{ and } A^g \leq T\rangle$. Then Lemma 9.3 and (IV) give that $N_G(T_1) \leq M$.
Also, let $g \in G$ and $A_0 \max B$ be such that $A_0^g \leq T$. Then for every maximal subgroup A of A_0^g and $h \in G$ such that $A^h \leq M$, we have that $|B^{gh} : B^{gh} \cap M| \leq |B^{gh} : A^h| = 4$. Thus (V) yields that $B^{gh} \leq M$. In particular $A_0^{gh} \leq M$ and so (III) implies that $A_0^g \leq O_2(H)$. In particular $T_1 \leq O_2(H)$.
Finally let $h \in Q$. Then $B^g \cap T \leq T_1 \leq O_2(H)$ and so $(B^g \cap T)^h \leq O_2(H) \leq T$. Hence it follows that $(B^g \cap T)^h = B^{gh} \cap T^h \cap T \leq B^{gh} \cap T$ and so $|B^{gh} : B^{gh} \cap T| \leq |B^{gh} : (B^g \cap T)^h| = |B^g : B^g \cap T| \leq 2$. This means that $B^{gh} \leq T_1$ and so $Q \leq N_G(T_1) \leq M$. This is a contradiction. □

Lemma 9.6. *Suppose that Hypothesis 8.2 holds. Whenever $H \leq G$ is a proper subgroup of G that contains $T\langle x\rangle$ and such that $O_2(H) \neq 1$, then $H \leq M$.*

Proof. Assume that this is false. Then there exists some proper subgroup H of G such that $T\langle x\rangle \leq H \nleq M$ and $O_2(H) \neq 1$. We choose H of minimal order.

(I) *H is soluble, $T\langle x\rangle$ is a Hall $\{2,3\}$-subgroup of H, and $H \cap M$ is a maximal subgroup of H.*

Proof. Let U be a proper subgroup of H that contains $\langle x\rangle T$. Then $O_2(H) \leq T \leq U$ and so $O_2(U) \neq 1$. It follows from the minimal choice of H that $U \leq M$. Therefore we may apply Lemma 9.4. It yields that H is soluble, that it has a normal 3-complement and that its Sylow 3-subgroups have order 3. In particular we see that $\langle x\rangle \in \mathrm{Syl}_3(H)$. Moreover $H \nleq M$ by assumption, thus $H \cap M$ is a maximal subgroup of H, and moreover $|H : M \cap H|$ is coprime to 2 and 3. ∎

(II) $|B| \leq 4$ *and* M *is soluble. In addition* $\Omega_1(Z(T))^{\#} \subseteq B^{\#} \subseteq I^*(M)$ *and* $N_H(\Omega_1(Z(T))) = H \cap M$.

Proof. Lemma 9.5 shows that $|B| \leq 4$. In particular $\bar{M}$ is isomorphic to a subgroup of $\mathcal{S}_3$. As $C_M(B)$ is soluble by Hypothesis 7.2 (a), we see that M is soluble.
From $F^*(M) = O_2(M)$ by Lemma 7.4 and from (32.4) of [3] we obtain that $\langle \Omega_1(Z(T))^M\rangle$ is an elementary abelian normal subgroup of M that is 2-reduced. It follows from Lemma 8.4 (d) that $\Omega_1(Z(T)) \leq B$. We recall that $B \trianglelefteq M$ and that $|B| \leq 4$, moreover $r(P) \geq 3$ by Hypothesis 7.2 (b). Then we see that $C_G(B)$ has non-cyclic Sylow 3-subgroups. As $\Omega_1(Z(T))$ is a subgroup of B, we conclude that $\Omega_1(Z(T))^{\#} \subseteq B^{\#} \subseteq I^*(M)$ by Lemma 8.4 (f) and that $C_G(\Omega_1(Z(T)))$ has non-cyclic Sylow 3-subgroups. Also, since $\Omega_1(Z(T))$ is normalised by x, it follows that $N_G(\Omega_1(Z(T))) \leq M$, by Lemma 6.4 (e). In particular $N_H(\Omega_1(Z(T))) \leq H \cap M$.
Assume for a contradiction that $N_H(\Omega_1(Z(T))) \neq H \cap M$. As $T\langle x\rangle \leq N_H(\Omega_1(Z(T)))$ and $T\langle x\rangle$ is a Hall $\{2,3\}$-subgroup of H by (I), there is some q-element $h \in H \cap M$ for some prime $q \geq 5$ that does not normalise $\Omega_1(Z(T))$. On the other hand $h \in M$ normalises B. We recall that $|B| \leq 4$ and $q \geq 5$, and then we see that h centralises B. Now h centralises $\Omega_1(Z(T))$, which is impossible. ∎

(III) $F^*(H) = O_2(H)$.

Proof. As $1 \neq O_2(H) \trianglelefteq T$, we find some element $c \in O_2(H) \cap \Omega_1(Z(T))^{\#}$. In particular (II) yields that $c \in B$ and $C_G(c) \leq M$. Let q be an odd prime. Then $O_q(H) \leq C_H(O_2(H)) \leq C_G(c) \leq M$ and therefore $[O_q(H), O_2(M)] \leq O_2(H) \cap O_2(M) = 1$, because $O_2(M) \leq T \leq H$.
We conclude that $O_q(H) \leq C_M(O_2(M)) = C_M(F^*(M)) \leq F^*(M) = O_2(M)$ by Lemma 7.4. But H is soluble by (I), and therefore $O_q(H) = 1$ forces $F^*(H) = O_2(H)$. ∎

We set $V := \langle \Omega_1(Z(T))^H\rangle$. Then (32.4) of [3] implies that $O_2(H/C_H(V)) = 1$ and that V is elementary abelian. Let $\hat{}: H \to H/C_H(V)$ denote the natural epimorphism.

(IV) *The group* $\hat{H}$ *has a unique minimal normal subgroup* $\hat{N}$ *that is abelian. Moreover it is true that* $C_{\hat{H}}(\hat{N}) = \hat{N}$, $\hat{H} = \hat{N} \cdot (\widehat{H \cap M})$ *and* $\hat{N} \cap (\widehat{H \cap M}) = 1$.

Proof. The group $\hat{H}$ acts on $\Omega_1(Z(T))^H = \{\Omega_1(Z(T))^h \mid h \in H\}$ in the following way:
$\hat{h} : \Omega_1(Z(T))^H \to \Omega_1(Z(T))^H$ maps $\Omega_1(Z(T))^g$ to $\Omega_1(Z(T))^{gh}$ for all $g \in H$ and $\hat{h} \in \hat{H}$. We observe that the action is transitive. The stabiliser of $\Omega_1(Z(T))$ in this action is $N_H(\Omega_1(Z(T)))C_H(V)/C_H(V)$. As $N_H(\Omega_1(Z(T))) = H \cap M$ by (II) and $H \cap M$ is a maximal subgroup of H by (I), we deduce that the action is also primitive (see for example II 1.4 of [16]). Since H is soluble, $\hat{H}$ has an abelian minimal normal subgroup $\hat{N}$. Then we apply Satz II 3.2 in [16] to obtain all the assertions. ∎

The group $\hat{N}$ from (IV) is a q-group for some odd prime q, because $O_2(\hat{H}) = 1$ (see before (IV)). Moreover $\hat{N} \cap (\widehat{M \cap H}) = 1$, so we deduce that $\hat{N} \nleq (\widehat{M \cap H})$ and then $q \neq 3$ by (I). Let $N \leq H$ be a pre-image of $\hat{N}$ such that N is a q-group for some odd prime $q \geq 5$. We investigate the natural action of $\hat{H}$ on V. Let $1 \neq V_0$ be a subgroup of $[V, \hat{N}]$ such that $\hat{H}$ acts irreducibly on V_0, and let $\hat{N}_1$,..., $\hat{N}_n$ be the maximal subgroups of $\hat{N}$. By (2.1) of [4] we see that $V_0 = V_1 \times .... \times V_n$, where $V_i = C_{V_0}(\hat{N}_i)$ for all $i \in \{1, ..., n\}$.

We let $\mathcal{O}_1$,...,$\mathcal{O}_m$ be the orbits of $\{V_1, ..., V_n\}$ under T and set $W_i = \bigtimes_{V_j \in \mathcal{O}_i} V_j$ for all $i \in \{1, ..., m\}$. Then $V_0 = W_1 \times ... \times W_m$ and, for all $j \in \{1, ..., m\}$, the 2-group W_j is invariant under T and $\hat{N}$.

(V) *$m = 1$ and n is a power of 2.*

Proof. For every $j \in \{1, ..., m\}$ there is some $v_j \in C_{W_j}(T)^{\#}$, since T and W_j are both 2-groups. We see that $C_{W_j}(T) \leq C_V(T) \leq C_T(T) = Z(T)$ and, as V is elementary abelian, it follows that $v_j \in \Omega_1(Z(T))$. The group $\Omega_1(Z(T))$ has at most three involutions by (II), and $W_i W_k \cap W_j = 1$ for all $i, k \in \{1, ..., m\}$ such that $i \neq j \neq k$. This implies that $m \leq 2$.
Assume for a contradiction that $m = 2$. Then, as x normalises T and V, we deduce that x stabilises the set $\{W_1, W_2\}$. This means that W_1 and W_2 are x-invariant. Thus W_1 and W_2 are invariant under $\langle \hat{x}, \hat{T}, \hat{N} \rangle = \hat{H}$. This is a contradiction, because $\hat{H}$ acts irreducibly on V_0. Consequently $m = 1$ and the 2-group T acts transitively on $\mathcal{O}_1 = \{V_1, ..., V_n\}$. This shows that n is a power of 2. ∎

Now $\hat{x}$ acts on the set of maximal subgroups of $\hat{N}$, so it also acts on $\{V_1, ..., V_n\}$. We recall that n is a power of 2, which means that $3 \nmid n$. Thus there is some $i \in \{1, ..., n\}$ such that $\hat{x}$ leaves V_i invariant. Without loss we may suppose that V_1 is $\hat{x}$-invariant and we set $S_1 := N_T(V_1)$. Then we have that $n = |V_1^T| = |T : S_1|$.

(VI) $|C_{V_1}(S_1)| \geq 8$.

Proof. The group $\hat{N}_1$ acts trivially on V_1 and is a $\langle \hat{x} \rangle \hat{S}_1$-invariant subgroup of $\hat{N}$. As $\langle \hat{x} \rangle \hat{S}_1$ acts coprimely on $\hat{N}$, we may apply Maschke's Theorem (8.4.6 of [17]). It provides an $\langle \hat{x} \rangle \hat{S}_1$-invariant complement $\hat{K}$ of $\hat{N}_1$ in $\hat{N}$.
Next we apply Hilfssatz II 3.11 of [16] to V_1 and $\hat{K} \langle \hat{x} \rangle \hat{S}_1 =: \hat{H}_1$. The group $\hat{K}$ is an abelian normal subgroup of $\hat{H}_1$ and we may decompose V_1 into the direct product of s subgroups, $s \in \mathbb{N}$, such that they are $\hat{K}$-isomorphic $\hat{K}$-modules. Let $l \in \mathbb{N}$ be such that $|V_1| = 2^l$ and let $r := \frac{l}{s}$. Then the Hilfssatz (from [16], see above) yields that $\hat{H}_1 / C_{\hat{H}_1}(V_1)$ is isomorphic to a subgroup of semi-linear mappings of an s-dimensional vector space over a field with 2^r elements. The elements of $C_{\hat{H}_1}(\hat{K}) / C_{\hat{H}_1}(V_1)$ are those that induce linear mappings. In particular $C_{\hat{T}_1}(\hat{K})$ centralises a 1-dimensional subspace V_1^*, which is a subgroup of order 2^r of V_1. Moreover V_1^* is $\hat{S}_1 \langle \hat{x} \rangle$-invariant. This group induces field automorphisms on V_1^*. Thus we have that $|C_{V_1}(S_1)| \geq |C_{V_1^*}(\hat{S}_1)| \geq 2^{o(x)} = 2^3$. ∎

From (VI) we obtain elements $v_1, v_2, v_3 \in C_{V_1}(S_1)$ such that $|\langle v_1, v_2, v_3 \rangle| \geq 8$. Let $S := \{s_1, ..., s_n\}$ be a set of coset representatives of T/S_1. We may choose S such that $V_1^{s_k} = V_k$ for all $k \in \{1, ..., n\}$ and such that $s_1 = 1$. Then we see for all $k \in \{1, ..., n\}$ and $j \in \{1, 2, 3\}$ that $v_j^{s_k} \in V_k$.
For all $j \in \{1, 2, 3\}$ we define $w_j := \prod_{k=1}^n v_j^{s_k} \in V$. Then the projection of $\langle w_1, w_2, w_3 \rangle$ on V_1 is equal to $\langle v_1, v_2, v_3 \rangle$. In particular $\langle w_1, w_2, w_3 \rangle$ has order at least 8. Moreover, if $t \in T$, then we calculate for all $j \in \{1, 2, 3\}$ that $w_j^t = (\prod_{k=1}^n v_j^{s_k})^t = \prod_{k=1}^n v_j^{s_k t}$. The set $S_0 := \{s_1 t, ..., s_n t\}$ is also a set of coset representatives of T/S_1. If $S_1 s_k t = S_1 s_l$ for some $k, l \in \{1, ..., n\}$, then we obtain some $t_1 \in S_1 \subseteq C_H(v_j)$ such that $s_k t = t_1 s_l$ and so $v_j^{s_k t} = v_j^{t_1 s_l} = v_j^{s_l}$. This shows that $w_j^t = \prod_{k=1}^n v_j^{s_k t} = \prod_{l=1}^n v_j^{s_l} = w_j$. Altogether $\langle w_1, w_2, w_3 \rangle$ is centralised by T. Now we obtain our final contradiction that $8 \leq |\langle w_1, w_2, w_3 \rangle| \leq |C_V(T)| \leq |\Omega_1(Z(T))| \leq |B| \leq 4$. □

10. The proof of Theorem A

Let us recall first what the aim is:

Theorem A.

Let G be a finite group and let $x \in G$ be an isolated element of order 3 such that $C_G(x)$ is soluble. Suppose further that $r_3(G) \geq 3$ and that the centraliser of every involution in every section of G is soluble. If the Z_3^*-Theorem holds in all sections of G of 3-rank 2, then $x \in Z_3^*(G)$.

Proof. Assume for a contradiction that the theorem is false and let G be a minimal counterexample. Then $x \in G$ has order 3 and is isolated in G, but $x \notin \mathrm{Z}_3^*(G)$. We show that G satisfies Hypothesis 4.1. Let $x \in H \lneq G$. Then Lemma 3.4 (a) implies that x is an isolated element of order 3 of H. If $r_3(H) \leq 2$, then Lemma 3.5 or our hypothesis gives that $x \in \mathrm{Z}_3^*(H)$. If $r_3(H) \geq 3$, then $C_H(x)$ is soluble, because $C_G(x)$ is soluble. In addition the centraliser of every involution in every section of H is soluble, because a section of H is also a section of G. Together the minimal choice of G implies that $x \in \mathrm{Z}_3^*(H)$. Suppose now that $1 \neq N \trianglelefteq G$. Then Lemma 3.4 (d) implies that Nx is an isolated element of order 3 in G/N. Thus again Lemma 3.5 or our hypothesis gives that $Nx \in \mathrm{Z}_3^*(G/N)$ if $r_3(G/N) \leq 2$. If $r_3(G/N) \geq 3$, then let D be the full pre-image of $C_{G/N}(Nx)$ in G. Then a Frattini argument shows that $D = N\langle x\rangle \cdot N_D(R) = N \cdot N_D(R)$ for some Sylow 3-subgroup R of $N\langle x\rangle$ containing x. From Lemma 3.4 (b) we moreover see that $N_G(R) \leq C_G(x)$ and so $D = N \cdot C_D(x)$. This implies that $C_{G/N}(Nx) = D/N \cong C_D(x)/C_N(x)$ is soluble. In addition the centraliser of every involution in every section of G/N is soluble, because every section of G/N is isomorphic to a section of G. Thus the minimal choice of G implies that $Nx \in \mathrm{Z}_3^*(G/N)$.
It follows that Hypothesis 4.1 is satisfied for $p = 3$ and hence Hypothesis 5.1 holds. Moreover our additional hypotheses guarantee that Hypothesis 6.1 and Hypothesis 7.2 hold. In conclusion we see from Lemma 6.3 and Lemma 6.4 (c) that we may suppose that Hypothesis 8.2 holds. By Corollary 8.14, the group T from Hypothesis 8.2 is a Sylow 2-subgroup of G, and then Lemma 9.6 implies that (*) of Satz 3 of [21] is true. Now this theorem forces G to be isomorphic to $\mathrm{PSL}_3(3)$, to $\mathrm{PSL}_2(r)$ for some prime power r, or to $\mathrm{PSU}_3(q)$ or $\mathrm{Sz}(q)$ for some power q of 2. In Theorem 4.4 we have seen that this is impossible. □

Concluding remarks

The solubility hypotheses can be relaxed in many cases, unfortunately at the price of much more complexity and technical arguments. On the other hand, it might be replaced by a stronger hypothesis for sections of a minimal counterexample in future work. At the moment we believe that, for more substantial steps towards a general Z_p^*-Theorem, it will be necessary to invoke some kind of hypothesis relying on the full Classification of Finite Simple Groups. We have stated and proved many general statements at the beginning of this article that we hope will be useful to anyone attempting to attack the Z_p^*-Theorem in more generality. In the meantime we aim for progress by extending and refining existing methods.

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